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The γ-coefficients of Brändén's (p, q)-Eulerian polynomials and André permutations

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Abstract. In 2008 Brändén proved a (p, q)-analogue of the γ-expansion formula for Eulerian polynomials and conjectured the divisibility of the γ-coefficient $\gamma_{n,k}(p, q)$ by $(p+q)^k$. As a follow-up, in 2012 Shin and Zeng showed that the fraction $\gamma_{n,k}(p, q)/(p + q)^k$ is a polynomial in $\mathbb{N}[p, q]$. The aim of this paper is to give a combinatorial interpretation of the latter polynomial in terms of André permutations, a class of objects first defined and studied by Foata, Schützenberger and Strehl in the 1970s. It turns out that our result provides an answer to a recent open problem of Han, which was the impetus of this paper.

Keywords: Eulerian polynomials, γ-coefficients, André permutations.

1 Introduction

The Euler number E_n, namely the coefficient of $x^n/n!$ in the expansion of $\sec(x) + \tan(x)$, is well studied and has many combinatorial interpretations and different refinements; see [4, 18, 7, 16, 9]. It was André who first proved that E_n is the number of alternating permutations $a_1 \dots a_n$ of $12 \dots n$, i.e., $a_1 > a_2 < \cdots$. Among the many remarkable identities for the Euler numbers there is the less known J-type continued fraction

$$\sum_{n=0}^{\infty} E_{n+1} x^n = \cfrac{1}{1 - x - \cfrac{x^2}{1 - 2x - \cfrac{3x^2}{1 - 3x - \cfrac{6x^2}{1 - 4x - \cfrac{10x^2}{1 - \cdots}}}}}. \tag{1.1}$$

Recently, Han [8] considered a q-version of (1.1) and asked for a combinatorial interpretation for the corresponding q-Euler numbers $E_n(q)$ (see (1.3) below). Motivated by

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Han's question, we shall study the more general polynomials $D_n(p,q,t)$ defined by the following continued fraction, which is a (p,q)-analogue of (1.1):

$$\sum_{n=0}^{\infty} D_{n+1}(p,q,t)x^n = \cfrac{1}{1-x-\cfrac{\binom{2}{2}_{p,q}t\,x^2}{1-[2]_{p,q}x-\cfrac{\binom{3}{2}_{p,q}t\,x^2}{1-[3]_{p,q}x-\cfrac{\binom{4}{2}_{p,q}t\,x^2}{1-[4]_{p,q}x-\cfrac{\binom{5}{2}_{p,q}t\,x^2}{1-\cdots}}}}}, \tag{1.2}$$

where the (p,q)-analogue of n is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p-q} = \sum_{i+j=n-1} p^i q^j \qquad (n \in \mathbb{N})$$

and the (p,q)-analogue of the binomial coefficient $\binom{n}{k}$ is defined by

$$\binom{n}{k}_{p,q} = \frac{[n]_{p,q}\dots[n-k+1]_{p,q}}{[1]_{p,q}\dots[k]_{p,q}} \qquad (0 \le k \le n).$$

Comparing (1.1) and (1.2) yields that

$$D_n(1,1,1) = E_n \qquad (n \ge 1).$$

The q-Euler number $E_n(q)$ of Han [8] can be expressed as

$$E_n(q) := D_n(1,q,1) = D_n(q,1,1) \qquad (n \ge 1). \tag{1.3}$$

The first few values of $D_n(p,q,t)$ are $D_1(p,q,t) = D_2(p,q,t) = 1$. It turns out that the polynomials $D_n(p,q,t)$ are related to the γ-coefficients of Brändén's (p,q)-analogue of Eulerian polynomials [2]. In this paper we shall interpret $D_n(p,q,t)$ in terms of *André permutations*, which were introduced and studied by Foata, Schützenberger and Strehl [4, 6, 5] in the 1970s. There are three ingredients in our proof: the connection of these polynomials with the γ-coefficients of Brändén's (p,q)-analogue of Eulerian polynomials [2], Shin-Zeng's continued fraction expansion of the γ-coefficients of generalized Eulerian polynomials [16] and a new action on the permutations without double descents.

