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A characteristic map for the symmetric space of symplectic forms over a finite field

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Abstract. The characteristic map for the symmetric group is an isomorphism relating the representation theory of the symmetric group to symmetric functions. An analogous isomorphism is constructed for the symmetric space of symplectic forms over a finite field, with the spherical functions being sent to Macdonald polynomials with parameters (q, q^2). An analogue of parabolic induction is interpreted as a certain multiplication of symmetric functions. Applications are given to Schur-positivity of skew Macdonald polynomials with parameters (q, q^2).

Keywords: Characteristic map, symmetric space, Macdonald polynomials

There is a well-known isomorphism, the characteristic map, between the class functions on the symmetric groups and the ring of symmetric functions. This has been extended to $\mathrm{GL}_n(\mathbf{F}_q)$ and the Gelfand pair S_{2n}/B_n (B_n denotes the hyperoctahedral group).

We develop an analogous theory for $\mathrm{GL}_{2n}(\mathbf{F}_q)/\mathrm{Sp}_{2n}(\mathbf{F}_q)$ and use it to study the Schur expansion of Macdonald polynomials. The symmetric space $\mathrm{GL}_{2n}(\mathbf{F}_q)/\mathrm{Sp}_{2n}(\mathbf{F}_q)$ is a natural q-analogue of the Gelfand pair S_{2n}/B_n; the former can be seen as the Weyl group version of the latter.

This extended abstract summarizes the results in [8] and outlines some of the ideas in the proofs. See [8] for details and full proofs, as well as another application to combinatorial formulas for spherical function values and a connection to work of Henderson [9], Grojnowski [7], and Shoji and Sorlin [11, 12, 13] on character sheaves on symmetric spaces.

The main result is to construct a characteristic map

$$\mathrm{ch} : \bigoplus_n \mathbf{C}[\mathrm{Sp}_{2n}(\mathbf{F}_q)\backslash \mathrm{GL}_{2n}(\mathbf{F}_q)/\mathrm{Sp}_{2n}(\mathbf{F}_q)] \to \bigotimes \Lambda$$

from the space of bi-invariant functions to a ring of symmetric functions and establish some basic properties. In particular, the spherical functions are shown to map to Macdonald polynomials with parameters (q, q^2). A"mixed product" can be defined using parabolic induction which takes a bi-invariant function on $\mathrm{GL}_{2n}(\mathbf{F}_q)$ and a class function on $\mathrm{GL}_m(\mathbf{F}_q)$ and produces a bi-invariant function on $\mathrm{GL}_{2(n+m)}(\mathbf{F}_q)$ and this corresponds to the product of symmetric functions under the characteristic map, up to a twist.

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The characteristic map is then applied to Schur-positivity of certain Macdonald polynomials. If $P_{\lambda/\mu}(x;q,t)$ denotes the skew Macdonald polynomials, define

$$C^{\nu}_{\lambda/\mu}(q,t) := \langle P_{\lambda/\mu}(q,t), s_\nu \rangle.$$

These are the coefficients of the expansion of $P_{\lambda/\mu}$ in terms of Schur functions. They can be considered a deformed Littlewood-Richardson coefficient.

The following theorem is proven using the characteristic map.

Theorem 0.1. *Let $q = p^e$ denote an odd prime power. Then for any partitions μ, ν, and λ,*

$$C^{\nu}_{\lambda/\mu}(q,q^2) \geq 0.$$

This theorem is related to a conjecture of Haglund [15], which states that

$$\frac{\langle J_\lambda(q,q^k), s_\mu \rangle}{(1-q)^{|\lambda|}} \in \mathbf{N}[q].$$

Taking $\mu = 0$ in Theorem 0.1 shows

$$\frac{\langle J_\lambda(q,q^2), s_\mu \rangle}{(1-q)^{|\lambda|}} \geq 0$$

when q is an odd prime power and gives further evidence for the conjecture.

In addition to the examples of characterstic maps already mentioned for S_n, $\mathrm{GL}_n(\mathbf{F}_q)$ and S_{2n}/B_n (all of which are explained in detail in [10], there are further examples of characteristic maps appearing in the literature. In particular, Thiem and Vinroot constructed such a map for $\mathrm{U}_n(\mathbf{F}_{q^2})$ [14] and in [1] a map is constructed between the supercharacters of the unipotent upper triangular matrices over a finite field and symmetric functions in non-commuting variables.

The Schur expansion of integral Macdonald polynomials has been previously studied by Yoo who gave combinatorial formulas for the expansion coefficients in some special cases showing that they were polynomials in q with positive integer coefficients [15, 16]. Some related coefficients in the Jack case when $q \to 1$ were studied in [2].

The $q \to 1$ case was previously studied by Bergeron and Garsia [4] (see also [10]), where the Gelfand pair S_{2n}/B_n played a similar role. In this case the multiplication on symmetric functions has a representation-theoretic interpretation giving results on the structure coefficients of Jack polynomials for $\alpha = 2$.

