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Strict log-concavity of the Kirchhoff polynomial and its applications

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Abstract. Anari, Gharan, and Vinzant showed that the basis generating functions for all matroids are log-concave. In this paper, we show that Kirchhoff polynomials, i.e. the basis generating functions for simple graphic matroids, are strictly log-concave. Our key observation is that the Kirchhoff polynomial of a complete graph can be seen as the irreducible relative invariant of a certain prehomogeneous vector space. Furthermore, we prove that an algebra associated to a graphic matroid satisfies the strong Lefschetz property and Hodge–Riemann bilinear relation at degree one.

Keywords: the complete graph, graphic matroids, Artinian Gorenstein algebras, the strong Lefshcetz property, Hodge–Riemann relation, prehomogeneous vector spaces

1 Introduction

Recently, in [1], Anari, Gharan, and Vinzant showed that, for any matroid M, the basis generating function F_M satisfies log-concavity (more precisely, complete log-concavity) on $\mathbb{R}^n_{\geq 0}$. In other words, $\log F_M$ is concave on $\mathbb{R}^n_{\geq 0}$, that is the Hessian matrix H_{F_M} and the gradient vector ∇F_M of F_M satisfy

$$\left(-F_M H_{F_M} + (\nabla F_M)^\top \nabla F_M\right)\Big|_{\boldsymbol{x}=\boldsymbol{a}} \text{ is positive semidefinite}$$

for any $\boldsymbol{a} \in \mathbb{R}^n_{\geq 0}$.

The basis generating functions for graphic matroids are called Kirchhoff polynomials. In our main theorem, we show that, the Kirchhoff polynomial F_Γ is strictly log-concave on $(\mathbb{R}_{>0})^n$ for any simple graph Γ with n edges. In other words, for any $\boldsymbol{a} \in (\mathbb{R}_{>0})^n$, $\log F_\Gamma$ is strictly concave at $\boldsymbol{a}$. In particular, the Hessian matrix $H_{F_\Gamma}|_{\boldsymbol{x}=\boldsymbol{a}}$ is non-degenerate with $n-1$ negative eigenvalues and one positive eigenvalue (see Theorem 3.2).

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Our main theorem is proved in two steps. First, we reduce our claim to the following determinantal identity of the Hessian of the Kirchhoff polynomial $F_{K_{r+1}}$ of complete graphs K_{r+1} (cf. Theorem 3.1):

$$\det H_{F_{K_{r+1}}} = (-1)^{n-1} c_r (F_{K_{r+1}})^{n-r-1},$$

where $c_r > 0$ is a constant, and $n := \binom{r+1}{2}$. Second, we show the above equality not through direct computation, but rather by identifying $F_{K_{r+1}}$ with the unique irreducible polynomial associated to a special $GL_r(\mathbb{C})$ representation or the so-called prehomogeneous vector space. Then, based on the general theory of prehomogeneous vector spaces [10], the Hessian $\det H_F$ of the relative invariant F is also a relative invariant of the same representation. Hence it follows from the uniqueness of the relative invariant that

$$\exists c \in \mathbb{C} \text{ such that } \det H_F = cF^m.$$

We also apply the main theorem to the strong Lefschetz property and the Hodge–Riemann bilinear relation of the graded Artinian Gorenstein algebra $R^*_\Gamma = \bigoplus_{\ell=0}^r R^\ell_\Gamma = \mathbb{R}[x_1, \ldots, x_n] / \operatorname{Ann}(F_\Gamma)$ associated to any simple graph Γ. This algebra is defined for any matroid M by Maeno and Numata. They conjectured that R^*_M has the strong Lefschetz property for any matroid M in an extended abstract [4] of the paper [5]. As an application of our main theorem, we prove that this conjecture at degree one when M is a graphic matroid. Since the Hodge–Riemann bilinear form of R^1_Γ is given by the Hessian H_{F_Γ}, we show that the Hodge–Riemann relation holds at degree one (see Theorem 4.4).

This paper is organized as follows. In Section 2, we introduce some concepts to give our main theorem. In Section 3, we define the Kirchhoff polynomials of simple graphs, and then prove our main result. In the last half of this section, we see that the connection between the Kirchhoff polynomials of complete graphs and certain prehomogeneous vector spaces. Finally, in Section 4, we conclude that our main result applies to algebras associated to graphic matroids.

