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Extended Schur functions and 0-Hecke modules

Dominic Searles*[1]

[1]*Department of Mathematics and Statistics, University of Otago, Dunedin, New Zealand*

Abstract. Three bases of noncommutative symmetric functions have been described as Schur-like: the immaculate symmetric functions, the noncommutative Schur functions, and the shin functions. Each of these has a dual basis in quasisymmetric functions. Dual bases of the former two have been given a representation-theoretic interpretation in terms of 0-Hecke modules. We complete the picture by constructing 0-Hecke modules whose quasisymmetric characteristics are the extended Schur functions, the dual basis to the shin functions. These modules are indecomposable.

Keywords: dual immaculate quasisymmetric functions, shin functions, quasisymmetric Schur functions, extended Schur functions, extended tableaux, 0-Hecke algebra.

1 Introduction

The algebra Sym of symmetric functions has many bases of interest in algebraic combinatorics. Of central importance is the basis of Schur functions, due to its myriad applications including to geometry of Grassmannians and representation theory of the symmetric and general linear groups.

The algebra NSym of noncommutative symmetric functions generalizes Sym. Three bases of NSym of particular interest due to their connection with Schur functions are the *immaculate basis* introduced in [2], the *noncommutative Schur basis* introduced in [4], and the *shin basis* introduced in [6]. These three bases are described in [5] as the *canonical Schur-like* bases of NSym.

The algebra QSym of quasisymmetric functions is dual (as Hopf algebras) to NSym, and contains Sym as a subalgebra. As is the case for NSym, there has been significant interest in finding and studying bases of QSym that generalize the Schur basis of Sym. The dual bases to the immaculate and noncommutative Schur bases are, respectively, the *dual immaculate* [2] and *quasi-Schur* [10] bases, which have been widely studied. In [1], the *extended Schur* basis of QSym was constructed as the stable limits of the polynomials arising from Kohnert's algorithm [11] applied to right-justified cell-diagrams. (In comparison, the polynomials arising from Kohnert's algorithm applied to left-justified cell-diagrams are exactly the *key polynomials* (Demazure characters), whose stable limits

*dominic.searles@otago.ac.nz.

form the Schur basis of Sym.) The extended Schur basis is exactly the dual basis to the shin basis of NSym.

In [3], modules of the 0-*Hecke algebra* were constructed whose *quasisymmetric characteristics* [7] are the dual immaculate quasisymmetric functions. Moreover in [14], 0-Hecke modules were constructed whose quasisymmetric characteristics are the quasisymmetric Schur functions. In this extended abstract, we outline how in [13] we construct 0-Hecke modules whose quasisymmetric characteristics are the extended Schur functions, giving a representation-theoretic interpretation of these functions and completing this picture for the three canonical Schur-like bases of NSym. We also show that these 0-Hecke modules are indecomposable. In comparison, the modules for dual immaculate symmetric functions are also indecomposable [3], but the modules for quasisymmetric Schur functions are not in general indecomposable [14].

2 Background

2.1 Quasisymmetric functions

A *composition* $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a finite sequence of positive integers. We call $\alpha_1, \ldots, \alpha_k$ the *parts* of α, and the *length* $\ell(\alpha)$ of α is k, the number of parts. We say that α is a *composition of n* when $\sum_{i=1}^{\ell(\alpha)} \alpha_i = n$.

For a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ of n, define $\mathbb{S}(\alpha)$ to be the subset $\{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\}$ of $[n-1]$. Via the map $\alpha \mapsto \mathbb{S}(\alpha)$, compositions of n are in bijection with subsets of $[n-1]$.

Example 2.1. Consider the composition $\alpha = (2,3,2)$ of $n = 7$. Then $\mathbb{S}(\alpha) = \{2,5\} \subset [6]$.