The paper is organized as follows. In Section 1, notation and preliminary background is reviewed. Section 2 develops the theory of parabolic induction for bi-invariant functions in an elementary way. In Section 3, the characteristic map for $\mathrm{GL}_{2n}(\mathbf{F}_q)/\mathrm{Sp}_{2n}(\mathbf{F}_q)$ is constructed. Section 4 gives an application to positivity and vanishing of the Schur expansion of skew Macdonald polynomials with parameters (q,q^2).

1 Preliminaries

If $f(q)$ is a rational function in q, define $f(q)_{q\mapsto q^2} := f(q^2)$. For a symmetric function f with rational coefficients in q when written in terms of p_μ, write $f_{q\mapsto q^2}$ to denote the symmetric function obtained by replacing q with q^2 in each coefficient.

Let G be a finite group. If $S \subseteq G$ is some subset, I_S will denote the indicator function for that set. If $H \subseteq G$ is a subgroup, let $\mathbf{C}[G]^G$ denote the set of class functions on G and let $\mathbf{C}[H\backslash G/H]$ denote the set of H bi-invariant functions on G. If S, H are subgroups of any group G, let $H_S = H \cap S$.

Given an element x of some finite field extension of $\mathbf{F}_q$, let f_x denote its minimal polynomial. As a matter of convention, algebraic groups G will be defined over $\overline{\mathbf{F}_q}$ while $G(\mathbf{F}_q)$ or G^F (F being the Frobenius endomorphism) will denote the points over $\mathbf{F}_q$.

1.1 Macdonald polynomials

For details on Macdonald polynomials including a construction and proofs, see [10]. Let Λ denote the ring of symmetric functions. Consider the inner product

$$\langle p_\lambda, p_\mu\rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod \frac{q^{\lambda_i}-1}{t^{\lambda_i}-1},$$

where $z_\lambda = \prod m_i(\lambda)! i^{m_i(\lambda)}$ and $m_i(\lambda)$ denotes the number of parts of size i in λ. When the context is clear, the dependence on (q,t) may be dropped.

The *Macdonald polynomials* $P_\lambda(x;q,t)$ are defined by the fact that they are orthogonal with respect to this inner product, and the change of basis to the monomial basis is upper triangular with 1 along the diagonal. When $q = 0$, the Macdonald polynomials are known as *Hall-Littlewood polynomials*, and are written $P_\lambda(x;t) = P_\lambda(x;0,t)$.

Define

$$c_\lambda(q,t) := \prod_{s\in\lambda}(1 - q^{a(s)} t^{l(s)+1}),$$

where $a(s)$ and $l(s)$ denote the arm and leg lengths respectively (so $a(s) + l(s) + 1 = h(s)$). Similarly define

$$c'_\lambda(q,t) := \prod_{s\in\lambda}(1 - q^{a(s)+1} t^{l(s)}).$$

The dual basis to the $P_\lambda(x;q,t)$ under $\langle,\rangle_{q,t}$ are denoted $Q_\lambda(x;q,t)$. The Macdonald polynomials have an integral form $J_\lambda(x;q,t) = c_\lambda(q,t)P_\lambda(x;q,t)$ and $\langle J_\lambda, J_\lambda\rangle = c_\lambda(q,t)c'_\lambda(q,t)$.

If λ/μ is a skew shape, the skew Macdonald polynomial $P_{\lambda/\mu}(x;q,t)$ is defined by requiring $\langle P_{\lambda/\mu}(q,t), f\rangle = \langle P_\lambda(q,t), Q_\mu(q,t) f\rangle$ for all symmetric functions f.

Let $\omega : \Lambda \to \Lambda$ denote the involution taking p_n to $(-1)^{n-1}p_n$ (and extending to make it an algebra homomorphism). Let $\omega_{q,t} : \Lambda \to \Lambda$ denote the involution defined by

$$\omega_{q,t} p_n = (-1)^{n-1}\frac{q^n-1}{t^n-1} p_n.$$

The involution ω satisfies $\omega s_\lambda = s_{\lambda'}$ while the involution $\omega_{q,t}$ satisfies

$$\omega_{q,t} P_\lambda(x;q,t) = Q_{\lambda'}(x;t,q)$$
$$\omega_{q,t} Q_\lambda(x;q,t) = P_{\lambda'}(x;t,q).$$

1.2 Representation theory of $\mathrm{GL}_n(\mathbf{F}_q)$

To fix notation, the representation theory of $\mathrm{GL}_n(\mathbf{F}_q)$ is briefly reviewed. The representation theory of $\mathrm{GL}_n(\mathbf{F}_q)$ was originally developed by Green in [6] but this section follows the conventions in [10]. Let $\mathbf{M}$ denote the group of units of $\overline{\mathbf{F}}_q$ and let $\mathbf{M}_n$ denote the fixed points of F^n with F the Frobenius endomorphism $F(x) = x^q$. Note that $\mathbf{M}_n$ may be identified with $\mathbf{F}_{q^n}^*$. Let $\mathbf{L}$ be the character group of the inverse limit of the $\mathbf{M}_n$ with norm maps between them. The Frobenius endomorphism F acts on $\mathbf{L}$ in a natural manner so let $\mathbf{L}_n$ denote the F^n fixed points in $\mathbf{L}$, and note there is a pairing of $\mathbf{L}_n$ with $\mathbf{M}_n$.