This article is a research announcement or extended abstract for the paper [8]. We omit proofs and details that can be found in the main paper.

2 Preliminaries

In this section, we introduce some concepts that will be useful for our main theorem. In Section 2.1, we define the strict log-concavity of a homogeneous polynomial. Then, we recall a relationship between strict log-concavity and the Hessian of the polynomial. Next, in Section 2.2, we introduce prehomogeneous vector spaces as a way of computing the Hessian. Finally, in Section 2.3, we introduce the matroids, which are the main objects of our theorem.

2.1 Strict log-concavity and Hessians

Let F be a homogeneous polynomial of degree r in n variables with real coefficients, where $r \geq 3$. For F, H_F denotes the Hessian matrix of F, and ∇F denotes the gradient vector of F. We call $\det H_F$ the Hessian of F.

Definition 2.1 (Strictly log-concave/log-concave)**.** We say that F is *log-concave* (*resp. strictly log-concave*) at $\boldsymbol{a} \in \mathbb{R}^n$ if

$$(-FH_F + (\nabla F)^\top \nabla F)|_{\boldsymbol{x}=\boldsymbol{a}}$$

is positive semidefinite (*resp.* positive definite).

The relation between strict log-concavity and the Hessian can be seen as follows.

Remark 2.2. By easy arguments, we obtain

$$\det\left(-FH_F + (\nabla F)^\top \nabla F\right) = (-1)^{n-1}\frac{1}{r-1}F^n \det H_F.$$

Therefore F is strictly log-concave at $\boldsymbol{a}$ if and only if $F(\boldsymbol{a}) \neq 0$, $\det H_F\big|_{\boldsymbol{x}=\boldsymbol{a}} \neq 0$, and F is log-concave at $\boldsymbol{a}$.

We assume that F is a polynomial with positive coefficients. Then $\operatorname{tr} H_F \geq 0$. If F is strictly log-concave, then its Hessian is non-degenerate. It follows from Cauchy's interlacing theorem that its Hessian has only one positive eigenvalue.

Theorem 2.3 (Cauchy's interlacing theorem [2, Corollary 4.3.9])**.** *For a real symmetric $n \times n$ matrix A with eigenvalues $\alpha_1 \geq \cdots \geq \alpha_n$, a vector $\boldsymbol{v} \in \mathbb{R}^n$, and the eigenvalues $\beta_1 \geq \cdots \geq \beta_n$ of $-A + \boldsymbol{v}^\top \boldsymbol{v}$, they satisfy*

$$\beta_1 \geq -\alpha_n \geq \beta_2 \geq \cdots \geq -\alpha_2 \geq \beta_n \geq -\alpha_1.$$

Corollary 2.4. *If F is strictly log-concave at $\boldsymbol{a} \in (\mathbb{R}_{>0})^n$, then $H_F\big|_{\boldsymbol{x}=\boldsymbol{a}}$ has exactly $n-1$ negative eigenvalues and exactly one positive eigenvalue. In particular,*

$$(-1)^{n-1}(\det H_F)|_{\boldsymbol{x}=\boldsymbol{a}} > 0.$$

A *multi-affine polynomial* is a linear combination of square-free monomials. The following is used for our main theorem:

Lemma 2.5. *Let $F \in \mathbb{R}[x_1, \ldots, x_n]$ be a multi-affine homogeneous polynomial of* $\deg F = r \geq 3$ *with positive coefficients. For a subset I of $[n]$ and $0 \leq k \leq n$, we define*

$$C^{n-k}_{I>0} = \left\{ (z_{k+1}, \ldots, z_n) \in \mathbb{R}^{n-k}_{\geq 0} \;\middle|\; z_j \geq 0 \; (j \notin I),\; z_i > 0 \; (i \in I) \right\}.$$

We assume that F is strictly log-concave on $C^n_{I>0}$. If

$$\frac{\partial F}{\partial x_1} \not\equiv 0, \frac{\partial F|_{x_1=0}}{\partial x_2} \not\equiv 0, \ldots, \frac{\partial F|_{x_1=\cdots=x_{k-1}=0}}{\partial x_k} \not\equiv 0$$

holds for some $0 \leq k \leq n-r$, then $F|_{x_1=\cdots=x_k=0} \in \mathbb{R}[x_{k+1}, \ldots, x_n]$ is strictly log-concave on $C^{n-k}_{I>0}$.

