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Complexes of graphs with bounded independence number

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Abstract. Let $G = (V, E)$ be a graph and n a positive integer. Let $I_n(G)$ be the simplicial complex whose simplices are the subsets of V that do not contain an independent set of size n in G. We study the collapsibility numbers of the complexes $I_n(G)$ for various classes of graphs, focusing on the class of graphs with maximum degree bounded by Δ. As an application, we obtain the following result:

Let G be a claw-free graph with maximum degree at most Δ. Then, every collection of $\left\lfloor \left(\frac{\Delta}{2}+1\right)(n-1)\right\rfloor + 1$ independent sets of size n in G has a rainbow independent set of size n.

Keywords: d-collapsibility, homology, rainbow independent sets

1 Introduction

Let X be a simplicial complex and d a non-negative integer. Let $\sigma \in X$ be a simplex that is contained in a unique maximal face τ of X. If $|\sigma| \leq d$, the operation of removing from X the face σ and all faces containing it is called an *elementary d-collapse*. A complex X is called *d-collapsible* if there exists a sequence of elementary d-collapses that reduces X to the void complex $\emptyset$. The *collapsibility number* of X, denoted by $C(X)$, is the minimum integer d such that X is d-collapsible.

The notion of d-collapsibility of a simplicial complex was introduced by Wegner in [17]. His motivation was the study of intersection patterns of convex sets in Euclidean space: Let $\mathcal{K} = \{K_1, \ldots, K_n\}$ be a family of convex sets in $\mathbb{R}^d$. The *nerve* of $\mathcal{K}$ is the simplicial complex $N(\mathcal{K}) = \{I \subset \{1, 2, \ldots, n\} : \cap_{i \in I} K_i \neq \emptyset\}$. Wegner proved the following:

Theorem 1.1 (Wegner [17])**.** *Let $\mathcal{K}$ be a finite family of convex sets in $\mathbb{R}^d$. Then $C(N(\mathcal{K})) \leq d$.*

In recent years, the notion of d-collapsibility was further investigated (see e.g. [14, 15, 16]), and several combinatorial applications were established (see e.g. [7]). One such combinatorial consequence of d-collapsibility is the following result, due to Kalai and Meshulam:

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Theorem 1.2 (Kalai and Meshulam [12])**.** *Let X be a d-collapsible simplicial complex on vertex set V. Let $V_1, \ldots, V_{d+1}$ be a partition of V into $d+1$ non-empty sets. If $\{v_1, v_2, \ldots, v_{d+1}\} \in X$ for every choice of vertices $v_1 \in V_1, \ldots, v_{d+1} \in V_{d+1}$, then there exists some $1 \leq i \leq d+1$ such that $V_i \in X$.*

Theorem 1.2 is a special case of [12, Theorem 2.1]. In the case where X is the nerve of a family of convex sets in $\mathbb{R}^d$, Theorem 1.2 specializes to Lovász's well known Colorful Helly Theorem.

In this paper, we study the collapsibility of certain simplicial complexes associated to graphs, defined as follows:

Let $G = (V, E)$ be a (simple) graph. A set $I \subset V$ is called an *independent set* in G if no two vertices in I are adjacent in G. The *independence number* of G, denoted by $\alpha(G)$, is the maximal size of an independent set in G. For $U \subset V$, we denote by $G[U]$ the subgraph of G induced by U. For every integer $n \geq 1$, we define the simplicial complex

$$I_n(G) = \{U \subset V : \alpha(G[U]) < n\}.$$

For example, $I_2(G)$ is the *clique complex* of G, i.e. $U \in I_2(G)$ if and only if $G[U]$ is a complete graph. For any graph G, the complex $I_1(G)$ is just the empty complex $\{\emptyset\}$.

Our main motivation for the study of the complexes $I_n(G)$ is the following problem, presented by Aharoni, Briggs, Kim, and Kim in [5]:

Let G be a graph, and let $\mathcal{F} = \{A_1, \ldots, A_m\}$ be a family of (not necessarily distinct) independent sets in G. An independent set A of size $n \leq m$ in G is called a *rainbow independent set* with respect to $\mathcal{F}$ if it can be written as $A = \{a_{i_1}, \ldots, a_{i_n}\}$, where $1 \leq i_1 < i_2 < \cdots < i_n \leq m$ and $a_{i_j} \in A_{i_j}$ for each $1 \leq j \leq n$.