Let $\mathbb{C}[[x_1, x_2, \ldots]]$ denote the algebra of formal power series of bounded degree in infinitely many commuting variables, over the complex numbers. The algebra QSym of quasisymmetric functions [9] is the subalgebra of $\mathbb{C}[[x_1, x_2, \ldots]]$ that consists of those formal power series f satisfying

$$[x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k} \mid f] = [x_{j_1}^{\alpha_1} \cdots x_{j_k}^{\alpha_k} \mid f]$$

for every composition $\alpha = (\alpha_1, \ldots \alpha_k)$ and any two sequences $1 \le i_1 < \cdots < i_k$ and $1 \le j_1 < \cdots < j_k$, where $[x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k} \mid f]$ denotes the coefficient of the monomial $x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$ when f is expanded in monomials.

The *monomial* and *fundamental quasisymmetric functions* $\{M_\alpha\}$ and $\{F_\alpha\}$ are two important additive bases of quasisymmetric functions, indexed by compositions. They were introduced in [9], and are defined by

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k} \qquad \text{and} \qquad F_\alpha = \sum_{\beta \text{ refines } \alpha} M_\beta,$$

where a composition β *refines* a composition α if α can be obtained by summing consecutive entries of β.

Example 2.2. Let $\alpha = (2,1,2)$. We have

$$M_{(2,1,2)} = \sum_{i<j<k} x_i^2 x_j x_k^2$$

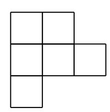
and

$$F_{(2,1,2)} = M_{(2,1,2)} + M_{(1,1,1,2)} + M_{(2,1,1,1)} + M_{(1,1,1,1,1)}.$$

2.2 The extended Schur basis of QSym

Define the *diagram* $D(\alpha)$ of a composition α to be an array of boxes with α_i boxes in row i, left-justified. We use French notation for composition diagrams, i.e., rows are numbered from bottom to top.

Example 2.3. The diagram $D(\alpha)$ of $\alpha = (1,3,2)$ is shown below.



Given a composition α of n, define a *standard extended tableau* of shape α to be a filling of the boxes of $D(\alpha)$ with the integers $1, 2, \ldots n$, each used exactly once, such that the entries in each row of $D(\alpha)$ increase from left to right and the entries in each column of $D(\alpha)$ increase from bottom to top. Denote the collection of all standard extended tableaux of shape α by $\mathrm{SET}(\alpha)$.

Remark 2.4. If α is a *partition*, i.e., $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{\ell(\alpha)}$, then the standard extended tableaux of shape α are exactly the *standard Young tableaux* of shape α.

We say an entry i of a standard extended tableau T is a *descent* of T if i is weakly to the right of $i+1$ in T. Define the *descent composition* $\mathrm{Des}(T)$ of T to be the composition α such that $\mathbb{S}(\alpha)$ is the set of all descents of T.

Example 2.5. The standard extended tableaux of shape $(1,3,2)$, along with their descent compositions, are shown below.

$$\begin{array}{ccccc}
\begin{array}{lll} 5 & 6 & \\ 2 & 3 & 4 \\ 1 & & \end{array} &
\begin{array}{lll} 4 & 6 & \\ 2 & 3 & 5 \\ 1 & & \end{array} &
\begin{array}{lll} 4 & 5 & \\ 2 & 3 & 6 \\ 1 & & \end{array} &
\begin{array}{lll} 3 & 5 & \\ 2 & 4 & 6 \\ 1 & & \end{array} &
\begin{array}{lll} 3 & 6 & \\ 2 & 4 & 5 \\ 1 & & \end{array} \\
\\
(1,3,2) & (1,2,2,1) & (1,2,3) & (1,1,2,2) & (1,1,3,1)
\end{array}$$

For compositions α, the *extended Schur functions* $\mathcal{E}_\alpha$ were defined in [1] as the stable limits of polynomials obtained by applying Kohnert's algorithm [11] to certain right-justified cell diagrams. Standard extended tableaux were also defined in terms of right-justified diagrams in [1]; our definition is a vertical reflection of this definition.