2.2 Prehomogeneous vector spaces

For a special polynomial, we have the following identity:

$$\det H_F = cF^{\frac{n(r-2)}{r}},$$

where c is non-zero (see Corollary 2.7). To prove this identity, we introduce the concept of prehomogeneous vector spaces (cf. [10]).

Let (G, ρ, V) be a triplet of a connected linear algebraic group G, a finite dimensional vector space V, and a rational representation ρ of G on V, all defined over $\mathbb{C}$. We call (G, ρ, V) a *prehomogeneous vector space* if there exists an algebraic G-invariant proper subset $S \subset V$ such that $V \setminus S$ is a single open dence G-orbit. Then, we say that S is the *singular set* of (G, ρ, V), and that (G, ρ, V) is *irreducible* when ρ is an irreducible representation.

Let (G, ρ, V) be a prehomogeneous vector space. A not identically zero rational function $F \in \mathbb{C}(V)$ is called a *relative invariant (with respect to χ) of* (G, ρ, V) if there exists a rational character $\chi \in \mathrm{Hom}(G, \mathbb{C}^*)$ which satisfies the following:

$$F(\rho(g)\boldsymbol{x}) = \chi(g)F(\boldsymbol{x}) \quad (g \in G, \boldsymbol{x} \in V).$$

If F is a relative invariant corresponding to some character χ, then $\det H_F$ is a relative invariant corresponding to the character $\chi^N \cdot (\det)^{-2}$, where $N = \dim V$ and $\det : G \to \mathbb{C}^* : g \mapsto \det(\rho(g))$.

The following is a fundamental proposition.

Proposition 2.6 (cf. [10, Proposition 12 in Section 4])**.** *Let (G, ρ, V) be an irreducible prehomogeneous vector space. Then, there is at most one irreducible relative invariant polynomial F up to constant multiple. In particular, any relative invariant has the form cF^m for $c \in \mathbb{C}$ and $m \in \mathbb{Z}$.*

We say that a prehomogeneous vector space (G, ρ, V) is *regular* if there exists a relative invariant $F \in \mathbb{C}(V)$ such that its Hessian $\det H_F$ is not identically zero on V. Then, we have the following key identity of the Hessian of the relative invariant when (G, ρ, V) is regular.

Corollary 2.7. *Let (G, ρ, V) be a regular irreducible prehomogeneous vector space of dimension n. If the degree of the relative invariant F is r, then, there exists a constant $c \in \mathbb{C}^*$ such that*

$$\det H_F = cF^{\frac{n(r-2)}{r}}.$$

2.3 Matroids

Here, we provide the basic terms of a matroid. A *matroid* M is a pair $(E, \mathcal{B})$ of a finite set E and a nonempty collection $\mathcal{B}$ of subsets of E satisfying the so-called basis exchange axioms: If B_1 and B_2 are in $\mathcal{B}$ and $x \in B_1 \setminus B_2$, then there is an element $y \in B_2 \setminus B_1$ such that $\{y\} \cup (B_1 \setminus \{x\}) \in \mathcal{B}$. See [9] for details. In this case, we call each $B \in \mathcal{B}$ a *basis* of M, and each subset of a basis of M an *independent set* of M. We call $e \in E$ a *loop* (*resp. coloop*) if every basis does not contain e (*resp.* every basis contains e). We say that a matroid M is *simple* if every subset of E with cardinality less than or equal to two is independent.

Example 2.8 (Graphic matroid)**.** For any finite connected graph $\Gamma = (V, E)$ with the vertex set V and the edge set E, we call a subgraph $T \subseteq \Gamma$ a *spanning tree* in Γ if T does not contain any cycles and T passes through all vertices of Γ. Let $\mathcal{B}_\Gamma$ be the set of all spanning trees in Γ. Then $M(\Gamma) = (E, \mathcal{B}_\Gamma)$ is a matroid. These matroids are called *graphic matroids*.