Denote by $O(\mathbf{M})$ and $O(\mathbf{L})$ the F-orbits in $\mathbf{M}$ and $\mathbf{L}$ respectively, with $f \in O(\mathbf{M})$ identified with irreducible polynomials. Use $\mathcal{P}$ to denote the set of partitions. Then the conjugacy classes of $\mathrm{GL}_n(\mathbf{F}_q)$, denoted C_μ, are indexed by functions $\mu : O(\mathbf{M}) \to \mathcal{P}$ with

$$\|\mu\| := \sum_{f \in O(\mathbf{M})} d(f)|\mu(f)| = n,$$

where $d(f)$ denotes the degree of f. Use q_f to denote $q^{d(f)}$. The irreducible characters of $\mathrm{GL}_n(\mathbf{F}_q)$, denoted χ_λ, are indexed by functions $\lambda : O(\mathbf{L}) \to \mathcal{P}$ with

$$\|\lambda\| := \sum_{\varphi \in O(\mathbf{L})} d(\varphi)|\lambda(\varphi)| = n,$$

where $d(\varphi)$ denotes the size of the orbit φ. The dimension of the irreducible representation corresponding to λ is given by

$$d_\lambda = \psi_n(q) \prod_{\varphi \in O(\mathbf{L})} q_\varphi^{n(\lambda(\varphi)')} \prod_{x \in \lambda} (q_\varphi^{h(x)} - 1)^{-1},$$

where $\psi_n(q) = \prod_{i=1}^n (q^i - 1)$, $q_\varphi = q^{d(\varphi)}$, $n(\lambda) = \sum (i-1)\lambda_i$, and h denotes hook length.

As a matter of convention, μ will always be used to denote partition-valued functions $O(\mathbf{M}) \to \mathcal{P}$ while λ will be used to denote partition-valued functions $O(\mathbf{L}) \to \mathcal{P}$. In general, if there is some expression involving a partition μ, $F(\mu)$, with $F(0) = 1$, then the same expression with $\mu : O(\mathbf{M}) \to \mathcal{P}$ will be defined as the product $\prod_{f \in O(\mathbf{M})} F(\mu(f))$. Thus, write $z_\mu = \prod_{f \in O(\mathbf{M})} z_{\mu(f)}$. If the expression contains q, in each factor it should be replaced by q_f. A similar convention is used for $\lambda : O(\mathbf{L}) \to \mathcal{P}$.

The *Deligne-Lusztig characters* give another basis for the space of class functions on $\mathrm{GL}_n(\mathbf{F}_q)$. For a more detailed overview of Deligne-Lusztig characters, including their construction and properties, see the book of Carter [5].

Given any rational maximal torus $T \subseteq \mathrm{GL}_n$, a (virtual) character $\zeta_T^{\mathrm{GL}_n}(\cdot|\theta)$ of $\mathrm{GL}_n(\mathbf{F}_q)$ associated to some irreducible character θ of T^F can be constructed. The character constructed depends only on the $\mathrm{GL}_n(\mathbf{F}_q)$-conjugacy class of the pair (T, θ).

Deligne-Lusztig characters are rational functions in q in the following sense. Given an element $g \in \mathrm{GL}_n(\mathbf{F}_q)$ with Jordan decomposition $g = su$, the Deligne-Lusztig characters can be computed as

$$\zeta_T^{\mathrm{GL}_n}(g|\theta) = \sum_{\substack{x \in (\mathrm{GL}_n / Z(s))^F \\ xsx^{-1} \in T^F}} \theta(xsx^{-1}) Q_{x^{-1}Tx}^{Z(s)}(u),$$

where $Q_{x^{-1}Tx}^{Z(s)}(u)$ is a rational function of q, known as the *Green function*. The Green functions can be computed as $Q_{x^{-1}Tx}^{Z(s)}(u) = \prod_{f \in O(\mathbf{M})} Q_{\gamma(f)}^{\mu(f)}(q_f)$, where $Q_\rho^\mu(q)$ denotes the *Green polynomials*, μ is a partition valued function indexing the conjugacy class of g and $\gamma(f)$ is the partition given by taking $s \in x^{-1}T^F x \cong \prod \mathbf{M}_{k_i}$ and including as parts the $k_i/d(f)$ for which f kills s restricted to $\mathbf{M}_{k_i}$.

The Green polynomials give the change of basis from power sum to Hall-Littlewood polynomials and so satisfy $p_\rho(x) = \sum_\mu Q_\rho^\mu(t) t^{-n(\mu)} P_\mu(x; t^{-1})$.

There is a correspondence between G^F orbits of pairs (T, θ) and functions $\lambda : O(\mathbf{L}) \to \mathcal{P}$ with $\|\lambda\| = n$. Call the function λ the *combinatorial data* associated to (T, θ).