The extended Schur functions form a basis of QSym and in fact expand positively in the fundamental basis of QSym [1]. We may take the formula for their fundamental expansion as our definition for the extended Schur functions.

Theorem 2.6 ([1])**.** *Let α be a composition. Then*

$$\mathcal{E}_\alpha = \sum_{T \in \mathrm{SET}(\alpha)} F_{\mathrm{Des}(T)}.$$

Example 2.7. By Example 2.5 we have

$$\mathcal{E}_{(1,3,2)} = F_{(1,3,2)} + F_{(1,2,2,1)} + F_{(1,2,3)} + F_{(1,1,2,2)} + F_{(1,1,3,1)}.$$

Every Schur function is in fact an extended Schur function. We may use this result to define the celebrated Schur basis of symmetric functions:

Proposition 2.8 ([1])**.** *If α is a partition, then the extended Schur function $\mathcal{E}_\alpha$ is equal to the Schur function s_α.*

The extended Schur basis of QSym thus contains the Schur basis of symmetric functions. Other well-studied bases of QSym such as the fundamental and monomial quasisymmetric functions, the quasisymmetric Schur functions, and the dual immaculate quasisymmetric functions do not contain the Schur functions as a subset.

2.3 Noncommutative symmetric functions

The algebra NSym of noncommutative symmetric functions [8] is a noncommutative analogue of the symmetric functions, generated by elements $H_1, H_2, \ldots$ with no relations. It has an additive basis $\{H_\alpha\}$ indexed by compositions $\alpha = (\alpha_1, \ldots, \alpha_k)$, where the *complete homogeneous function* H_α is defined to be the product $H_{\alpha_1} \cdots H_{\alpha_k}$.

Three other important bases of NSym are the *immaculate basis* $\{\mathfrak{S}_\alpha\}$ introduced in [2], the *noncommutative Schur basis* $\{S^*_\alpha\}$ introduced in [4], and the *shin* basis $\{\mathcal{E}^*_\alpha\}$ introduced in [6]. These three bases are described in [5] as the *canonical Schur-like* bases of NSym.

There is a natural projection χ from noncommutative symmetric functions to symmetric functions, defined by

$$\chi(H_\alpha) = h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_{\ell(\alpha)}},$$

where h_i is the degree i *complete homogeneous symmetric function*: the sum of all monomials of degree i.

When α is a partition, we have

$$\chi(\mathfrak{S}_\alpha) = s_\alpha, \qquad \chi(S^*_\alpha) = s_\alpha \qquad \text{and} \qquad \chi(\mathcal{E}^*_\alpha) = s_\alpha,$$

by [2], [4], and [6] respectively. In this sense, all three bases are "Schur-like".

When α is not a partition, the image under χ of these basis elements is in general different. In [2] it is proved that $\chi(\mathfrak{S}_\alpha)$ is the determinant of a matrix whose entries are complete homogeneous symmetric functions; this reduces to the famous Jacobi-Trudi identity in the case α is a partition. In [4] it is proved that $\chi(S^*_\alpha) = s_{\mathtt{sort}(\alpha)}$, where $\mathtt{sort}(\alpha)$ is the partition obtained by rearranging the entries of α into weakly decreasing order. In [6] it is proved that $\chi(\mathcal{E}^*_\alpha) = 0$ whenever α is not a partition.

The algebra NSym is dual to QSym via the pairing $\langle H_\alpha, M_\beta\rangle = \delta_{\alpha,\beta}$. Each of these algebras inherit a coalgebra and Hopf algebra structure via this pairing. The basis of NSym that is dual to the fundamental basis $\{F_\beta\}$ of QSym is the *ribbon Schur* functions $\{\mathbf{r}_\alpha\}$. Under this pairing, the extended Schur basis of QSym is dual to the shin basis of NSym. In [6] it was proved that complete homogeneous functions expand positively in the shin basis, which then implies via duality that extended Schur functions expand positively into the monomial basis $\{M_\beta\}$ of QSym. Since extended Schur functions expand positively into the fundamental basis $\{F_\beta\}$ of QSym (Theorem 2.6), duality implies the following result for shin functions.