For a permutation $\sigma := \sigma_1\sigma_2\dots\sigma_n$ of $[n]$, the *descent number* $\operatorname{des}\sigma$ is the number of descent positions, i.e. $i < n$ such that $\sigma_i > \sigma_{i+1}$, and the *excedance number* $\operatorname{exc}\sigma$ is the number of excedance positions, i.e. $i \in [n]$ such that $\sigma_i > i$. Let $\mathfrak{S}_n$ be the set of permutations of $[n] := \{1,\dots,n\}$. Thanks to the work of MacMahon [10] and Riordan [14] we can define the Eulerian polynomials $A_n(t)$ by

$$A_n(t) = \sum_{\sigma\in\mathfrak{S}_n} t^{\operatorname{des}\sigma} = \sum_{\sigma\in\mathfrak{S}_n} t^{\operatorname{exc}\sigma}.$$

The following γ-decompositions for $A_n(t)$ are well-known [4, Section 4].

Theorem 1.1 (Foata and Schützenberger)**.**

$$A_n(t) = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_{n,k} t^k (1+t)^{n-1-2k} \tag{1.4}$$

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} 2^k d_{n,k} t^k (1+t)^{n-1-2k}, \tag{1.5}$$

where $\gamma_{n,k} = 2^k d_{n,k}$ and $d_{n,k}$ are positive integers satisfying the recurrence

$$and_{1,0} = 1 \quad and\ for\ n \geq 2,\ k \geq 0,$$
$$\mathrm{d}_{n,k} = (k+1)\mathrm{d}_{n-1,k} + (n-2k)\mathrm{d}_{n-1,k-1}. \tag{1.6}$$

Moreover, the sum $\sum_k d_{n,k}$ is precisely the Euler number E_n.

In the last two decades even though many refinements of (1.4) have been found in combinatorics and geometry (see [13, 12, 1, 15]), similar extension of (1.5) does not seem to be known. In this paper we will provide two refinements of (1.5) (see (1.11) and (1.15)).

Definition 1.2. For a permutation $\sigma = \sigma_1 \ldots \sigma_n$ of $[n]$ with $\sigma_0 = \sigma_{n+1} = 0$, the entry σ_i is

- a *peak* if $\sigma_{i-1} < \sigma_i$ and $\sigma_i > \sigma_{i+1}$;
- a *valley* if $\sigma_{i-1} > \sigma_i$ and $\sigma_i < \sigma_{i+1}$;
- a *double ascent* if $\sigma_{i-1} < \sigma_i$ and $\sigma_i < \sigma_{i+1}$;
- a *double descent* if $\sigma_{i-1} > \sigma_i$ and $\sigma_i > \sigma_{i+1}$.

Let $\operatorname{pk}\sigma$ (resp. $\operatorname{val}\sigma$, $\operatorname{da}\sigma$, $\operatorname{dd}\sigma$) denote the number of peaks (resp. valleys, double ascents, double descents) in σ. Note that $\operatorname{des}\sigma = \operatorname{val}\sigma + \operatorname{dd}\sigma$ and $\operatorname{pk}\sigma = \operatorname{val}\sigma + 1$. Let $\mathcal{G}_{n,k} = \{\sigma \in \mathfrak{S}_n : \operatorname{val}\sigma = k, \operatorname{dd}\sigma = 0\}$.

Definition 1.3. For a permutation σ of $[n]$, let $\sigma_{[k]}$ be the *subword* of σ consisting of $1, \ldots, k$ in the order they appear in σ. Then, the permutation σ is an André permutation if $\sigma_{[k]}$ has no double descents (and ends with an ascent) for all $1 \leq k \leq n$.