The correspondence is as follows. Given a torus T with rational points isomorphic to $\prod \mathbf{M}_{k_i}$, and a character θ of T^F, consider the partition-valued function sending φ to the partition with parts $k_i/d(\varphi)$ for all i such that θ restricted to $\mathbf{M}_{k_i}$ lies in the orbit φ. Conversely, given λ, T can be constructed as a torus with rational points isomorphic to $\prod_{\varphi,i} \mathbf{M}_{\lambda(\varphi)_i d(\varphi)}$ and θ is given by picking for each factor $\mathbf{M}_{\lambda(\varphi)_i d(\varphi)}$ an element of the orbit φ (which determines T and θ up to conjugacy by G^F).

1.3 The symmetric space $\mathrm{GL}_{2n}(\mathbf{F}_q) / \mathrm{Sp}_{2n}(\mathbf{F}_q)$

Let J be a matrix defining a symplectic form on $\mathbf{F}_q^{2n}$ and define the involution

$$\iota(X) = -J(X^T)^{-1}J$$

of GL_{2n}. For convenience, take J such that the subgroups of diagonal matrices and upper triangular matrices are stable under ι.

Now for any ι-stable subgroup S, let S^ι denote the subgroup of ι fixed points and $S^{-\iota}$ denote the set of *ι-split* elements, which are elements $s \in S$ such that $\iota(s) = s^{-1}$. Then $\mathrm{Sp}_{2n} = \mathrm{GL}_{2n}^\iota$ and $\mathrm{GL}_{2n} / \mathrm{Sp}_{2n} \cong \mathrm{GL}_{2n}^{-\iota}$.

The space $\mathrm{GL}_n(\mathbf{F}_q) / \mathrm{Sp}_n(\mathbf{F}_q)$ is a Gelfand pair, and the spherical functions, denoted ϕ_λ, are indexed by partition-valued function $\lambda : O(\mathbf{L}) \to \mathcal{P}$ with $\|\lambda\| = n$ (see [3], although note a different convention is used in this paper so all partitions labeling representations are transposed). Then ϕ_λ is the spherical function corresponding to the representation with character $\chi_{\lambda\cup\lambda}$ of $\mathrm{GL}_{2n}(\mathbf{F}_q)$.

Let $M_\mu \in \mathrm{GL}_n(\mathbf{F}_q)$ denote a conjugacy class representative of C_μ. Let $g_\mu \in \mathrm{GL}_{2n}(\mathbf{F}_q)$ denote the matrix acting only on the first n coordinates by M_μ. The $\mathrm{Sp}_{2n}(\mathbf{F}_q)$-double cosets of are indexed by $\mu : O(\mathbf{M}) \to \mathcal{P}$, with $\|\mu\| = n$, with the g_μ being double coset representatives.

A key result from [3] gives a formula for the values of spherical functions in terms of Deligne-Lusztig characters on $\mathrm{GL}_n(\mathbf{F}_q)$.

The function $\zeta_T^{\mathrm{GL}_n}(\cdot|\theta)$ (or any other class function on $\mathrm{GL}_n(\mathbf{F}_q)$) may be turned into an H^F-bi-invariant function on $\mathrm{GL}_{2n}(\mathbf{F}_q)$ by defining $\zeta_T^{\mathrm{GL}_n}(g_\mu|\theta) = \zeta_T^{\mathrm{GL}_n}(M_\mu|\theta)$ and extending to the double-coset. It turns out that the correct analogue for Deligne-Lusztig characters are given by the *basic functions*

$$\zeta_T^{G/H}(\cdot|\theta) = \zeta_T^{\mathrm{GL}_n}(\cdot|\theta)_{q\mapsto q^2}.$$

The following theorem relates the spherical functions to the basic functions.

Theorem 1.1 ([3, Theorem 6.6.1]). *Let $\lambda : O(\mathbf{L}) \to \mathcal{P}$ with $\|\lambda\| = n$. Then*

$$\phi_\lambda = \frac{(-1)^{|\lambda|}}{|W(\lambda)|} \sum_{w\in W(\lambda)} \left(\prod_{\varphi\in O(\mathbf{L})} q_\varphi^{-n(\lambda(\varphi)')} d_{\lambda(\varphi)'}(w)(q_\varphi) \right) \operatorname{sgn}(w) \frac{\zeta_{T_w}^{G/H}(\cdot|\theta_w)}{|T_w|\zeta_{T_w}^{G/H}(1|\theta_w)},$$

where $d_\lambda(w)(q)$ denotes the change of basis from power sum to $J_\lambda(q^2, q)$ and sgn *denotes the sign character of $W(\lambda)$.*

2 Induction and restriction

This section develops parabolic induction on symmetric spaces in an elementary way. It was originally defined by Grojnowski in [7] for character sheaves and most results in this section follow easily from more general results proved in [7] or [9].

Fix a rational ι-stable pair (T, B) of a maximal torus T and Borel subgroup B containing T. For convenience take T to be the diagonal matrices and B to be the upper triangular matrices in GL_n. Then the standard parabolic subgroups are the block upper-triangular ones, while the ι-stable ones are those for which the block structure is symmetric about the main anti-diagonal. All pairs of rational ι-stable (T, B) with $T \subseteq B$ are conjugate under H^F (see [11] for example).