Proposition 2.9 ([13]). *The ribbon Schur functions expand positively in the shin basis of* NSym *via the formula*

$$\mathbf{r}_\beta = \sum_\beta K_{\alpha,\beta}\mathcal{E}^*_\alpha$$

where $K_{\alpha,\beta}$ is the number of $T \in \mathrm{SET}(\alpha)$ such that $\mathrm{Des}(T) = \beta$.

2.4 0-Hecke algebras

The 0-Hecke algebra $H_n(0)$ is the algebra over $\mathbb{C}$ with generators $T_1, \ldots, T_{n-1}$, subject to the following relations:

$$\begin{aligned} T_i^2 &= T_i \quad \text{for all } 1 \le i \le n-1 \\ T_iT_j &= T_jT_i \quad \text{for all } i,j \text{ with } |i-j| \ge 2 \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} \quad \text{for all } 1 \le i \le n-2. \end{aligned}$$

An additive basis of $H_n(0)$ is given by $\{T_\sigma : \sigma \in S_n\}$, where $T_\sigma = T_{i_1}T_{i_2}\cdots T_{i_r}$ for any reduced word $s_{i_1}s_{i_2}\cdots s_{i_r}$ for σ. The second and third relations above ensure T_σ is well-defined.

The *Grothendieck group* $\mathcal{G}_0(H_n(0))$ is the quotient of the linear span of the isomorphism classes of the finite-dimensional representations of $H_n(0)$, by the relation $[Y] =$

$[X] + [Z]$ whenever there is a short exact sequence $0 \to X \to Y \to Z \to 0$ of $H_n(0)$-representations X, Y, Z.

The irreducible representations of $H_n(0)$ are indexed by the 2^{n-1} compositions of n. Let $\mathcal{F}_\alpha$ denote the irreducible representation corresponding to the composition α. By [12], $\mathcal{F}_\alpha$ is one-dimensional and therefore equal to the span of some nonzero vector v_α. The structure of $\mathcal{F}_\alpha$ as a $H_n(0)$-representation is given by the following action of the generators T_i of $H_n(0)$:

$$T_i(v_\alpha) = \begin{cases} v_\alpha & \text{if } i \notin \mathbb{S}(\alpha) \\ 0 & \text{if } i \in \mathbb{S}(\alpha). \end{cases} \tag{2.1}$$

Define

$$\mathcal{G} = \bigoplus_{n \geq 0} \mathcal{G}_0(H_n(0)).$$

The set $\{\mathcal{F}_\alpha\}$ as α ranges over all compositions is an additive basis of $\mathcal{G}$. Moreover, $\mathcal{G}$ has a ring structure via the induction product, and there is a ring isomorphism $ch : \mathcal{G} \to$ QSym ([7]) defined by $ch([\mathcal{F}_\alpha]) = F_\alpha$. For any $H_n(0)$-module X, $ch([X])$ is called the *quasisymmetric characteristic* of X.

3 Modules for extended Schur functions

Our main goal is to interpret the extended Schur functions as quasisymmetric characteristics of certain $H_n(0)$-modules. In particular, we construct a $H_n(0)$-module X_α for each composition α of n such that the quasisymmetric characteristic $ch([X_\alpha])$ is equal to the extended Schur function $\mathcal{E}_\alpha$. We additionally show that for any composition α, the module X_α is indecomposable.

Recall the extended Schur functions are the dual basis of QSym to the shin basis of NSym. Interpretations of the dual bases of the immaculate and noncommutative Schur bases as quasisymmetric characteristics of certain $H_n(0)$-modules are given in [3] and [14] respectively. Providing such an interpretation for the extended Schur functions completes this picture for the three canonical Schur-like bases of NSym.