For a positive integer n, let $f_G(n)$ be the minimum integer t such that every collection of t independent sets of size n in G has a rainbow independent set of size n.

The connection between the complexes $I_n(G)$ and the parameters $f_G(n)$ is given by the following result, which follows by a standard application of Theorem 1.2 (see e.g. [6]).

Proposition 1.3. *Let G be a graph. Then $f_G(n) \leq C(I_n(G)) + 1$.*

Thus, any upper bound on the collapsibility number of the complex $I_n(G)$ provides a corresponding bound for $f_G(n)$.

The study of rainbow independent sets originated as a generalization of the "rainbow matching problem" in graphs (note that a matching in a graph is an independent set in its line graph); see e.g. [3, 4, 8]. The study of rainbow matchings via collapsibility numbers was initiated in [6], and further developed in [10]. In [11], the Leray number, a homological variant of the collapsibility number, of complexes of graphs with bounded matching number is studied, and some applications to rainbow matching problems are found.

Here, we study the collapsibility numbers of the complexes $I_n(G)$ for several classes of graphs. We obtain tight upper bounds for the classes of chordal graphs and k-colorable graphs. Our main focus is on the class of graphs with maximum degree bounded by Δ. For graphs in this class we obtain tight upper bounds in the cases $n = 2$ and $n = 3$, and in the case $\Delta \leq 2$ (for general n). Furthermore, in the case $\Delta = 3$, we find new lower bounds for the collapsibility numbers, implying that a certain natural extension of a conjecture in [5] does not hold in general. The proof of the lower bounds is topological; it follows by bounding the Leray numbers of our complexes.

By combining Proposition 1.3 with our different bounds on the collapsibility numbers of the complexes $I_n(G)$, we obtain new proofs for several of the results on rainbow independent sets appearing in [5]. Moreover, we obtain a new upper bound for $f_G(n)$ in the case of bounded degree claw-free graphs.

Remark. This manuscript is an extended abstract. Full proofs and details can be found in the full paper, [13].

2 Upper bounds on collapsibility numbers

Although the concept of d-collapsibility has been relatively well studied, little emphasis has been placed on finding general methods for bounding the collapsibility number of a simplicial complex. In [6] and [10], a strategy analogous to Wegner's proof in [17] of the d-collapsibility of nerves of convex sets in $\mathbb{R}^d$ was applied. Here, we present a set of bounds of a different nature.

The bounds appearing in this section are the main technical tools used for our results on the collapsibility of the complexes $I_n(G)$, presented in Sections 3 and 4. Moreover, we believe these bounds may be of independent interest, and expect them to be useful in the study of other families of simplicial complexes as well.

Let X be a simplicial complex on vertex set V. For $U \subset V$, the subcomplex of X *induced* by U is the complex $X[U] = \{\sigma \in X : \sigma \subset U\}$.

For any vertex $v \in V$, we define the *deletion* of v in X to be the subcomplex

$$X \setminus v = \{\sigma \in X : v \notin \sigma\} = X[V \setminus \{v\}].$$

Let $\tau \subset V$. We define the *link* of τ in X to be the subcomplex

$$\text{lk}(X, \tau) = \{\sigma \in X : \sigma \cap \tau = \emptyset, \sigma \cup \tau \in X\}.$$

Note that $\text{lk}(X, \tau) = \emptyset$ unless $\tau \in X$. If $\tau = \{v\}$, we write $\text{lk}(X, v) = \text{lk}(X, \{v\})$.

Our starting point is the following basic bound, due to Tancer:

Lemma 2.1 (Tancer [16, Prop. 1.2])**.** *Let X be a simplicial complex on vertex set V, and let $v \in V$. Then*

$$C(X) \leq \max\{C(X \setminus v), C(\text{lk}(X, v)) + 1\}.$$

Based on Lemma 2.1, we obtain the following results, which bound the collapsibility number of a simplicial complex in terms of the collapsibility numbers of certain of its subcomplexes:

Lemma 2.2. *Let X be a simplicial complex, and let $\sigma = \{v_1, \dots, v_k\} \in X$. For every $0 \leq i \leq k-1$, define $\sigma_i = \{v_j : 1 \leq j \leq i\}$. Let $d \geq k$. If for all $0 \leq i \leq k-1$,*

$$C(\mathrm{lk}(X \setminus v_{i+1}, \sigma_i)) \leq d - i,$$

and

$$C(\mathrm{lk}(X, \sigma)) \leq d - k,$$

then $C(X) \leq d$.