Note that if M is a graphic matroid, then there exists a connected graph Γ such that $M(\Gamma)$ is isomorphic to M.

Example 2.9 (Submatroid)**.** Let $M = (E, \mathcal{B})$ be a matroid. For $E' \subset E$, we define $\mathcal{B}'$ by $\mathcal{B}' = \{ B \in \mathcal{B} \mid B \subset E' \}$. Then $M' = (E', \mathcal{B}')$ is a matroid. We call M' a submatroid of M.

Definition 2.10 (Basis generating function)**.** For any matroid $M = (E, \mathcal{B})$, we define the *basis generating function* $F_M(\boldsymbol{x})$ of M by

$$F_M(\boldsymbol{x}) = \sum_{B \in \mathcal{B}} \prod_{i \in B} x_i.$$

Example 2.11 (Kirchhoff polynomial)**.** The basis generating function $F_{M(\Gamma)}$ for a graphic matroid $M(\Gamma)$ is called the Kirchhoff polynomial of Γ. In this case, we write $F_\Gamma = F_{M(\Gamma)}$.

By basis exchange axioms, if B and B' are bases of M, then $|B| = |B'|$. We say that a matroid M has *rank* r if the number of elements of a basis of M is r. For a matroid $M = (E, \mathcal{B})$ of rank r, its basis generating function $F_M(\boldsymbol{x})$ is a multi-affine homogeneous polynomial of degree r in $|E|$ variables with coefficients equal to one. Hence, we have

$$F_M(\boldsymbol{x}) = F_M(\boldsymbol{x})\big|_{x_e=0} + x_e \frac{\partial}{\partial x_e} F_M(\boldsymbol{x}).$$

Moreover

$$F_M(\boldsymbol{x})\big|_{x_e=0} = \begin{cases} 0 & \text{if } e \text{ is a coloop,} \\ F_{M\setminus e}(\boldsymbol{x}) & \text{otherwise,} \end{cases}$$

$$\frac{\partial}{\partial x_e} F_M(\boldsymbol{x}) = \begin{cases} 0 & \text{if } e \text{ is a loop,} \\ F_{M/e}(\boldsymbol{x}) & \text{otherwise,} \end{cases}$$

where $M \setminus e$ (M/e) is the deletion (contraction) of M with respect to e. Hence, for any $e \in E$ that is not a loop or a coloop, we have

$$F_M(\boldsymbol{x}) = F_{M\setminus e}(\boldsymbol{x}) + x_e F_{M/e}(\boldsymbol{x}).$$

Let $B \in \mathcal{B}$. Every element $e \in E \setminus B$ is not a coloop. Hence, for the matroid M_0 obtained by deleting some elements $e_1, \dots, e_k \in E \setminus B$ from M, we have

$$F_{M_0} = F_M|_{x_{e_1} = \cdots = x_{e_k} = 0}.$$

Example 2.12. Every Kirchhoff polynomial is obtained from the Kirchhoff polynomial of the complete graph with the same number vertices by substituting zero for some variables. In other words, every simple graphic matroid is a submatroid of the simple graphic matroid of the complete graph.

Note that, for any matroid M on $[n] = \{1, 2, \dots, n\}$, log-concavity of $F_M(\boldsymbol{x})$ on $\mathbb{R}^n_{\geq 0}$ is already known in [1, Theorem 4.2].

Theorem 2.13 ([1, Theorem 4.2])**.** *For any matroid M, $F_M(\boldsymbol{x})$ is log-concave on $\mathbb{R}^n_{\geq 0}$.*

If M is not simple, then $\det(-F_M H_{F_M} + (\nabla F_M)^T (\nabla F_M))$ is identically zero, in particular, it cannot be positive definite at any point in $\mathbb{R}^n$. See [8] for more details.

Below, we prepare some lemmas for our main theorem.