We will construct the 0-Hecke module X_α as a quotient of a 0-Hecke module spanned by a certain family of tableaux of shape α. It is not actually necessary to construct X_α as a quotient, but doing so simplifies the proofs; see Remark 3.5. Given a composition α of n, define a *standard row-increasing tableau* of shape α to be a filling of the boxes of $D(\alpha)$ with the integers $1, \ldots, n$, each used exactly once, such that entries increase from left to right along rows. Note that there are no conditions imposed on columns. Let $\mathrm{SRIT}(\alpha)$ denote the set of standard row-increasing tableaux of shape α.

Given any $T \in \mathrm{SRIT}(\alpha)$ and any $1 \le i \le n-1$, define

$$\pi_i(T) = \begin{cases} T & \text{if } i \text{ is weakly above } i+1 \text{ in } T \\ s_i(T) & \text{otherwise} \end{cases}$$

where $s_i(T)$ denotes the filling of $D(\alpha)$ obtained from T by swapping the entries i and $i+1$. Notice that $\pi_i(T) \in \mathrm{SRIT}(\alpha)$, since π_i cannot exchange i and $i+1$ when they are in the same row.

Example 3.1. Let $\alpha = (1,3,2)$ and let

$$T = \begin{array}{|c|c|c} \hline 3 & 6 & \multicolumn{1}{|c}{} \\ \hline 1 & 4 & \multicolumn{1}{|c|}{5} \\ \hline 2 & \multicolumn{2}{|c}{} \\ \cline{1-1} \end{array} \in \mathrm{SRIT}(\alpha).$$

Then $\pi_1(T) = \pi_3(T) = \pi_4(T) = T$, while

$$\pi_2(T) = s_2(T) = \begin{array}{|c|c|c} \hline 2 & 6 & \multicolumn{1}{|c}{} \\ \hline 1 & 4 & \multicolumn{1}{|c|}{5} \\ \hline 3 & \multicolumn{2}{|c}{} \\ \cline{1-1} \end{array} \in \mathrm{SRIT}(\alpha).$$

and

$$\pi_5(T) = s_5(T) = \begin{array}{|c|c|c} \hline 3 & 5 & \multicolumn{1}{|c}{} \\ \hline 1 & 4 & \multicolumn{1}{|c|}{6} \\ \hline 2 & \multicolumn{2}{|c}{} \\ \cline{1-1} \end{array} \in \mathrm{SRIT}(\alpha).$$

Let V_α denote the $\mathbb{C}$-vector space spanned by $\mathrm{SRIT}(\alpha)$. In fact V_α is a 0-Hecke module:

Proposition 3.2 ([13])**.** *The operators π_i define a $H_n(0)$-action on V_α. Specifically, we have $\pi_i(T) \in V_\alpha$ for all $T \in V_\alpha$ and all $1 \le i \le n-1$, and the π_i satisfy the relations for the generators T_i of the 0-Hecke algebra.*

In [13], it is proved by direct case-checking that the π_i satisfy the 0-Hecke relations. Proposition 3.2 can also be proved more simply by assigning to each SRIT a reading word by reading the entries along rows from right to left, starting at the bottom row and proceeding upwards. Interpreting the reading word as a permutation in one-line notation, it is straightforward to observe this defines an injective $H_n(0)$-module isomorphism from $\mathrm{SRIT}(\alpha)$ to the left regular representation of $H_n(0)$. We further note the action in Proposition 3.2 is in fact equivalent to the $H_n(0)$-action defined on *words of content α* in [3], but we prefer to work directly with tableaux of shape $D(\alpha)$.