Lemma 2.3. *Let X be a complex on vertex set V, and let $B \subset V$. Let $<$ be a linear order on the vertices of B. Let $\mathcal{P} = \mathcal{P}(X, B)$ be the family of partitions (B_1, B_2) of B satisfying:*

- *$B_2 \in X$.*
- *For any $v \in B_2$, the complex*

$$\mathrm{lk}(X[V \setminus \{u \in B_1 : u < v\}], \{u \in B_2 : u < v\})$$

  is not a cone over v.

If $C(\mathrm{lk}(X[V \setminus B_1], B_2)) \leq d - |B_2|$ for every $(B_1, B_2) \in \mathcal{P}$, then $C(X) \leq d$.

The proofs of Lemmas 2.2 and 2.3 consist of straightforward inductive applications of Lemma 2.1.

A *missing face* of a complex X is a set $\tau \subset V$ such that $\tau \notin X$, but $\sigma \in X$ for any $\sigma \subsetneq \tau$. The following bound follows by an inductive application of Lemma 2.2:

Proposition 2.4. *Let X be a simplicial complex on vertex set V. If all the missing faces of X are of dimension at most d, then*

$$C(X) \leq \left\lfloor \frac{d|V|}{d+1} \right\rfloor.$$

Moreover, equality $C(X) = \frac{d|V|}{d+1}$ is obtained if and only if X is the join of $r = \frac{|V|}{d+1}$ disjoint copies of the boundary of a d-dimensional simplex (or equivalently, if the set of missing faces of X consists of r disjoint sets of size $d+1$).

Proposition 2.4 can be seen as the "collapsibility version" of [1, Proposition 5.4].

3 Collapsibility numbers of the complexes $I_n(G)$

In this section we present our main results, upper bounds on the collapsibility numbers of $I_n(G)$ for different families of graphs G. The proofs are mostly omitted, but some remarks are made about the methods applied in each case.

3.1 Chordal graphs and k-colorable graphs

Recall that a graph G is *chordal* if it contains no cycle of length at least 4 as an induced subgraph.

Theorem 3.1. *Let $G = (V, E)$ be a chordal graph and $n \geq 1$ an integer. Then $C(I_n(G)) \leq n-1$. Moreover, if $\alpha(G) \geq n$, then $C(I_n(G)) = n-1$.*

It is a well known fact that any chordal graph contains a simplicial vertex; that is, a vertex whose set of neighbours forms a clique in the graph. The proof of Theorem 3.1 relies on the application of Lemma 2.1 with respect to such a simplicial vertex v.

A graph $G = (V, E)$ is said to be *k-colorable* if its vertex set V can be partitioned into k parts $V_1, \ldots, V_k$ so that each V_i is independent in G.

Proposition 3.2. *Let G be a k-colorable graph and $n \geq 1$ an integer. Then $C(I_n(G)) \leq k(n-1)$.*

Proposition 3.2 follows from the stronger fact that for a k-colorable graph G, the dimension of the complex $I_n(G)$ is at most $k(n-1)-1$. The proof is simple: Any simplex $\sigma \in I_n(G)$ contains no independent set of size n. In particular, given a proper k-coloring of G, σ contains at most $n-1$ vertices from each color class. So, $|\sigma| \leq k(n-1)$.

The sharpness of the bound can be demonstrated by taking G to be the complete k-partite graph $G = K_{n,\ldots,n}$ (see Section 5).

3.2 Graphs with bounded maximum degree

One of the main conjectures in [5] is the following.

Conjecture 3.3 (Aharoni, Briggs, Kim, and Kim [5]). *Let G be a graph with maximum degree at most Δ, and let n be a positive integer. Then*

$$f_G(n) \leq \left\lceil \frac{\Delta+1}{2} \right\rceil (n-1) + 1.$$

It is shown in [5] that Conjecture 3.3 is true for $\Delta \leq 2$ and for $n \leq 3$. Moreover, for every n and Δ, an example of a graph G achieving equality is presented. In the general case, the best bound observed by Aharoni et al. is $f_G(n) \leq \Delta(n-1) + 1$.