Let $\mathfrak{D}_n$ be the set of André permutations of $[n]$ and let $\mathfrak{D}_{n,k}$ be the set of André permutations of $[n]$ with k descents.

Proposition 1.4 ([4, 5])**.** *The coefficients $\gamma_{n,k}$ and $d_{n,k}$ equal the cardinalities of $\mathcal{G}_{n,k}$ and $\mathfrak{D}_{n,k}$, respectively.*

For $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$, the statistic $(31\text{-}2)\,\sigma$ is the number of pairs (i,j) such that $2 \le i < j \le n$ and $\sigma_{i-1} > \sigma_j > \sigma_i$. Similarly, the statistic $(2\text{-}13)\,\sigma$ is the number of pairs (i,j) such that $1 \le i < j \le n-1$ and $\sigma_{j+1} > \sigma_i > \sigma_j$. In 2008 Brändén [2] defined a (p,q)-analogue of Eulerian polynomials and proved a (p,q)-analogue of (1.4). In this paper we shall use the following variant of Brändén's (p,q)-Eulerian polynomials in [16]

$$A_n(p,q,t) := \sum_{\sigma \in \mathfrak{S}_n} p^{(2\text{-}13)\,\sigma} q^{(31\text{-}2)\,\sigma} t^{\operatorname{des}\sigma}. \tag{1.7}$$

For $0 \le k \le (n-1)/2$ define the (p,q)-analogue of $\gamma_{n,k}$ and $d_{n,k}$ in (1.4) and (1.5) by

$$\gamma_{n,k}(p,q) = \sum_{\sigma \in \mathcal{G}_{n,k}} p^{(2\text{-}13)\,\sigma} q^{(31\text{-}2)\,\sigma}, \tag{1.8}$$

$$d_{n,k}(p,q) = \sum_{\sigma \in \mathfrak{D}_{n,k}} p^{(2\text{-}13)\,\sigma} q^{(31\text{-}2)\,\sigma - k}. \tag{1.9}$$

Our main results are the following two theorems.

Theorem 1.5. *We have*

$$A_n(p,q,t) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,k}(p,q) t^k (1+t)^{n-1-2k} \tag{1.10}$$

$$= \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (p+q)^k \mathrm{d}_{n,k}(p,q) t^k (1+t)^{n-1-2k}. \tag{1.11}$$

Remark 1.6. An equivalent γ-expansion of (1.10) was proved by Brändén [2] using the *modified Foata-Stehl* action. The divisibility of $\gamma_{n,k}(p,q)$ by $(p+q)^k$ was conjectured by Brändén (*op.cit.*) and proved by Shin and Zeng [16] using the combinatorial theory of continued fractions.

Theorem 1.7. *We have*

$$D_n(p,q,t) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \mathrm{d}_{n,k}(p,q) t^k \tag{1.12}$$

$$= \sum_{\sigma \in \mathfrak{D}_n} p^{(2\text{-}13)\,\sigma} q^{(31\text{-}2)\,\sigma - \operatorname{des}\sigma} t^{\operatorname{des}\sigma}. \tag{1.13}$$

Remark 1.8. It is not difficult to see that $(31\text{-}2)\,\sigma \ge \operatorname{des}\sigma$ for any $\sigma \in \mathfrak{D}_n$, see (iii) of Proposition 2.2.

Combining Theorem 1.5 with Theorem 1 in [17], which is (1.14) below, we derive a q-analogue of (1.4) and (1.5).

Corollary 1.9. *We have*

$$\sum_{\sigma\in\mathfrak{S}_n} q^{(\mathrm{inv}-\mathrm{exc})\sigma} t^{\mathrm{exc}\,\sigma} = \sum_{k=0}^{\lfloor (n-1)/2\rfloor} \gamma_{n,k}(q^2,q) t^k (1+t)^{n-1-2k} \tag{1.14}$$

$$= \sum_{k=0}^{\lfloor (n-1)/2\rfloor} (1+q)^k \mathrm{d}_{n,k}(q) t^k (1+t)^{n-1-2k}, \tag{1.15}$$

where

$$\mathrm{d}_{n,k}(q) = \sum_{\sigma\in\mathfrak{D}_{n,k}} q^{2\,(\text{2-13})\,\sigma + (\text{31-2})\,\sigma}.$$

By (1.3) and Theorem 1.7 we derive two interpretations for Han's q-Euler numbers[8].

