Séminaire Lotharingien de Combinatoire **84B** (2020)
Article #46, 11 pp.

Proceedings of the 32nd Conference on Formal Power Series and Algebraic Combinatorics (Online)

Noncover complexes, Independence complexes, and domination numbers of hypergraphs

Jinha Kim*[1] and Minki Kim†[1]

[1] *Department of Mathematics, Technion – Israel Institute of Technology, Haifa, Israel*

Abstract. Let $\mathcal{H}$ be a hypergraph on a finite set V. An *independent set* of $\mathcal{H}$ is a set of vertices that does not contain an edge of $\mathcal{H}$. The *indepenence complex* of $\mathcal{H}$ is the simplicial complex on V whose faces are independent sets of $\mathcal{H}$. A *cover* of $\mathcal{H}$ is a vertex subset which meets all edges of $\mathcal{H}$. The *noncover complex* of $\mathcal{H}$ is the simplicial complex on V whose faces are noncovers of $\mathcal{H}$. In this extended abstract, we study homological properties of the independence complexes and the noncover complexes of hypergraphs. In particular, we obtain a lower bound on the homological connectivity of independence complexes and an upper bound on the Leray number of noncover complexes. The bounds are in terms of hypergraph domination numbers. Our proof method is applied to compute the reduced Betti numbers of the independence complexes of certain uniform hypergraphs, called *tight paths* and *tight cycles*. This extends to hypergraphs known results on graphs.

Keywords: Domination numbers, Noncover complexes, Independence complexes, Homological connectivity, Leray numbers

1 Introduction

A hypergraph $\mathcal{H}$ on a vertex set V is a collection of non-empty subsets of V called *edges*. The set $V = V(\mathcal{H})$ is called the *vertex set* of $\mathcal{H}$. A singleton edge $\{v\} \in \mathcal{H}$ is called a loop. For a positive integer k, a hypergraph is said to be *k-uniform* if every edge has size k. For example, the usual graphs are viewed as 2-uniform hypergraphs. Throughout this extended abstract, we assume every hypergraph has a non-empty vertex set, and no two edges in a hypergraph are identical.

Let $\mathcal{H}$ be a hypergraph on V. A subset W of V is said to be *independent* if it contains no edge of $\mathcal{H}$. An *abstract simplicial complex* on V is a family of subsets of V that is closed under the operation of taking subsets. The set $\mathcal{I}(\mathcal{H})$ of independent sets of $\mathcal{H}$ is clearly an abstract simplicial complex. It is called the *independence complex* of $\mathcal{H}$.

A *cover* of $\mathcal{H}$ is a subset W of V that meets all edges of $\mathcal{H}$. Observe that W is a cover of $\mathcal{H}$ if and only if $V \setminus W$ is an independent set of $\mathcal{H}$. Let $\mathcal{NC}(\mathcal{H})$ be the complex of

*jinhakim@campus.technion.ac.il.
†kimminki@campus.technion.ac.il.

noncovers of $\mathcal{H}$. Note that every maximal face of $\mathcal{NC}(\mathcal{H})$ is the complement of an edge of $\mathcal{H}$.

For a simplicial complex K, the *(combinatorial) Alexander dual* is the complex $D(K) := \{\sigma \subset V : V \setminus \sigma \notin K\}$. Observe that the noncover complex of a hypergraph $\mathcal{H}$ is the Alexander dual of the indepedence complex of $\mathcal{H}$. Let $\tilde{H}_i(K)$ be the i-dimensional reduced homology group of K. In this extended abstract the coefficients of homology groups are taken in $\mathbb{Z}_2$. The homology groups of a simplicial complex K and those of its dual $D(K)$ are related by a duality theorem. (See [4].)

Theorem 1.1 (The duality theorem)**.** *If K be a simplicial complex on V. Then $\tilde{H}_i(D(K)) \cong \tilde{H}_{|V|-i-3}(K)$ for all i.*

In this extended abstract, we study relations between domination numbers for hypergraphs and homological properties of noncover complexes and indepenence complexes of hypergraphs.

1.1 Homological connectivity and Leray numbers of simplicial complexes

A simplicial complex K on V is said to be *d-Leray* if $\tilde{H}_i(K[W]) = 0$ for all $i \geq d$ and $W \subset V$, where $K[W] = \{\sigma \subset W : \sigma \in K\}$ is the subcomplex of K induced on W. The Leray number $L(K)$ of K is the minimal integer d such that K is d-Leray. For example, the boundary of an n-simplex is n-Leray.