It is natural to ask whether the following extension of Conjecture 3.3 holds:

Question 3.4 (Aharoni [2]). *Let G be a graph with maximum degree at most Δ, and let n be a positive integer. Does the following bound hold?*

$$C(I_n(G)) \leq \left\lceil \frac{\Delta+1}{2} \right\rceil (n-1).$$

Our main results for this family of graphs are the following:

Theorem 3.5. *Let $G = (V, E)$ be a graph with maximum degree at most Δ, and let $n \geq 1$ be an integer. Then $C(I_n(G)) \leq \Delta(n-1)$.*

The bound in Theorem 3.5 is tight only for $\Delta \leq 2$. In the case $n \leq 3$ we can prove the following tight bounds, for general Δ:

Theorem 3.6. *Let $G = (V, E)$ be a graph with maximum degree at most Δ. Then*

$$C(I_2(G)) \leq \left\lceil \frac{\Delta+1}{2} \right\rceil.$$

Theorem 3.7. *Let $G = (V, E)$ be a graph with maximum degree at most Δ. Then*

$$C(I_3(G)) \leq \begin{cases} \Delta+2 & \text{if } \Delta \text{ is even,} \\ \Delta+1 & \text{if } \Delta \text{ is odd.} \end{cases}$$

The proof of Theorem 3.5 divides into two cases: The more difficult case is when $\Delta \leq 2$; the proof in this case proceeds by first applying Lemma 2.2 in order to reduce the problem to the case of chordal graphs, and then using Theorem 3.1. The case $\Delta > 2$ is easier, and follows from Brooks' Theorem on the chromatic number of graphs with bounded degree and Proposition 3.2.

Theorem 3.6 follows quite simply by an application of Lemma 2.1 and Proposition 2.4, based on the observation that for any vertex $v \in V$, all the vertices in the subcomplex $\text{lk}(I_2(G), v)$ are neighbours of v in G, and therefore $\text{lk}(I_2(G), v)$ contains at most Δ vertices.

Our most technically challenging result is Theorem 3.7. The main ingredient in the proof is the following claim:

Proposition 3.8. *Let $G = (V, E)$ be a graph with maximum degree at most Δ. For a vertex $v \in V$, let $N_G(v)$ be the set of neighbours of v in G. Let $A = \{a_1, a_2\}$ be an independent set of size 2 in G. Assume that there exists an independent set in G of the form $\{a_1, w, w'\}$, where $w, w' \in N_G(a_2)$, or there exists an independent set of the form $\{a_2, v, v'\}$, where $v, v' \in N_G(a_1)$. Then*

$$C(\text{lk}(I_3(G), A)) \leq \begin{cases} \Delta & \text{if } \Delta \text{ is even,} \\ \Delta-1 & \text{if } \Delta \text{ is odd.} \end{cases}$$

We give a sketch of the proof:

Sketch of proof. Let $v \notin N_G(a_1) \cup N_G(a_2)$. Then $A \cup \{v\}$ is an independent set of size 3 in G; hence, $v \notin \text{lk}(I_3(G), A)$. So, we may assume without loss of generality that $V = A \cup N_G(a_1) \cup N_G(a_2)$.

Let $B = N_G(a_1) \cap N_G(a_2)$ and $U = (N_G(a_1) \cup N_G(a_2)) \setminus B$. Since the maximum degree of a vertex in G is at most Δ, we have

$$|N_G(a_1) \cup N_G(a_2)| = |N_G(a_1)| + |N_G(a_2)| - |N_G(a_1) \cap N_G(a_2)| \leq 2\Delta - |B|.$$

So, $|U| \leq 2\Delta - 2|B|$.

Write $B = \{u_1, \ldots, u_k\}$. Let $\mathcal{P} = \mathcal{P}(\mathrm{lk}(I_3(G), A), B)$ be the family of partitions (B_1, B_2) of B satisfying:

- $B_2 \in \mathrm{lk}(I_3(G), A)$.
- For any $u_i \in B_2$, the complex

  $$\mathrm{lk}(I_3(G)[V \setminus \{u_j \in B_1 : j < i\}], A \cup \{u_j \in B_2 : j < i\})$$

  is not a cone over u_i.