Corollary 1.10. *We have*

$$E_n(q) = \sum_{\sigma\in\mathfrak{D}_n} q^{(\text{2-13})\,\sigma} \tag{1.16}$$

$$= \sum_{\sigma\in\mathfrak{D}_n} q^{(\text{31-2})\,\sigma - \mathrm{des}\,\sigma}. \tag{1.17}$$

In Section 3 we shall give a simple sum formula for $D_n(1,-1,t)$ (cf. Theorem 3.6).

2 Proof outlines of main Theorems

2.1 Some basic definitions and results

The following definition was given as a lemma in [6, Lemma 1].

Definition 2.1. Let $w = x_1x_2...x_n$ ($n > 0$) be a permutation and x be one of the letters x_i ($1 < i < n$). Then w has a unique factorization (w_1, w_2, x, w_4, w_5) of length 5, called its x-factorization, which is characterized by the three properties

(i) w_1 is empty or its last letter is less than x;

(ii) w_2 (resp. w_4) is empty or all its letters are greater than x;

(iii) w_5 is empty or its first letter is less than x.

We can charactrize André permutations in terms of x-factorization [4].

Proposition 2.2. *A permutation $\sigma \in \mathfrak{S}_n$ is an André permutation if it is empty or satisfies the following:*

(i) *σ has no double descents,*

(ii) *$n-1$ is not a descent position, i.e. $\sigma_{n-1} < \sigma_n$,*

(iii) *If σ_i is a valley of σ with σ_i-factorization $(w_1, w_2, \sigma_i, w_4, w_5)$, then $\min(w_2) > \min(w_4)$, i.e., the minimum letter of w_2 is larger than the minimum letter of w_4.*

The next theorem follows from the work of [2, 5].

Theorem 2.3. *For any $\tilde{\sigma} \in \mathfrak{S}_n$ without double decent, we have*

$$\sum_{\sigma\in \mathsf{Orb}(\tilde{\sigma})} p^{(2\text{-}13)\,\sigma} q^{(31\text{-}2)\,\sigma} t^{\operatorname{des}\sigma} = p^{(2\text{-}13)\,\tilde{\sigma}} q^{(31\text{-}2)\,\tilde{\sigma}} t^{\operatorname{des}\tilde{\sigma}} (1+t)^{n-1-2\operatorname{des}\tilde{\sigma}}.$$

Let $A_n(p,q,t,u,v,w)$ be the generalized Eulerian polynomials defined by

$$A_n(p,q,t,u,v,w) := \sum_{\sigma\in\mathfrak{S}_n} p^{(2\text{-}13)\,\sigma} q^{(31\text{-}2)\,\sigma} t^{\operatorname{des}\sigma} u^{\operatorname{da}\sigma} v^{\operatorname{dd}\sigma} w^{\operatorname{val}\sigma}. \tag{2.1}$$

As $\operatorname{des} = \operatorname{val} + \operatorname{dd}$ we derive the following generalization of (1.10) from Theorem 2.3. This was first proved in [16] by using combinatorial theory of continued fractions.