Let P be a rational ι-stable parabolic subgroup with rational ι-stable Levi factor L and unipotent radical U. Then define a function

$$\mathrm{Ind}_{L\subseteq P}^{G/H} : \mathbf{C}[H_L^F\backslash L^F/H_L^F] \to \mathbf{C}[H^F\backslash G^F/H^F]$$

with the formula

$$\operatorname{Ind}_{L\subseteq P}^{G}(f)(x) = |H^F \cap P^F|^{-2} \sum_{\substack{h,h'\in H^F \\ hxh'\in P^F}} f(\overline{hxh'}).$$

The next two results show that induction is unchanged under conjugation by H^F and that it is transitive, like for regular parabolic induction.

Lemma 2.1. *For any $h_0 \in H^F$ and $f \in \mathbf{C}[H_L^F\backslash L^F/H_L^F]$,*

$$\operatorname{Ind}_{h_0Lh_0^{-1}\subseteq h_0Ph_0^{-1}}^{G/H}({}^{h_0}f) = \operatorname{Ind}_{L\subseteq P}^{G/H}(f)$$

where ${}^{h_0}f$ is the function ${}^{h_0}f(x) = f(h_0^{-1}xh_0)$.

Proposition 2.2. *If $M \subseteq L$ are rational ι-stable Levi subgroups with $Q \subseteq P$ rational ι-stable parabolics with Levi factors M, L respectively, then*

$$\operatorname{Ind}_{M\subseteq Q}^{G/H} = \operatorname{Ind}_{L\subseteq P}^{G/H} \circ \operatorname{Ind}_{M\subseteq Q\cap L}^{L/H_L}.$$

Similar to the definition of induction, define a function

$$\operatorname{Res}_{L\subseteq P}^{G/H} : \mathbf{C}[H^F\backslash G^F/H^F] \to \mathbf{C}[H_L^F\backslash L^F/H_L^F]$$

with the formula

$$\operatorname{Res}_{L\subseteq P}^{G/H}(f)(x) := \sum_{p\in P^F, \overline{p}=x} f(p).$$

Note that this definition coincides with the usual definition of parabolic restriction.

Proposition 2.3. *If f is H^F bi-invariant on G^F and g is H_L^F bi-invariant on L^F, then*

$$\langle f, \operatorname{Ind}_{L\subseteq P}^{G/H} g\rangle = |H^F|^2 |H^F \cap P^F|^{-2} \langle \operatorname{Res}_{L\subseteq P}^{G/H} f, g\rangle.$$

Let L_0 denote a rational Levi subgroup of the form $\mathrm{GL}_n \times \mathrm{GL}_n$ in GL_{2n} such that $L_0 \cap H \cong \mathrm{GL}_n$ and $L_0/(L_0 \cap H) \cong \mathrm{GL}_n$. Let P_0 denote a rational ι-stable parabolic for L_0.

Then bi-invariant functions on L_0 can be identified with class functions on GL_n, ι-stable Levi and parabolic subgroups can be identified with Levi and parabolic subgroups of GL_n and bi-invariant parabolic induction corresponds to usual parabolic induction.

For any ι-stable rational Levi subgroup $L \subseteq G$, with ι-stable rational parabolic, $L^F \cong \mathrm{GL}_{2n_0}(\mathbf{F}_q) \times \prod \mathrm{GL}_{n_i}(\mathbf{F}_q) \times \mathrm{GL}_{n_i}(\mathbf{F}_q)$ where $\sum n_i = n$ and $L/H_L \cong \mathrm{GL}_{2n_0}/\mathrm{Sp}_{2n_0} \times \prod \mathrm{GL}_{n_i}$. Let $T = \prod T_i$ be a rational maximal torus of $\prod \mathrm{GL}_{n_i}$. Then if $\theta = \prod \theta_i$ is an irreducible character of T^F, let $\zeta_T^L(\cdot|\theta)$ be the function on L^F defined by

$$\zeta_T^{L/H_L}(l|\theta) = \zeta_{T_0}^{\mathrm{GL}_{2n_0}/\mathrm{Sp}_{2n_0}}(l_0|\theta_0)_{q\mapsto q^2} \prod \zeta_{T_i}^{\mathrm{GL}_{n_i}}(l_i|\theta_i)$$

where $l = (l_0, l_1, \cdots)$ and $\zeta_{T_i}^{\mathrm{GL}_{n_i}}(\cdot|\theta_i)$ is viewed as a function on $\mathrm{GL}_{n_i}(\mathbf{F}_q) \times \mathrm{GL}_{n_i}(\mathbf{F}_q)$ under the correspondence described above. These are called *basic functions* and form a basis for the bi-invariant functions on L^F.

The next proposition extends [3, Theorem 5.3.2] from $L_0 \subseteq P_0$ to any maximal ι-stable Levi and parabolic subgroup. It follows from the special case of $L_0 \subseteq P_0$ and transitivity of induction given by Proposition 2.2.