A closely related parameter is the (homological) connectivity. A simplicial complex K on V is said to be *(homologically) k-connected* if $\tilde{H}_i(K) = 0$ for all $-1 \leq i \leq k$. We denote by $\eta(K)$ the maximum integer k where K is $(k-2)$-connected. For example, any non-empty complex K has $\eta(K) \geq 1$ and the boundary of an n-simplex $2^{[n+1]}$ has $\eta(\partial 2^{[n+1]}) = n$ for any positive integer n. If there is no such k then we write $\eta(K) = \infty$. Theorem 1.1 implies that any complex K has $L(K) \leq d$ if and only if $\eta(D(K[W])) \geq |W| - d - 1$ for every $W \subset V$.

1.2 Domination numbers of hypergraphs

We define three domination parameters of hypergraphs.

Let $\mathcal{H}$ be a hypergraph on V. We say $W \subset V$ *strongly dominates* a vertex $v \in V$ if there exists $W' \subset W$ such that $W' \cup \{v\}$ is an edge of $\mathcal{H}$. In particular, the empty set strongly dominates v if v is a loop. For a subset A of V, if $W \subset V$ strongly dominates every vertex in A, then we say W *strongly dominates* A. The *strong domination number of A in $\mathcal{H}$* is the integer

$$\gamma_0(\mathcal{H}; A) := \min\{|W| : W \subset V,\ W \text{ strongly dominates } A\}.$$

The *strong domination number* $\tilde{\gamma}(\mathcal{H})$ of $\mathcal{H}$ is the strong domination number of the whole vertex set, i.e. $\tilde{\gamma}(\mathcal{H}) = \gamma_0(\mathcal{H}; V)$. Similar definitions were introduced in [1] and [5], but all of those are little different from our definition.

$A \subset V$ is said to be *strongly independent* in $\mathcal{H}$ if it is independent and every edge of $\mathcal{H}$ contains at most one vertex of A. The *strong independence domination number* of $\mathcal{H}$ is the integer

$$\gamma_{si}(\mathcal{H}) := \max\{\gamma_0(\mathcal{H}; A) : A \text{ is a strongly independent set of } \mathcal{H}\}.$$

The *edgewise-domination number* of $\mathcal{H}$ is the minimum number of edges whose union strongly dominates the whole vertex set V, i.e.

$$\gamma_E(\mathcal{H}) := \min\{|\mathcal{F}| : \mathcal{F} \subset \mathcal{H}, \bigcup_{F \in \mathcal{F}} F \text{ strongly dominates } V\}.$$

Clearly, if $\mathcal{H}$ is k-uniform, then $\gamma_E(\mathcal{H}) \geq \left\lceil \frac{\tilde{\gamma}(\mathcal{H})}{k} \right\rceil$.

Note that if $\binom{V}{1} \subset \mathcal{H}$, then $\tilde{\gamma}(\mathcal{H}) = \gamma_{si}(\mathcal{H}) = \gamma_E(\mathcal{H}) = 0$. If $\mathcal{H}$ has an *isolated vertex* v, i.e. if no edge of $\mathcal{H}$ contains v, then there does not exist $W \subset V$ that strongly dominates v. In this case $\tilde{\gamma}(\mathcal{H})$, $\gamma_{si}(\mathcal{H})$ and $\gamma_E(\mathcal{H})$ are defined as $\tilde{\gamma}(\mathcal{H}), \gamma_{si}(\mathcal{H}), \gamma_E(\mathcal{H}) = \infty$.

2 Homological connectivity of Independence complexes

Bounding $\eta(\mathcal{I}(\mathcal{H}))$ in terms of domination parameters when $\mathcal{H}$ is a (2-uniform) graph has been studied extensively. The following theorem summarizes such results in [2, 3, 7]. (See also [14, 13].)

Theorem 2.1. *Let G be a graph. Then $\eta(\mathcal{I}(G)) \geq \max\{\left\lceil \frac{\tilde{\gamma}(G)}{2} \right\rceil, \gamma_{si}(G), \gamma_E(G)\}$.*

Note that an immediate application of Theorem 1.1 to Theorem 2.1 gives us

$$\tilde{H}_i(\mathcal{NC}(G)) = 0 \text{ for all } i \geq |V(G)| - \max\{\left\lceil \frac{\tilde{\gamma}(G)}{2} \right\rceil, \gamma_{si}(G), \gamma_E(G)\} - 1. \tag{2.1}$$

Our first result is a hypergraph analogue of Theorem 2.1.