Corollary 2.4. *For the γ-coefficients $\gamma_{n,k}(p,q)$ in (1.8) we have*

$$A_n(p,q,t,u,v,w) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,k}(p,q)(tw)^k (u+vt)^{n-1-2k}. \tag{2.2}$$

2.2 New action on permutations without double descent

Let DD_n be the set of permutations of $[n]$ without double descent. For any permutation $\sigma \in \mathrm{DD}_n$ and $x \in [n]$ we shall identify σ with its x-factorization, i.e., $\sigma = (w_1, w_2, x, w_4, w_5) = w_1\, w_2\, x\, w_4\, w_5$, and let $y_1 := \min(w_2)$, $y_2 := \min(w_4)$. A valley x of σ is said to be

- *good* (resp. *bad*) if $y_1 > y_2$ (resp. $y_1 < y_2$);
- of *type I* if $\min(y_1, y_2)$ is a peak or double ascent,
- of *type II* if $\min(y_1, y_2)$ is a valley.

We denote by $\mathsf{Val}\,\sigma$ the set of valleys of σ.

Proposition 2.5. *Let $\sigma \in \mathrm{DD}_n$ and $x \in \mathsf{Val}\,\sigma$ with $y = \min(y_1, y_2)$.*

(i) *If y is a peak, then $w_4 = y$ (resp. $w_2 = y$) if $y_1 > y_2$ (resp. $y_1 < y_2$).*

(ii) If y is a double ascent, then $w_4 = yw_4''$ (resp. $w_2 = yw_2''$) with $w_2'', w_4'' \neq \epsilon$ if $y_1 > y_2$ (resp. $y_1 < y_2$).

(iii) If y is a valley, then $w_4 = w_4'yw_4''$ (resp. $w_2 = w_2'yw_2''$) with $w_2', w_2'', w_4', w_4'' \neq \epsilon$ if $y_1 > y_2$ (resp. $y_1 < y_2$).

Definition 2.6. For $\sigma \in \mathrm{DD}_n$ and each $x \in \mathsf{Val}\,\sigma$ with $y = \min(y_1, y_2)$, we define its transform $\varphi(\sigma, x)$ as follows:

(i) If y is a peak, then

$$\varphi(\sigma,x) = \begin{cases} (w_1, y, x, w_2, w_5) & \text{if} \quad y = y_2, \\ (w_1, w_4, x, y, w_5) & \text{if} \quad y = y_1. \end{cases}$$

(ii) If y is a double ascent, then

$$\varphi(\sigma,x) = \begin{cases} (w_1, yw_2, x, w'', w_5) & \text{if} \quad y = y_2 \text{ and } w_4 = yw'', \\ (w_1, w'', x, yw_4, w_5) & \text{if} \quad y = y_1 \text{ and } w_2 = yw''. \end{cases}$$

(iii) If y is a valley, then

$$\varphi(\sigma,x) = \begin{cases} (w_1, w_2yw', x, w'', w_5) & \text{if} \quad y = y_2 \text{ and } w_4 = w'yw'', \\ (w_1, w', x, w''yw_4, w_5) & \text{if} \quad y = y_1 \text{ and } w_2 = w'yw'' \end{cases}$$

with $w', w'' \neq \epsilon$.

Obviously this transformation switches y from left to right or right to left of x and $\varphi(\varphi(\sigma, x), x) = \sigma$. We record the basic properties of this transformation in the following proposition.

Proposition 2.7. *If $\sigma \in \mathrm{DD}_{n,k}$ and $x \in \mathsf{Val}\,\sigma$, then $\varphi(\sigma, x) \in \mathrm{DD}_{n,k}$ and*

$$\begin{aligned} (\text{2-13})\,\varphi(\sigma,x) &= \begin{cases} (\text{2-13})\,\sigma + 1 & \textit{if } x \textit{ is good} \\ (\text{2-13})\,\sigma - 1 & \textit{if } x \textit{ is bad;} \end{cases} \\ (\text{31-2})\,\varphi(\sigma,x) &= \begin{cases} (\text{31-2})\,\sigma - 1 & \textit{if } x \textit{ is good} \\ (\text{31-2})\,\sigma + 1 & \textit{if } x \textit{ is bad.} \end{cases} \end{aligned} \tag{2.3}$$

Next we define the transform $\varphi(\sigma, S)$ for any subset S of $\mathsf{Val}(\sigma)$ with $\sigma \in \mathrm{DD}_n$.