Lemma 2.14. *Let M be a matroid on E. Fix a basis $B \in \mathcal{B}$ of M. For $S = \{j_1, \dots, j_k\} \subset E \setminus B$, and $j \in E \setminus (B \cup S)$,*

$$\frac{\partial F_M|_{x_{j_1} = \cdots = x_{j_k} = 0}}{\partial x_j} = F_{(M \setminus S)/j} \neq 0 \tag{2.1}$$

if j is not a loop.

Since the basis generating function $F_M(\boldsymbol{x})$ of any simple matroid M satisfies the condition (2.1) by Lemmas 2.5 and 2.14, we have the following.

Lemma 2.15. *Let M be a simple matroid on $[n]$ of rank $r \geq 3$. For any basis B, we assume that F_M is strictly log-concave on $C^n_{B>0}$, where $C^n_{B>0}$ is the same as Lemma 2.5. Then for any submatroid $M_0 := M \setminus \{j_1, \dots, j_k\}$ of rank r, F_{M_0} is strictly log-concave on $C^{n-k}_{B_0>0}$ for any basis B_0 of M_0.*

Let $\Gamma = (V, E)$ be a simple graph with $|E| = n$. For a spanning tree T, we define

$$C^n_{T>0} = \left\{ \boldsymbol{a} \in \mathbb{R}^n_{\geq 0} \;\middle|\; z_i > 0 \ (i \in T),\ z_j \geq 0 \ (j \notin T) \right\} \ (\supset (\mathbb{R}_{>0})^n).$$

Then we can find the following corollary to Lemma 2.15.

Corollary 2.16. *Let $\Gamma = (V, E)$ be a simple connected graph with $|V| = r + 1 \geq 3$ and $|E| = n \geq 3$. For each spanning tree T in Γ, we assume that F_Γ is strictly log-concave on $C^n_{T>0}$ $(\supset (\mathbb{R}_{>0})^n)$. Then for any connected subgraph $\Gamma' = (V', E')$ with $|V'| = r + 1$ and $|E'| = n - k$, $F_{\Gamma'}$ is strictly log-concave on $C^{n-k}_{T'>0}$ $(\supset (\mathbb{R}_{>0})^{n-k})$ for any spanning tree T' in Γ'.*

3 Main result

In this section, we will prove our main result that the Kirchhoff polynomial of each simple graph is strictly log-concave on $\mathbb{R}^n_{>0}$ (see Theorem 3.2).

3.1 Main result

First, we consider the Kirchhoff polynomial of the complete graph. As stated in Section 2.2, for the relative invariant of an irreducible prehomogeneous vector space, its Hessian has the form cF^m. We can show that the Kirchhoff polynomial of the complete graph can be realized as the relative invariant. We will prove Theorem 3.1 in Section 3.2.

Theorem 3.1. *Let $n = \binom{r+1}{2}$. We have*

$$\det H_{F_{K_{r+1}}} = (-1)^{n-1} c_r (F_{K_{r+1}})^{n-r-1},$$

where $c_r = 2^{n-r}(r-1)$.

The Kirchhoff polynomial is the basis generating function for a graphic matroid by Example 2.11. By Theorem 2.13, we know that the Kirchhoff polynomial is log-concave. Note that $F_\Gamma(\boldsymbol{x}) > 0$ on $C^n_{T>0}$ for each spanning tree T. Based on Remark 2.2, the Kirchhoff polynomial is strictly log-concave on $C^n_{T>0}$ if and only if its Hessian does not vanish on $C^n_{T>0}$. Hence Theorem 3.1 tells us that, for any spanning tree T, the Kirchhoff polynomial of the complete graph is strictly log-concave on $C^n_{T>0}$. By Example 2.12 and Corollary 2.16, we obtain the following.

Theorem 3.2 (Main result)**.** *For any simple connected graph $\Gamma = (V, E)$ with $|V| = r+1 \geq 3$ and $|E| = n \geq 3$, the Kirchhoff polynomial $F_\Gamma(\boldsymbol{x})$ is strictly log-concave on $(\mathbb{R}_{>0})^n$. In other words,*

$$(-F_\Gamma H_{F_\Gamma} + (\nabla F_\Gamma)^T \nabla F_\Gamma)|_{\boldsymbol{x}=\boldsymbol{a}}$$

is positive definite at any $\boldsymbol{a} \in (\mathbb{R}_{>0})^n$. In particular, $H_{F_\Gamma}|_{\boldsymbol{x}=\boldsymbol{a}}$ is non-degenerate, with $n-1$ negative eigenvalues and exactly one positive eigenvalue. Thus,

$$(-1)^{n-1}(\det H_{F_\Gamma})|_{\boldsymbol{x}=\boldsymbol{a}} > 0.$$

Moreover, for each spanning tree T in Γ, F_Γ is strictly log-concave on $C^n_{T>0}$ ($\supset (\mathbb{R}_{>0})^n$).