Let $(B_1, B_2) \in \mathcal{P}$, and let $X = \mathrm{lk}(I_3(G)[V \setminus B_1], A \cup B_2)$. Note that the vertex set of X is contained in U. Since X is a subcomplex of $I_3(G)$, its missing faces are either independent sets of size 3 in G, or subsets of size 2 of such independent sets. We will show that all the missing faces are of the latter kind: Let $\tau \subset U$ be a missing face of X. Assume for contradiction that $|\tau| = 3$. Then, τ forms an independent set in G, and for any $\tau' \subset \tau$ of size 2, $\tau' \in X$; that is, $\tau' \cup (A \cup B)$ does not contain an independent set of size 3 in G.

The vertex a_1 is adjacent to at least 2 vertices of τ. Otherwise, there would be a set $\tau' \subset \tau$ of size 2 such that $\tau' \cup \{a_1\}$ forms an independent set in G, a contradiction to τ being a missing face. Similarly, a_2 is adjacent to at least 2 vertices of τ. Therefore, there exists a vertex in $u \in \tau$ that is adjacent to both a_1 and a_2. That is, $u \in B$. But this is a contradiction to $\tau \subset U$ and $U \cap B = \emptyset$.

Since all the missing faces of X are of size 2, by Proposition 2.4 we obtain

$$C(X) \leq \frac{|U|}{2} \leq \frac{2\Delta - 2|B|}{2} = \Delta - |B| \leq \Delta - |B_2|. \tag{3.1}$$

Therefore, by Lemma 2.3, $C(\mathrm{lk}(I_3(G), A)) \leq \Delta$.

This finishes the proof for the case of even Δ. The odd Δ case consists of considerably more work, so we only state the main steps in the proof:

Assume that Δ is odd. For $B_2 \neq B$, we have by (3.1), $C(X) \leq \Delta - |B| \leq \Delta - 1 - |B_2|$.

We are left to show that, whenever $(\emptyset, B) \in \mathcal{P}$, the complex $\mathrm{lk}(I_3(G), A \cup B)$ is $(\Delta - |B| - 1)$-collapsible. This is proved by showing that the complex does not satisfy the equality case in Proposition 2.4. Note that this is the only step where the additional conditions on G stated in the the proposition are needed. $\square$

Theorems 3.5 to 3.7 settle Question 3.4 affirmatively in the special cases where $\Delta \leq 2$ or $n \leq 3$. Unfortunately, the bound in Question 3.4 does not hold in general: In Section 5 we present a family of counterexamples to the case $\Delta = 3$.

4 Rainbow independent sets

Combining the bounds from Section 3 with Proposition 1.3, we can recover several of the upper bounds for $f_G(n)$ first proved by Aharoni, Briggs, Kim, and Kim in [5]:

Corollary 4.1 ([5, Theorem 3.20]). *Let $G = (V, E)$ be a chordal graph and $n \geq 1$ an integer. Then $f_G(n) \leq n$.*

Corollary 4.2 ([5, Theorem 4.1]). *Let G be a k-colorable graph and $n \geq 1$ an integer. Then $f_G(n) \leq k(n-1)+1$.*

Corollary 4.3 ([5]). *Let $G = (V, E)$ be a graph with maximum degree at most Δ and $n \geq 1$ an integer. Then $f_G(n) \leq \Delta(n-1)+1$.*

Corollary 4.4 ([5, Theorem 5.6]). *Let $G = (V, E)$ be a graph with maximum degree at most Δ. Then*

$$f_G(2) \leq \left\lceil \frac{\Delta+1}{2} \right\rceil + 1.$$

Corollary 4.5 ([5, Theorem 5.7]). *Let $G = (V, E)$ be a graph with maximum degree at most Δ. Then*

$$f_G(3) \leq \begin{cases} \Delta+3 & \text{if } \Delta \text{ is even,} \\ \Delta+2 & \text{if } \Delta \text{ is odd.} \end{cases}$$

The following bound, however, is new: Recall that a graph is called *claw-free* if it does not contain the complete bipartite graph $K_{1,3}$ as an induced subgraph.