Definition 2.8. Let $\sigma \in \mathrm{DD}_n$. For any $S \subseteq \mathsf{Val}\,\sigma$, let $\{S_1, S_2\}$ be the partition of S such that

(1) S_1 is the subset of S consisting of valleys of type I, say $i_1, \ldots, i_l$;

(2) S_2 is the subset of S consisting of valleys of type II, say $j_k < \cdots < j_2 < j_1$.

Define the transforms

$$\begin{aligned}\varphi(\sigma, S_1) &= \varphi(i_l, \ldots, \varphi(i_2, \varphi(i_1, \sigma))),\\ \varphi(\sigma, S_2) &= \varphi(j_k, \ldots, \varphi(j_2, \varphi(j_1, \sigma))),\\ \varphi(\sigma, S) &= \varphi(\varphi(\sigma, S_1), S_2).\end{aligned}$$

Remark 2.9. The image $\varphi(\sigma, S_1)$ is independent of the order of $i_1, \ldots, i_l$ while $\varphi(\sigma, S_2)$ is defined for the elements of S_2 in the decreasing order $j_1 > j_2 > \ldots > j_1$.

Proposition 2.10. *If $\sigma \in \mathfrak{D}_{n,k}$ and $S \subseteq \mathsf{Val}(\sigma)$, then $\tau := \varphi(\sigma, S) \in \mathrm{DD}_{n,k}$ is well defined and*

$$S = \{x \in \mathsf{Val}(\tau) \mid x \text{ is a bad guy}\}. \tag{2.4}$$

For any set S we denote by 2^S the set of all subsets of S. In what follows, for $\sigma \in \mathrm{DD}_{n,k}$ we will identify $\mathsf{Val}(\sigma)$ with $[k]$ under the map $a_i \mapsto i$ for $i \in [k]$ if $\mathsf{Val}(\sigma)$ consists of $a_1 < a_2 < \ldots < a_k$, and identify any subset $S \in \mathsf{Val}(\sigma)$ with its image $S' \in 2^{[k]}$. Thus we will use $2^{[k]}$ instead of $2^{\mathsf{Val}(\sigma)}$.

Proposition 2.11. *The map $\varphi : \mathfrak{D}_{n,k} \times 2^{[k]} \to \mathcal{G}_{n,k}$ is a bijection such that for $(\sigma, S) \in \mathfrak{D}_{n,k} \times 2^{[k]}$ we have*

$$\begin{aligned}(\text{2-13})\,\sigma + |S| &= (\text{2-13})\,\varphi(\sigma, S),\\ (\text{31-2})\,\sigma - |S| &= (\text{31-2})\,\varphi(\sigma, S).\end{aligned} \tag{2.5}$$

2.3 Proof of Theorem 1.5

Clearly (1.10) is a special case of Corollary 2.4, and (1.11) is equivalent to

$$(p+q)^k \sum_{\sigma \in \mathfrak{D}_{n,k}} p^{(\text{2-13})\,\sigma} q^{(\text{31-2})\,\sigma - k} = \sum_{\sigma \in \mathrm{DD}_{n,k}} p^{(\text{2-13})\,\sigma} q^{(\text{31-2})\,\sigma}. \tag{2.6}$$

As $(p+q)^k = \sum_{S \in 2^{[k]}} p^{|S|} q^{k-|S|}$ we can rewrite the above identity as

$$\sum_{(\sigma,S) \in \mathfrak{D}_{n,k} \times 2^{[k]}} p^{(\text{2-13})\,\sigma + |S|} q^{(\text{31-2})\,\sigma - |S|} = \sum_{\sigma \in \mathrm{DD}_{n,k}} p^{(\text{2-13})\,\sigma} q^{(\text{31-2})\,\sigma}.$$