Theorem 2.2. *Let $\mathcal{H}$ be a hypergraph. Then $\eta(\mathcal{I}(\mathcal{H})) \geq \max\{\left\lceil \frac{\tilde{\gamma}(\mathcal{H})}{2} \right\rceil, \gamma_{si}(\mathcal{H}), \gamma_E(\mathcal{H})\}$.*

As an application, Theorem 2.2 gives an alternative proof of the main result in [10]. Our proof method also can be applied to compute the reduced Betti numbers of the independence complexes of certain uniform hypergraphs, called *tight paths* and *tight cycles*. These are generalizations of (2-uniform) paths and cycles, respectively.

3 Leray numbers of noncover complexes

The second result strengthens Theorem 2.2 for some cases. We prove upper bounds of $L(\mathcal{NC}(\mathcal{H}))$ in terms of the domination parameters.

Theorem 3.1. *Let $\mathcal{H}$ be a hypergraph on V with no isolated vertices. Then*

1. *If $|e| \leq 3$ for every $e \in \mathcal{H}$, then $L(\mathcal{NC}(\mathcal{H})) \leq |V| - \left\lceil \frac{\tilde{\gamma}(\mathcal{H})}{2} \right\rceil - 1$.*
2. *If $|e| \leq 2$ for every $e \in \mathcal{H}$, then $L(\mathcal{NC}(\mathcal{H})) \leq |V| - \gamma_{si}(\mathcal{H}) - 1$.*
3. *$L(\mathcal{NC}(\mathcal{H})) \leq |V| - \gamma_E(\mathcal{H}) - 1$.*

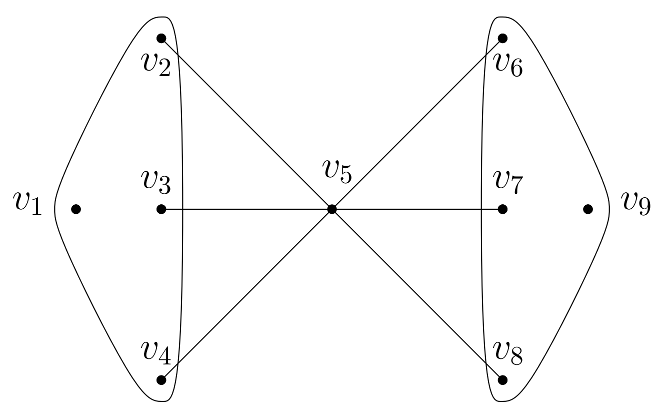
The case of (2-uniform) graphs in the second part of Theorem 3.1 was also proved in [8]. (See also [6] for a stronger version.) Note that if a hypergraph contains an isolated vertex v, then the noncover complex $\mathcal{NC}(\mathcal{H})$ is a cone with apex v, which is contractible. Hence we observe that $L(\mathcal{NC}(\mathcal{H})) = L(\mathcal{NC}(\mathcal{H}'))$ where $\mathcal{H}'$ is the hypergraph obtained from $\mathcal{H}$ by removing all isolated vertices.

Here are examples showing that the restrictions on the size of edges in the parts 1 and 2 of Theorem 3.1 are necessary.

1. Let $\mathcal{H}_r$ be a hypergraph on $V = \{v_1, \dots, v_{2r+1}\}$, whose edges are

$$\begin{aligned}\mathcal{H}_r = \{&\{v_1, \dots, v_r\}, \{v_2, v_{r+1}\}, \{v_3, v_{r+1}\}, \dots, \{v_r, v_{r+1}\},\\ &\{v_{r+1}, v_{r+2}\}, \{v_{r+1}, v_{r+3}\}, \dots, \{v_{r+1}, v_{2r}\}, \{v_{r+2}, \dots, v_{2r+1}\}\}.\end{aligned}$$

In this case, $\tilde{\gamma}(\mathcal{H}_r) = 2r - 1$ but $\mathcal{NC}(\mathcal{H}_r)$ is not $(|V| - \left\lceil \frac{\tilde{\gamma}(\mathcal{H}_r)}{2} \right\rceil - 1)$-Leray. See Figure 1 for the illustration when $r = 4$.