Theorem 4.6. *Let G be a claw-free graph with maximum degree at most Δ. Then*

$$f_G(n) \leq \left\lfloor \left(\frac{\Delta}{2}+1\right)(n-1) \right\rfloor + 1.$$

Theorem 4.6 shows that Conjecture 3.3 holds for the subclass of claw-free graphs with maximum degree at most Δ, in the case where Δ is even. The proof relies on the following result:

Proposition 4.7. *Let $G = (V, E)$ be a claw-free graph with maximum degree at most Δ, and let $n \geq 1$ be an integer. Let A be an independent set of size $n-1$ in G. Then,*

$$C(\text{lk}(I_n(G), A)) \leq \left\lfloor \frac{(n-1)\Delta}{2} \right\rfloor.$$

The main observation is that, since G is claw free, any vertex outside of the independent set A is adjacent to at most 2 vertices in A. This enables us to apply an argument very similar to the one used in the proof of Proposition 3.8.

5 Lower bounds on Leray numbers

For $i \geq -1$, let $\tilde{H}_i(X)$ be the i-th reduced homology group of X with real coefficients. We say that X is *d-Leray* if for any induced subcomplex Y of X, $\tilde{H}_i(Y) = 0$ for all $i \geq d$. The *Leray number* of X, denoted by $L(X)$, is the minimum integer d such that X is d-Leray.

The following observation, due to Wegner, relates the Leray number of a complex X to its collapsibility number:

Lemma 5.1 (Wegner [17])**.** *Let X be a simplicial complex. Then $C(X) \geq L(X)$.*

In this section we present some examples establishing the sharpness of our different bounds on the collapsibility of $I_n(G)$. Also, we present a family of counterexamples to the conjectural bound presented in Question 3.4, in the case of graphs with maximum degree at most 3.

5.1 Extremal examples

Let n be an integer, and k be an even integer. Let $G_{k,n}$ be the graph obtained from a cycle of length $\left(\frac{k}{2}+1\right)n$ by adding all edges connecting two vertices of distance at most $\frac{k}{2}$ in the cycle. Note that $G_{k,n}$ is a k-regular graph, i.e. every vertex has degree exactly k. Moreover, $G_{k,n}$ is claw-free.

In [5] it is shown that $f_{G_{k,n}}(n) \geq \left(\frac{k}{2}+1\right)(n-1)+1$. In particular, this shows the tightness of Theorem 4.6, in the case of even maximum degree. Moreover, by Proposition 1.3, we obtain

$$C(I_n(G_{k,n})) \geq f_{G_{k,n}}(n) - 1 \geq \left(\frac{k}{2}+1\right)(n-1).$$

This shows that the bound in Question 3.4, whenever it holds, is tight. A different way to show this is as follows.

Proposition 5.2.

$$\tilde{H}_i(I_n(G_{k,n})) = \begin{cases} \mathbb{R} & \text{if } i = \left(\frac{k}{2}+1\right)(n-1)-1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $L(I_n(G_{k,n})) \geq \left(\frac{k}{2}+1\right)(n-1)$.

The proof follows from the fact that the complex $I_n(G_{k,n})$ can be described as the join of $\frac{k}{2}+1$ disjoint copies of the boundary of an $(n-1)$-dimensional simplex; hence, it is homotopy equivalent to a $\left(\left(\frac{k}{2}+1\right)(n-1)-1\right)$-dimensional sphere.

Similarly, for $G = K_{n,\ldots,n}$, the complete k-partite graph with parts of size n, the complex $I_n(G)$ is the join of k disjoint copies of the boundary of an $(n-1)$-dimensional simplex. Therefore, $C(I_n(G)) \geq k(n-1)$, showing the bound in Proposition 3.2 is tight.

5.2 A negative answer to Question 3.4

Let $G = (V, E)$ be the dodecahedral graph. We can represent G as a generalized Petersen graph (see [18]), as follows:

$$V = \{a_1, \ldots, a_{10}, b_1, \ldots, b_{10}\}$$

and

$$E = \{\{a_i, b_i\}, \{a_i, a_{i+1}\}, \{b_i, b_{i+2}\} : i = 1, 2, \ldots, 10\},$$

where the indices are taken modulo 10.

Every vertex in G is adjacent to exactly 3 vertices; that is, G is 3-regular. The maximum independent sets in G are the sets

$$I_i = \{a_i, a_{i+2}, a_{i+5}, a_{i+7}, b_{i-2}, b_{i-1}, b_{i+3}, b_{i+4}\}$$

for $i = 1, \ldots, 5$ (again, the indices are to be taken modulo 10). In particular, $\alpha(G) = 8$.