The result follows from Proposition 2.11. □

2.4 Proof of Theorem 1.7

We shall use the J-type continued fraction as a formal power series defined by

$$\sum_{n=0}^{\infty} \mu_n t^n = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{1 - \cdots}}},$$

where (b_n) and (λ_{n+1}) $(n \geq 0)$ are two sequences in a commutative ring. When $b_n = 0$ we obtain the S-type continued fraction:

$$\sum_{n=0}^{\infty} \mu_n t^n = \cfrac{1}{1 - \cfrac{\lambda_1 t}{1 - \cfrac{\lambda_2 t}{1 - \cdots}}}.$$

Recall the following continued fraction expansion formula from [16, (28)]:

$$\sum_{n\geq 1} A_n(p,q,t,u,v,w) x^{n-1} = \cfrac{1}{1 - (u+tv)[1]_{p,q} x - \cfrac{[1]_{p,q}[2]_{p,q} twx^2}{1 - (u+tv)[2]_{p,q} x - \cfrac{[2]_{p,q}[3]_{p,q} twx^2}{\cdots}}} \tag{2.7}$$

with $b_n = (u+tv)[n+1]_{p,q}$ and $\lambda_n = [n]_{p,q}[n+1]_{p,q} tw$.

By Theorem 1.5 and substituting (t,u,v,w) with $(p+q,0,1,t)$ in (2.2), we obtain

$$A_n(p,q,p+q,0,1,t) = (p+q)^{n-1} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} d_{n,k}(p,q) t^k.$$

Thus, substituting (t,u,v,w) with $(p+q,0,1,t)$ in (2.7) and replacing x by $x/(p+q)$ we obtain the same continued fraction in (1.2). This proves (1.12). □

3 An explicit formula for $D_n(1,-1,t)$

A *Motzkin path* of length n is a sequence of points $\eta := (\eta_0, \ldots, \eta_n)$ in the integer plane $\mathbb{Z} \times \mathbb{Z}$ such that

- $\eta_0 = (0,0)$ and $\eta_n = (n,0)$,

- $\eta_i - \eta_{i-1} \in \{(1,0),(1,1),(1,-1)\}$,
- $\eta_i := (x_i, y_i) \in \mathbb{N} \times \mathbb{N}$ for $i = 0, \ldots, n$.

In other words, a Motzkin path of length n is a lattice path starting at $(0,0)$, ending at $(n,0)$, and never going below the x-axis, consisting of up-steps $\mathsf{U} = (1,1)$, level-steps $\mathsf{L} = (1,0)$, and down-steps $\mathsf{D} = (1,-1)$. Let $\mathcal{MP}_n$ be the set of Motzkin paths of length n. Clearly we can identify Motzkin paths of length n with words w on $\{\mathsf{U},\mathsf{L},\mathsf{D}\}$ of length n such that all prefixes of w contain no more D's than U's and the number of D's equals the number of $\mathsf{D}'s$. The height of a step (η_i, η_{i+1}) is the coordinate of the starting point η_i. Given a Motzkin path $p \in \mathcal{MP}_n$ and two sequences (b_i) and (λ_i) of a commutative ring R, we weight each up-step by 1, and each level-step (resp. down-step) at height i by b_i (resp. λ_i) and define the weight $w(p)$ of p by the product of the weights of all its steps. The following result of Flajolet [3] is our starting point.

Lemma 3.1 (Flajolet)**.** *We have*

$$\sum_{n=0}^{\infty}\left(\sum_{p\in\mathcal{MP}_n} w(p)\right)t^n = \cfrac{1}{1-b_0t-\cfrac{\lambda_1t^2}{1-b_1t-\cfrac{\lambda_2t^2}{1-b_2t-\cdots}}}.$$

A Motzkin path without level-steps is called a *Dyck path*, and a Motzkin path without level-steps at odd height is called an *André path*. We denote by $\mathcal{AP}_{n,k}$ the set of André paths of half-length n with k level-steps, and $\mathcal{DP}_n$ the set of Dyck paths of half length n.