3.2 Proof of Theorem 3.1

Here we study the Kirchhoff polynomials more precisely, and give a proof of Theorem 3.1.

We see that the Kirchhoff polynomial is realized as the determinant of some matrix. This is called the matrix-tree theorem (cf. [11, Theorem VI.29]): Let E_{ij} be the $r \times r$ matrix such that the (i, j)-component is one and the others are zero. For a graph $\Gamma = (V, E)$ with $|V| = r$, we associate a variable x_e to each edge $e \in E$, and define the *Laplacian* L_Γ *of* Γ indexed by vertices as

$$L_\Gamma = \sum_{e=\{i,j\}\in E} x_e(E_{ii} - E_{ij} - E_{ji} + E_{jj}).$$

Then the Kirchhoff polynomial F_Γ is equal to any cofactor of its Laplacian L_Γ. In other words, for a graph $\Gamma = (V, E)$ and any $1 \le i, j \le |V|$,

$$F_\Gamma = (-1)^{i+j} \det(L_\Gamma^{(ij)}),$$

where $L_\Gamma^{(ij)}$ denotes the submatrix of L_Γ obtained by removing the ith row and jth column.

For a graph, we associate $x_{ij} = x_{ji} = x_e$ to each edge $e = \{i, j\}$. For the complete graph K_{r+1}, the entries in Laplacian $L_{K_{r+1}} = \left(\ell_{ij}\right)_{1\le i,j\le r+1}$ are

$$\ell_{ij} = \begin{cases} \left(\sum_{k=1}^{r+1} x_{ik}\right) - x_{ii} & (\text{if } \ i = j), \\ -x_{ij} & (\text{otherwise}). \end{cases}$$

One can see that $L_{K_{r+1}}^{(11)}$ is a symmetric matrix and $\{x_{ij}\}_{1\le i<j\le r+1}$ gives a coordinate of the vector space $\mathrm{Sym}(r, \mathbb{C})$, which consists of all $r \times r$ symmetric matrices over $\mathbb{C}$.

Proposition 3.3. *We have*

$$\left\{ \left. L_{K_{r+1}}^{(11)} \right|_{\boldsymbol{x}=\boldsymbol{a}} \;\middle|\; \boldsymbol{a} = (a_{ij})_{i<j}, a_{ij} \in \mathbb{C} \right\} = \mathrm{Sym}(r, \mathbb{C}).$$

Therefore the Kirchhoff polynomial $F_{K_{r+1}}$ *can be regarded as a function from* $\mathrm{Sym}(r, \mathbb{C})$ *to* $\mathbb{C}$. *In other words, we can regard the Kirchhoff polynomial as the following function:*

$$F_{K_{r+1}} = \det : \mathrm{Sym}(r, \mathbb{C}) \to \mathbb{C}.$$

Example 3.4. The Laplacian matrix L_{K_4} of the complete graph K_4 is

$$L_{K_4} = \begin{pmatrix} x_{12} + x_{13} + x_{14} & -x_{12} & -x_{13} & -x_{14} \\ -x_{21} & x_{21} + x_{23} + x_{24} & -x_{23} & -x_{24} \\ -x_{31} & -x_{32} & x_{31} + x_{32} + x_{34} & -x_{34} \\ -x_{41} & -x_{42} & -x_{43} & x_{41} + x_{42} + x_{43} \end{pmatrix}.$$

The $(1,1)$ minor $L_{K_4}^{(11)}$ of L_{K_4} is

$$L_{K_4}^{(11)} = \begin{pmatrix} x_{21}+x_{23}+x_{24} & -x_{23} & -x_{24} \\ -x_{32} & x_{31}+x_{32}+x_{34} & -x_{34} \\ -x_{42} & -x_{43} & x_{41}+x_{42}+x_{43} \end{pmatrix}.$$