Let $\mathrm{NSET}(\alpha)$ denote $\mathrm{SRIT}(\alpha) \setminus \mathrm{SET}(\alpha)$, i.e., those elements of $\mathrm{SRIT}(\alpha)$ that have a column in which entries do not increase from bottom to top. Denote by Y_α the vector subspace of V_α spanned by $\mathrm{NSET}(\alpha)$. It is straightforward to check that $\pi_i(T) \in \mathrm{NSET}(\alpha)$ for any $T \in \mathrm{NSET}(\alpha)$ and any $1 \le i \le n-1$. Therefore we have

Lemma 3.3 ([13])**.** *The vector space Y_α is an $H_n(0)$-submodule of V_α.*

Now define X_α to be the quotient module V_α / Y_α. By definition, $\mathrm{SET}(\alpha)$ is a basis of X_α.

Theorem 3.4 ([13])**.** *For any $1 \le i \le n-1$ and any composition α of n, the action of π_i on X_α is given by*

$$\pi_i(T) = \begin{cases} T & \text{if } i \text{ is strictly left of } i+1 \text{ in } T \\ 0 & \text{if } i \text{ and } i+1 \text{ are in the same column of } T \\ s_i(T) & \text{if } i \text{ is strictly right of } i+1 \text{ in } T \end{cases}$$

for any $T \in \mathrm{SET}(\alpha)$.

Remark 3.5. Theorem 3.4 is proved by confirming that $\pi_i(T) = T$ in the first case, $\pi_i(T) \in \mathrm{NSET}(\alpha)$ in the second case, and $\pi_i(T) = s_i(T) \in \mathrm{SET}(\alpha)$ in the third case, all of which are quick to check. It is also possible to show directly that the operators π_i on $\mathrm{SET}(\alpha)$ as stated in Theorem 3.4 satisfy the 0-Hecke relations, but this requires laborious case-checking.

Example 3.6. Let $\alpha = (1,3,2)$ and let

$$T = \begin{array}{|c|c|c} \hline 3 & 6 & \multicolumn{1}{c}{} \\ \hline 2 & 4 & \multicolumn{1}{c|}{5} \\ \hline 1 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \in \mathrm{SET}(\alpha).$$

Then $\pi_1(T) = \pi_2(T) = 0$, $\pi_3(T) = \pi_4(T) = T$, while

$$\pi_5(T) = s_5(T) = \begin{array}{|c|c|c} \hline 3 & 5 & \multicolumn{1}{c}{} \\ \hline 2 & 4 & \multicolumn{1}{c|}{6} \\ \hline 1 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \in \mathrm{SET}(\alpha).$$

To show the quasisymmetric characteristic of this module is $\mathcal{E}_\alpha$, we find a filtration of X_α by $H_n(0)$-submodules. To this end, define a relation $\preceq$ on $\mathrm{SET}(\alpha)$ by setting $S \preceq T$ if S can be obtained by applying a (possibly empty) sequence of the π_i operators to T.

Lemma 3.7 ([13])**.** *The relation $\preceq$ is a partial order on* $\mathrm{SET}(\alpha)$.

Figure 1 below shows this partial order for $\mathrm{SET}(2,3,2)$, where the maximal element is at the top and the arrows give the covering relations.

Let $\preceq^t$ denote an arbitrary choice of extension of $\preceq$ to a total order, and suppose $\preceq^t$ orders the m elements of $\mathrm{SET}(\alpha)$ as $T_1 \preceq^t T_2 \preceq^t \cdots \preceq^t T_m$. For each $1 \le j \le m$, define X_j to be the $\mathbb{C}$-linear span of all $T_k \in \mathrm{SET}(\alpha)$ such that $k \le j$.

It follows immediately from the definitions of $\preceq^t$ and X_j that X_j is a $H_n(0)$-module for each $1 \le j \le m$. Therefore,

$$0 := X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_m = X_\alpha$$

is a filtration of X_α, in which each quotient module X_j / X_{j-1} is one-dimensional, spanned by $T_j \in \mathrm{SET}(\alpha)$.