Lemma 3.2. *Let $b_i = 0$ $(i \geq 0)$ and $\lambda_i = \lfloor \frac{i+1}{2} \rfloor$ $(i \geq 1)$. Then*

$$n! = \sum_{p\in\mathcal{MP}_n} w(p).$$

In other words, the factorial $n!$ is the generating polynomial of $\mathcal{DP}_n$.

Remark 3.3. A bijective proof of Euler's formula (3.2) is known, see [11, (4.9)].

Lemma 3.4. *Let $b_{2i} = 1$, $b_{2i+1} = 0$ $(i \geq 0)$ and $\lambda_k = \lfloor \frac{k+1}{2} \rfloor t$ $(i \geq 1)$. Then*

$$D_{n+1}(1,-1,t) = \sum_{p\in\mathcal{AP}_n} w(p).$$

In other words, the polynomial $D_{n+1}(1,-1,t)$ is the generating polynomial of André paths of length n.

Let

$$\mathcal{Y}_{n,k} := \{(y_1, \ldots, y_{k+1}) \in \mathbb{N}^{k+1} : y_1 + \cdots + y_{k+1} = n - 2k\}.$$

Lemma 3.5. *For $0 \le k \le \lfloor n/2 \rfloor$, there is an explicit bijection $\psi : \mathcal{AP}_{n,n-2k} \to \mathcal{Y}_{n,k} \times \mathcal{DP}_k$ such that if $\psi(u) = (y,p)$ with for $u \in \mathcal{AP}_{n,n-2k}$ and $(y,p) \in \mathcal{Y}_{n,k} \times \mathcal{DP}_k$ then $w(u) = w(p)$, where the weight is associated to the sequences (b_i) and (λ_i) with $b_{2i} = 1$, $b_{2i+1} = 0$ $(i \ge 0)$, and $\lambda_k = \lfloor \frac{k+1}{2} \rfloor t$ $(i \ge 1)$.*

Proof. Since an André path (word) on $\{\mathsf{U},\mathsf{D},\mathsf{L}\}$ has only level-steps at even height and starts from height 0, so the subword between two consecutive level-steps L's must be of even length and is a word on the alphabet $\{\mathsf{UU},\mathsf{DD},\mathsf{UD},\mathsf{DU}\}$. Thus, any André word $u \in \mathcal{AP}_{n,n-2k}$ can be written in a unique way as follows:

$$u = \mathsf{L}^{y_1} w_1 \mathsf{L}^{y_2} w_2 \dots w_k \mathsf{L}^{y_{k+1}} \quad \text{with} \quad w_i \in \{\mathsf{UU},\mathsf{DD},\mathsf{UD},\mathsf{DU}\}.$$

Let $y := (y_1, \dots, y_{k+1})$ and $p := w_1 \dots w_k$. As the path p is obtained by removing out all the level-steps L's from the André path u, each step in p keeps the same height in u, and $(y,p) \in \mathcal{Y}_{n,k} \times \mathcal{DP}_k$, Let $\psi(u) = (y,p)$. It is clear that this is the desired bijection. □

Theorem 3.6. *For $n \ge 1$ we have*

$$D_n(1,-1,t) = \sum_{k=0}^{n-1} \binom{n-1-k}{k} k! t^k. \tag{3.1}$$

Proof of Theorem 3.6. By Lemmas 3.4 and 3.5 we have

$$D_{n+1}(1,-1,t) = \sum_{k \ge 0} \sum_{(y,p) \in \mathcal{Y}_{n,k} \times \mathcal{DP}_k} w(p).$$

Since the cardinality of $\mathcal{Y}_{n,k}$ is $\binom{n-k}{k}$, and the generating polynomial of $\mathcal{DP}_k$ is equal to $k!t^k$ by Lemma 3.2, summing over all $0 \le k \le \lfloor n/2 \rfloor$ we obtain Equation (3.1). □

Remark 3.7. The full-length paper for this extended abstract is available at [11].

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