Note that $L_{K_4}^{(11)}$ is a symmetric matrix and $\{x_{ij}\}_{1\le i<j\le r+1}$ gives a coordinate of $\mathrm{Sym}(3,\mathbb{C})$. Hence we have

$$\left\{ L_{K_4}^{(11)}\big|_{\boldsymbol{x}=\boldsymbol{a}} \;\middle|\; \boldsymbol{a} = (a_{ij})_{i<j}, a_{ij} \in \mathbb{C} \right\} = \mathrm{Sym}(3,\mathbb{C}).$$

In [10], irreducible prehomogeneous vector spaces have already been classified. Here, we focus on the following prehomogeneous vector space, whose relative invariant is given by the Kirchhoff polynomial of the complete graphs. See [10, Proposition 3 in Section 5] or [10, Section 7, I-(2)] for details on Proposition 3.5.

Proposition 3.5 (cf. [10])**.** *Let ρ be the representation of $GL_r(\mathbb{C})$ on $\mathrm{Sym}(r,\mathbb{C})$ such that*

$$\rho(P)X = PXP^T \ \ (P \in GL_r(\mathbb{C})).$$

Then $(GL_r(\mathbb{C}), \rho, Sym(r,\mathbb{C}))$ is a regular irreducible prehomogeneous vector space. Moreover, the relative invariant is given by $\det : \mathrm{Sym}(r,\mathbb{C}) \to \mathbb{C}$.

As stated in Proposition 3.3, the Kirchhoff polynomial $F_{K_{r+1}}(\boldsymbol{x})$ of the complete graph K_{r+1} is the relative invariant of the prehomogeneous vector space in Proposition 3.5. Evaluation of $(\det H_{F_{K_{r+1}}})|_{\boldsymbol{x}=(1,1,\ldots,1)}$ was performed the second author [13]. Note that we used Cayley's theorem $F_{K_{r+1}}(1,1,\ldots,1) = (r+1)^{r-1}$ at the second equality in Proposition 3.6 (see [11, Theorem VI. 30] for details).

Proposition 3.6 ([13, Theorem 3.3])**.** *For the complete graph K_{r+1},*

$$\begin{aligned}(\det H_{F_{K_{r+1}}})|_{\boldsymbol{x}=(1,1,\ldots,1)} &= (-1)^{n-1}2^{n-(r+1)}(r+1)^{r+1+n(r-3)}(r-1) \\ &= (-1)^{n-1}2^{n-r}(r-1)(F_{K_{r+1}}(1,1,\ldots,1))^{n-r-1},\end{aligned}$$

where $n = \binom{r+1}{2}$.

By Corollary 2.7 and Propositions 3.5 and 3.6, we have Theorem 3.1.

4 Applications

In this section, we define a graded Artinian Gorenstein algebra R_Γ associated to a graph Γ (more generally, to a matroid), as introduced by Maeno–Numata [5]. Then, using strict log-concavity of F_Γ at any $\boldsymbol{a} \in (\mathbb{R}_{>0})^n$, we prove that $L_{\boldsymbol{a}} := a_1x_1 + \cdots + a_nx_n \in R^1_{F_\Gamma}$ satisfies the strong Lefschetz property at $R^1_{F_\Gamma}$.

First, we define an Artinian Gorenstein algebra associated to each homogeneous polynomial. For a homogeneous polynomial $F \in \mathbb{R}[x_1, \dots, x_n]$ of degree r, define an ideal $\operatorname{Ann}(F)$ and a quotient algebra R^*_F by

$$\operatorname{Ann}(F) = \left\{ P \in \mathbb{R}[x_1, \dots, x_n] \;\middle|\; P\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) F = 0 \right\},$$

$$R^*_F = \bigoplus_{\ell=0}^{r} R^\ell_F = \mathbb{R}[x_1, \dots, x_n] / \operatorname{Ann}(F).$$

Then R^*_F is a graded Artinian Gorenstein algebra. Conversely, every graded Artinian Gorenstein algebra can be represented as above by some homogeneous polynomial. See [6, Proposition 2.1, Theorem 2.1 and Remark 2.3] for more details.