Proposition 2.4. *Let L be a rational ι-stable Levi subgroup of the form $\mathrm{GL}_{2n} \times \mathrm{GL}_m \times \mathrm{GL}_m$ with $H_L \cong \mathrm{Sp}_{2n} \times \mathrm{GL}_m$ and P a rational ι-stable parabolic with L as its Levi factor. Let $T = T_1 \times T_2$ be a rational maximal torus of GL_{n+m} with T_1 and T_2 maximal tori in GL_n and GL_m respectively, and θ an irreducible character of T^F. Then*

$$\mathrm{Ind}_{L \subseteq P}^{G/H}(\zeta_T^{L/H_L}(\cdot|\theta)) = \frac{|T_2^F|_{q \mapsto q^2}}{|T_2^F|} \zeta_T^{G/H}(\cdot|\theta).$$

3 The Characteristic Map

In this section, a characteristic map

$$\mathrm{ch} : \mathbf{C}[\mathrm{Sp}_{2n}(\mathbf{F}_q) \backslash \mathrm{GL}_{2n}(\mathbf{F}_q) / \mathrm{Sp}_{2n}(\mathbf{F}_q)] \to \bigotimes_{f \in O(\mathbf{M})} \Lambda$$

is constructed from the $\mathrm{Sp}_{2n}(\mathbf{F}_q)$ bi-invariant functions on $\mathrm{GL}_{2n}(\mathbf{F}_q)$. The notational convention will be to drop the tensor and if $p \in \Lambda$ corresponds to the factor indexed by f, then write $p(f) = p(x_{1,f}, \ldots)$. See [10] for details on the $\mathrm{GL}_n(\mathbf{F}_q)$ theory and notation and definitions.

Define

$$\widetilde{p}_n(x) := \begin{cases} p_{n/d(f_x)}(f_x) & \text{if } d(f_x) | n \\ 0 & \text{else} \end{cases},$$

$$\widetilde{p}_n(\xi) := \begin{cases} (-1)^{n-1} \sum_{x \in \mathbf{M}_n} \xi(x) \widetilde{p}_n(x) & \text{if } \xi \in \mathbf{L}_n \\ 0 & \text{else} \end{cases},$$

$$p_n(\varphi) := \widetilde{p}_{nd(\varphi)}(\xi),$$

where $\xi \in \varphi$ for $\varphi \in O(\mathbf{L})$. Since they are algebraically independent, the $p_n(\varphi)$ may be viewed as power sum symmetric functions in "dual variables $x_{i,\varphi}$" and an arbitrary symmetric function in $x_{i,\varphi}$ is defined in terms of them.

As a matter of convention, μ will denote functions $O(\mathbf{M}) \to \mathcal{P}$ and λ will denote functions $O(\mathbf{L}) \to \mathcal{P}$, and symmetric functions indexed by μ will always be in variables $x_{i,f}$ for $f \in O(\mathbf{M})$ and symmetric functions indexed by λ will always be in the variables

$x_{i,\varphi}$. Symmetric functions labeled by a partition valued function are interpreted as a product. Thus, $p_\mu = \prod_{f \in O(\mathbf{M})} p_{\mu(f)}(f)$.

Define the inner product

$$\langle f, g \rangle = \sum_{x \in \mathrm{GL}_{2n}(\mathbf{F}_q)} f(x)\overline{g(x)}$$

on $\mathbf{C}[\mathrm{Sp}_{2n}(\mathbf{F}_q) \backslash \mathrm{GL}_{2n}(\mathbf{F}_q) / \mathrm{Sp}_{2n}(\mathbf{F}_q)]$ and the inner product on $\otimes \Lambda$ by

$$\langle p_\mu, p_\mu \rangle = z_\mu \prod_{f \in O(\mathbf{M})} \prod_i \frac{1}{q_f^{2\mu(f)_i} - 1}.$$

Define the characteristic map ch by

$$\mathrm{ch}(I_{H^F g_\mu H^F}) = \prod_{f \in O(\mathbf{M})} q_f^{-2n(\mu(f))} P_{\mu(f)}(f; q_f^{-2})$$

and extending linearly.

Proposition 3.1. *Let T be a rational maximal torus of GL_n with $T^F \cong \mathbf{M}_{k_1} \times \cdots \times \mathbf{M}_{k_r}$ and θ a character of T^F. Then*

$$\mathrm{ch}(\zeta_T^{\mathrm{GL}_{2n} / \mathrm{Sp}_{2n}}(\cdot|\theta)) = (-1)^{n-r} \prod_i p_{k_i/d(\varphi_i)}(\varphi_i) = (-1)^{n-l(\lambda)} p_\lambda,$$

where φ_i is the F-orbit of $\theta|_{\mathbf{M}_{k_i}} \in \mathbf{L}$ and λ is the combinatorial data associated to (T, θ).

Now $\mathrm{ch}(\phi_\lambda)$ may be computed using Theorem 1.1.