Proposition 5.3. *Let $G = (V, E)$ be the dodecahedral graph. Then,*

$$\tilde{H}_i(I_8(G)) = \begin{cases} \mathbb{R}^4 & \text{if } i = 15, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $L(I_8(G)) \geq 16$.

We obtain $C(I_8(G)) \geq L(I_8(G)) \geq 16 > 2 \cdot (8 - 1) = 14$. Therefore, $I_8(G)$ does not satisfy the bound in Question 3.4. However, it is not hard to check that $f_G(8) \leq 11$. So, G does not contradict Conjecture 3.3. The proof of Proposition 5.3 follows by a standard application of Alexander duality and the Nerve theorem (see e.g. [9]).

The next result will aid us in constructing more examples of complexes that do not satisfy the bound in Question 3.4:

Theorem 5.4. *Let G be the disjoint union of the graphs $G_1, \ldots, G_m$. For $1 \leq i \leq m$, let $t_i = \alpha(G_i)$ and let $\ell_i = L(I_{t_i}(G_i))$. Let $t = \sum_{i=1}^m t_i = \alpha(G)$ and $\ell = L(I_t(G))$. Then,*

$$\ell = \sum_{i=1}^{m} \ell_i + m - 1.$$

The proof relies on the following claim.

Proposition 5.5. *Let G be the disjoint union of the graphs $G_1, \ldots, G_m$. For $1 \leq i \leq m$, let $t_i = \alpha(G_i)$. Let $t = \sum_{i=1}^m t_i = \alpha(G)$. Then, $\tilde{H}_k(I_t(G)) = 0$ if and only if for every choice of integers $k_1, \ldots, k_m$ satisfying $\sum_{i=1}^m k_i = k - 2m + 2$, $\tilde{H}_{k_i}(I_{t_i}(G_i)) = 0$ for all $1 \leq i \leq m$.*

Combining Theorem 5.4 with Proposition 5.3, we obtain:

Corollary 5.6. *Let G_k be the union of k disjoint copies of the dodecahedral graph. Then*

$$L(I_{8k}(G_k)) \geq 17k - 1.$$

Note that the graphs G_k are 3-regular, and $\frac{L(I_{8k}(G_k))}{8k-1} \geq \frac{17k-1}{8k-1} > 2\frac{1}{8} > 2$. Thus, the complexes $I_{8k}(G_k)$ do not satisfy the bound in Question 3.4.

6 Open questions

In this paper we showed that the answer to Question 3.4 is positive in several cases, but negative in general. It would be interesting to decide for which values of Δ and n the bound in Question 3.4 holds. Alternatively, one could try to characterize the graphs satisfying the bound for all values of n.

The following is a weaker result, which may hold for general bounded degree graphs:

Conjecture 6.1. *Let $G = (V, E)$ be a graph with maximum degree at most Δ, and let $n \geq 1$ be an integer. Let A be an independent set of size $n - 1$ in G. Then,*

$$C(\mathrm{lk}(I_n(G), A)) \leq \left\lfloor \frac{(n-1)\Delta}{2} \right\rfloor.$$

For the subclass of claw-free graphs, this is proved in Proposition 4.7. Conjecture 6.1 would imply the bound $f_G(n) \leq \left\lfloor \left(\frac{\Delta}{2} + 1\right)(n-1) \right\rfloor + 1$ (by the same argument as the one used to prove Theorem 4.6), settling Conjecture 3.3 in the case of even Δ.

Another possible direction is to focus on the family of claw-free bounded degree graphs. We showed in Theorem 4.6 that Conjecture 3.3 holds for graphs in this family when Δ is even. In the case of odd Δ, although we obtain good upper bounds for $f_G(n)$, the question remains unsettled. It would also be interesting to prove the corresponding tight upper bound on the collapsibility number of $I_n(G)$, at least for the case of even Δ.

We know, by Proposition 5.3, that the bound in Question 3.4 does not hold for graphs with maximum degree at most 3. The following question arises:

Question 6.2. *What is the smallest positive integer $g(n)$ such that the following holds: For every graph G with maximum degree at most 3, $C(I_n(G)) \leq g(n)$?*

By Theorem 3.5 and Proposition 5.2 we have $2(n-1) \leq g(n) \leq 3(n-1)$ for all $n \geq 1$, and, by Corollary 5.6, $g(8k) \geq 17k - 1$ for all $k \geq 1$. Improving either the upper or lower bounds for $g(n)$ may be of interest.

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