We recall the concepts of the strong Lefschetz property and the Hodge–Riemann bilinear relation.

Definition 4.1 (Strong Lefschetz property)**.** We say that $L \in R^1_F$ satisfies the *strong Lefschetz property* at degree ℓ if the following multiplication map is bijective:

$$\begin{aligned} \times L^{r-2\ell} : R^\ell_F &\to R^{r-\ell}_F, \\ f &\mapsto L^{r-2\ell} f. \end{aligned}$$

Definition 4.2 (Hodge–Riemann relation)**.** We say that $L \in R^1_F$ satisfies the *Hodge–Riemann relation* at degree ℓ if the Hodge–Riemann bilinear form

$$\begin{aligned} Q^\ell_L : R^\ell_F \times R^\ell_F &\to \mathbb{R}, \\ (\xi_1, \xi_2) &\mapsto [\xi_1 L^{r-2\ell} \xi_2] \end{aligned}$$

is negative definite on $\operatorname{Ker}(L^{r-1})$, where $[-] : R^r_F \xrightarrow{\sim} \mathbb{R}$ is the isomorphism as

$$P \mapsto P\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) F.$$

We note that, for a graded Artinian Gorenstein algebra R^*_F associated to a homogeneous polynomial F, the Hodge–Riemann bilinear form Q^1_L at degree one is non-degenerate if and only if L satisfies the strong Lefschetz property at degree one. We also note that if $F(\boldsymbol{a}) > 0$, then $L_{\boldsymbol{a}}$ satisfies the Hodge–Riemann relation at R^1_F if and only if $Q^1_{L_{\boldsymbol{a}}}$ is non-degenerate and has only one positive eigenvalue. See the proof of [6, Theorem 3.1] for more details.

We will use the following criterion, which is a special case of the general criterion in [6, Theorem 3.1] and [12, Theorem 4].

Theorem 4.3 ([6, Theorem 3.1], [12, Theorem 4]). *Assume that $x_1, \dots, x_n \in R_F^1$ is a basis. An element $L_{\boldsymbol{a}} := a_1x_1 + \cdots + a_nx_n \in R_F^1$ satisfies the strong Lefschetz property at degree one if and only if $F(a_1, \dots, a_n) \neq 0$ and $\det H_F|_{\boldsymbol{x}=\boldsymbol{a}} \neq 0$, where H_F is the Hessian matrix of F.*

Next, we consider the Artinian Gorenstein algebra $R^*_{F_M}$ associated to the basis generating function F_M of a matroid M, in particular, the Kirchhoff polynomial F_Γ of a simple graph Γ. Let R^* be $R^*_{F_\Gamma}$.

By our main result Theorem 3.2, we have the following.

Theorem 4.4. *Consider a simple graph $\Gamma = (V, E)$ with $|V| = r + 1 \geq 3$ and $|E| = n \geq 3$. For $\boldsymbol{a} = (a_1, \dots, a_n) \in (\mathbb{R}_{>0})^n$, we define $L_{\boldsymbol{a}} = a_1x_1 + \cdots + a_nx_n \in R^1$. Then we have the following:*

1. *The linear form $L_{\boldsymbol{a}}$ satisfies the strong Lefschetz property at degree one.*

2. *The Hodge–Riemann bilinear form*

$$Q^1_{L_{\boldsymbol{a}}} : R^1 \times R^1 \to \mathbb{R}, \quad (\xi_1, \xi_2) \to [\xi_1 L_{\boldsymbol{a}}^{r-2} \xi_2]$$

is non-degenerate. Moreover, $Q^1_{L_{\boldsymbol{a}}}$ has $n - 1$ negative eigenvalues and one positive eigenvalue.

Remark 4.5. Related topics are studied in [3]. Huh and Wang study another class of algebras associated to matroids in the paper.

Remark 4.6. Recently, strict log-concavity of the basis generating functions for simple matroids has been proven in [7]. Murai, Nagaoka, and Yazawa use relations between strong Lefschetz property and Hodge–Riemann relation to prove.

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