Proposition 3.2. *Let $\lambda : O(\mathbf{L}) \to \mathcal{P}$ with $\|\lambda\| = n$. Then*

$$\mathrm{ch}(\phi_\lambda) = \frac{(-1)^{|\lambda|}}{\psi_n(q^2)} \prod_{\varphi \in O(\mathbf{L})} q_\varphi^{-n(\lambda'(\varphi))} J_{\lambda(\varphi)}(q_\varphi, q_\varphi^2).$$

The map ch is an isometry up to some constant depending only on n.

Lemma 3.3. *The map* ch *satisfies*

$$\langle \phi, \psi \rangle = q^{-n} |H^F|^2 \langle \mathrm{ch}(\phi), \mathrm{ch}(\psi) \rangle.$$

The inner product on symmetric functions in the dual variables is given by $\langle , \rangle_{q,q^2}$.The inner product on symmetric functions in the dual variables is given by the following.

Proposition 3.4. *For $\lambda : O(\mathbf{L}) \to \mathcal{P}$,*

$$\langle p_\lambda, p_\lambda \rangle = z_\lambda \prod_{\varphi \in O(\mathbf{L})} \prod_i \frac{q_\varphi^{\lambda(\varphi)_i} - 1}{q_\varphi^{2\lambda(\varphi)_i} - 1}.$$

Let $G = \mathrm{GL}_{2(n+m)}$ and let $H = \mathrm{Sp}_{2(n+m)}$. Let L be an ι-stable Levi subgroup with $L^F = \mathrm{GL}_{2n}(\mathbf{F}_q) \times \mathrm{GL}_m(\mathbf{F}_q) \times \mathrm{GL}_m(\mathbf{F}_q)$, such that $L^F \cap H^F = \mathrm{Sp}_{2n}(\mathbf{F}_q) \times \mathrm{GL}_m(\mathbf{F}_q)$. Let P be a rational ι-stable parabolic with L as its Levi factor. Then the function $\mathrm{Ind}_{L\subseteq P}^{G/H}$ may be viewed as taking f a $\mathrm{Sp}_{2n}(\mathbf{F}_q)$ bi-invariant function on $\mathrm{GL}_{2n}(\mathbf{F}_q)$ and g a class function on $\mathrm{GL}_m(\mathbf{F}_q)$ (since there is an isomorphism $\mathrm{GL}_m \times \mathrm{GL}_m \,/\, \mathrm{GL}_m \cong \mathrm{GL}_m$) and producing a $\mathrm{Sp}_{2(n+m)}(\mathbf{F}_q)$ bi-invariant function on $\mathrm{GL}_{2(n+m)}(\mathbf{F}_q)$, denoted by $f * g$. This can be done for any n, m and so defines a graded bilinear map.

If f_1, f_2 are functions on G_1 and G_2 respectively, let $f_1 \times f_2$ denote the function on $G_1 \times G_2$ given by $(f_1 \times f_2)(x_1, x_2) = f_1(x_1) f_2(x_2)$. Then

$$f * g := \mathrm{Ind}_{L\subseteq P}^{G/H}(f \times g).$$

Theorem 3.5. *Let f be an $\mathrm{Sp}_{2n}(\mathbf{F}_q)$ bi-invariant function on $\mathrm{GL}_{2n}(\mathbf{F}_q)$ and let g be a $\mathrm{GL}_m(\mathbf{F}_q)$ bi-invariant function on $\mathrm{GL}_m(\mathbf{F}_q) \times \mathrm{GL}_m(\mathbf{F}_q)$ (or equivalently a class function on $\mathrm{GL}_m(\mathbf{F}_q)$). Then*

$$\mathrm{ch}(f * g) = \mathrm{ch}(f)\omega\omega_{q^2,q}\,\mathrm{ch}_{\mathrm{GL}_m}(g).$$

Proof. It suffices to check the equation holds for basic functions, where it follows from Proposition 2.4 and transitivity of induction given by Proposition 2.2. □

4 Schur expansion of skew Macdonald polynomials

This section outlines a proof of Theorem 0.1. The strategy will be to use the characteristic map and the mixed product to reinterpret $C^{\nu}_{\lambda/\mu}(q, q^2)$ in terms of bi-invariant functions on $\mathrm{GL}_{2n}(\mathbf{F}_q)$ and then utilize general facts about Gelfand pairs and the formula for parabolic induction to establish the result.

For any finite group G, a *positive-definite* function on G is a function $f : G \to \mathbf{C}$ such that the matrix indexed by G whose x, y entry is $f(x^{-1}y)$ is a positive-definite matrix.

Proposition 4.1 ([10, VII, Section 1])**.** *Let G/H be a Gelfand pair. Any spherical function is positive-definite, and moreover if f is an H bi-invariant function on G that is positive-definite, then for any spherical function ϕ on G, $\langle f, \phi\rangle \geq 0$.*

If $\alpha : G \to H$ is a group homomorphism, then given functions $f : G \to \mathbf{C}$ and $g : H \to \mathbf{C}$, define the functions $\alpha^* g : G \to \mathbf{C}$, or the pullback, and $\alpha_* f : H \to \mathbf{C}$, or the pushforward, by $\alpha^* g(x) = g(\alpha(x))$ and $\alpha_* f(x) = \sum_{\alpha(y)=x} f(y)$. It's clear that $\langle \alpha_* f, g\rangle = \langle f, \alpha^* g\rangle$.