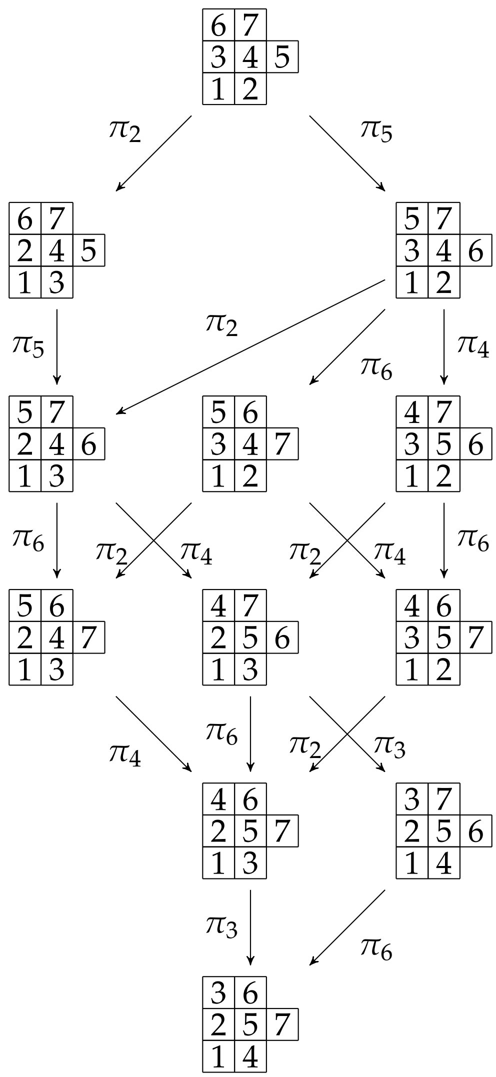


Figure 1: The partial order on $\mathrm{SET}(2,3,2)$, illustrating how elements of $\mathrm{SET}(2,3,2)$ are obtained from others via sequences of the operators $\{\pi_i\}$.

Lemma 3.8 ([13])**.** *For any* $1 \leq i \leq n-1$ *and any* $1 \leq j \leq m$*, we have*

$$\pi_i(T_j) = \begin{cases} T_j & \text{if } i \notin \mathbb{S}(\mathrm{Des}(T_j)) \\ 0 & \text{if } i \in \mathbb{S}(\mathrm{Des}(T_j)). \end{cases}$$

Therefore by (2.1), X_j/X_{j-1} is isomorphic as $H_n(0)$-modules to $\mathcal{F}_{\mathrm{Des}(T_j)}$, whence $ch([X_j/X_{j-1}]) = F_{\mathrm{Des}(T_j)}$. Using this and the formula in Theorem 2.6 for expanding extended Schur functions in fundamental quasisymmetric functions establishes the main result:

Theorem 3.9 ([13])**.** *Let α be a composition of n. The quasisymmetric characteristic of the $H_n(0)$-module X_α is the extended Schur function $\mathcal{E}_\alpha$.*

The modules X_α for the extended Schur functions are indecomposable, as is the case for the dual immaculate quasisymmetric functions but not the case for the quasisymmetric Schur functions. To establish indecomposability of X_α in [13], we follow the approach used in [3] and [14]. This involves first showing that X_α is cyclically generated [13]. In particular, it is generated by the *super-standard* extended tableau T_α^{sup}, the standard extended tableau of shape α whose entries in the ith row are the first α_i integers larger than $\alpha_1 + \cdots + \alpha_{i-1}$. (In Example 2.5, the leftmost SET is the super-standard extended tableau of shape $(1,3,2)$.) Figure 1 above shows how $X_{(2,3,2)}$ is generated by the super-standard element of $\mathrm{SET}(2,3,2)$, which is at the top.

The following theorem can then be proved using properties of T_α^{sup} and techniques from representation theory.

Theorem 3.10 ([13])**.** *Let α be a composition of n. Then the $H_n(0)$-module X_α is indecomposable.*

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