Lemma 4.2. *If $f : G \to \mathbf{C}$ is positive-definite, and $\alpha : G \to H$ and $\beta : H \to G$ are group homomorphisms, then $\alpha_* f$ and $\beta^* f$ are also positive-definite.*

Now $\operatorname{Res}^G_{L\subseteq P}(f) = pr_* i^*(f)$ where i is the inclusion of P^F into G^F and pr is the projection $P^F \to L^F$, so it follows that the parabolic restriction of a positive-definite function is positive-definite. Then as induction and restriction are adjoint, the following lemma holds.

Lemma 4.3. *If f is positive-definite bi-invariant on L^F, then $\operatorname{Ind}^G_{L\subseteq P}(f)$ is a positive-definite function.*

Theorem 0.1 follows from the lemma and Proposition 4.1, as $C^\nu_{\lambda/\mu}(q,q^2)$ is a positive scalar multiple of $\langle \phi_\lambda, \phi_\mu * \chi_\nu \rangle$ by Proposition 3.2 (here partitions are viewed as functions $O(\mathbf{L}) \to \mathcal{P}$ by sending the trivial character to the partition).

Remark 4.4. Let $J^\perp_\mu(q,q^2)$ denote the adjoint of multiplication by $J_\mu(q,q^2)$ with respect to the $\langle , \rangle_{q,q^2}$ inner product. Some computations for small partitions in Sage suggest that

$$\frac{\langle J^\perp_\mu(q,q^2) J_\lambda(q,q^2), s_\nu \rangle}{(1-q)^{|\lambda|+|\mu|}} \in \mathbf{N}[q]$$

extending the conjecture of Haglund. If this conjecture holds, it would be interesting to see what combinatorial interpretation the coefficients might have.

The characteristic map also gives a condition for vanishing of $C^\nu_{\lambda/\mu}(q,q^2)$.

Theorem 4.5. *If $\langle s_{\lambda\cup\lambda}, s_{\mu\cup\mu} s_\nu s_\nu \rangle = 0$, then $C^\nu_{\lambda/\mu}(q,q^2)$ vanishes as a function of q.*

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References

[1] M. Aguiar et. al. "Supercharacters, symmetric functions in noncommuting variables, and related Hopf algebras". *Adv. Math.* **229**.4 (2012), pp. 2310–2337. Link.

[2] P. Alexandersson, J. Haglund, and G. Wang. "On the Schur expansion of Jack polynomials". 2018. arXiv:1805.00511.

[3] E. Bannai, N. Kawanaka, and S.-Y. Song. "The character table of the Hecke algebra $\mathscr{H}(GL_{2n}(\mathbf{F}_q), Sp_{2n}(\mathbf{F}_q))$". *J. Algebra* **129**.2 (1990), pp. 320–366. Link.

[4] N. Bergeron and A. M. Garsia. "Zonal polynomials and domino tableaux". *Discrete Math.* **99**.1 (1992), pp. 3–15. Link.

[5] R. W. Carter. *Finite Groups of Lie Type: Conjugacy Classes and Complex Characters*. Wiley Classics Library. Wiley, 1993.

[6] J. A. Green. "The Characters of the Finite General Linear Groups". *Trans. Amer. Math. Soc.* **80**.2 (1955), pp. 402–447. Link.

[7] I. Grojnowski. "Character sheaves on symmetric spaces". Thesis (Ph.D.)–Massachusetts Institute of Technology. 1992.

[8] J. He. "A characteristic map for the symmetric space of symplectic forms over a finite field". 2019. arXiv:1906.05966.

[9] A. Henderson. "Character sheaves on symmetric spaces". Thesis (Ph.D.)–Massachusetts Institute of Technology. 2001.

[10] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Second. Oxford Mathematical Monographs. With contributions by A. Zelevinsky, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995, pp. x+475.

[11] T. Shoji and K. Sorlin. "Exotic symmetric space over a finite field, I". *Transform. Groups* **18**.3 (2013), pp. 877–929. Link.

[12] T. Shoji and K. Sorlin. "Exotic symmetric space over a finite field, II". *Transform. Groups* **19**.3 (2014), pp. 887–926. Link.

[13] T. Shoji and K. Sorlin. "Exotic symmetric space over a finite field, III". *Transform. Groups* **19**.4 (2014), pp. 1149–1198. Link.

[14] N. Thiem and C. R. Vinroot. "On the characteristic map of finite unitary groups". *Adv. Math.* **210**.2 (2007), pp. 707–732. Link.

[15] M. Yoo. "A combinatorial formula for the Schur coefficients of the integral form of the Macdonald polynomials in the two column and certain hook cases". *Ann. Comb.* **16**.2 (2012), pp. 389–410. Link.

[16] M. Yoo. "Schur coefficients of the integral form Macdonald polynomials". *Tokyo J. Math.* **38**.1 (2015), pp. 153–173. Link.