$|V(\mathcal{H}_4)| = 9,\ \tilde{\gamma}(\mathcal{H}_4) = 7,\ NC(\mathcal{H}_4)[\{v_2, v_3, v_4, v_6, v_7, v_8\}] \simeq \partial\Delta_5$

Figure 1: $|V(\mathcal{H}_4)| - \left\lceil \frac{\tilde{\gamma}(\mathcal{H}_4)}{2} \right\rceil - 1 = 4$ but $\mathcal{NC}(\mathcal{H}_4)$ is not 4-Leray.

2. For $r \geq 3$, consider an r-uniform hypergraph

$$\mathcal{F}_r := \{\{(i,1),\dots,(i,r))\} : i \in [r]\} \cup \{\{(1,i),\dots,(r,i)\} : i \in [r] \setminus \{1\}\}$$

defined on $[r] \times [r]$. In this case, $\gamma_{si}(\mathcal{F}_r) \geq (r-1)r$ but $\mathcal{NC}(\mathcal{F}_r)$ is not $(r-1)$-Leray whenever $r \geq 3$. See Figure 2 for the illustration when $r = 4$.

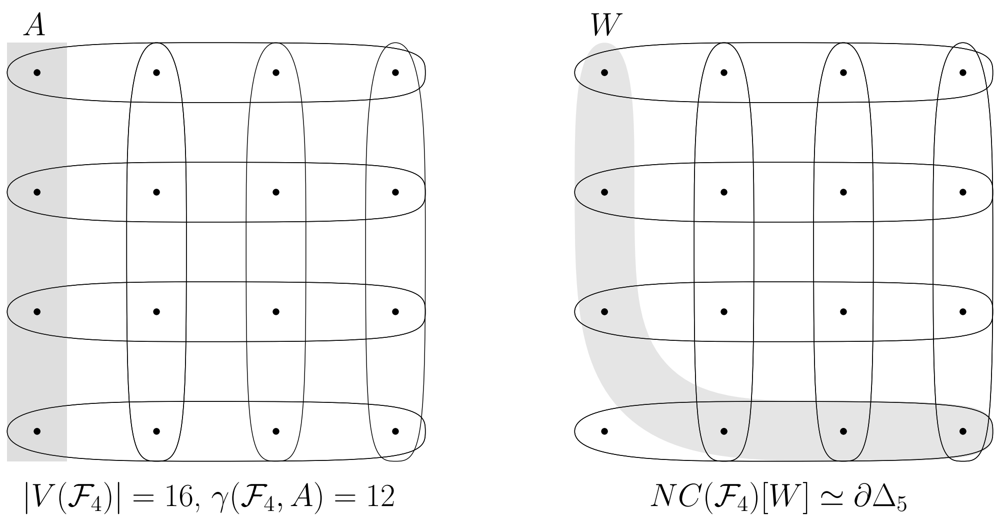


Figure 2: $|V(\mathcal{F}_4)| - \gamma_{si}(\mathcal{F}_4) - 1 \leq 3$ but $\mathcal{NC}(\mathcal{F}_4)$ is not 4-Leray.

4 Proof idea

4.1 Edge annihilation

Given a hypergraph $\mathcal{H}$ and an edge $e \in \mathcal{H}$, an *edge-annihilation* of e in $\mathcal{H}$ is

$$\mathcal{H} \neg\, e := \{f \setminus e : f \in \mathcal{H} \text{ and } f \nsubseteq e\}.$$

See Figure 3 for the illustration of an edge-annihilation.

We give some relations between the domination parameters of $\mathcal{H}$ and those of $\mathcal{H} \neg\, e$. This is a hypergraph analogue of Meshulam's observations for graphs [14].

Lemma 4.1. *Let $\mathcal{H}$ be a hypergraph with vertex set V. If $\mathcal{H}$ has no isolated vertices, then each of the following holds:*

1. $\tilde{\gamma}(\mathcal{H} \neg\, e) \geq \tilde{\gamma}(\mathcal{H}) - 2|e| + 2$ *for every edge $e \in \mathcal{H}$ with $|e| \geq 2$.*
2. *Suppose $\binom{V}{1} \nsubseteq \mathcal{H}$. Let A be a strongly independent set of $\mathcal{H}$ such that $\gamma_{si}(\mathcal{H}) = \gamma(\mathcal{H}; A)$. Take a vertex $v \in A$ and an edge $e_0 \in \mathcal{H}$ that contains the vertex v. Then*

$$\gamma_{si}(\mathcal{H} \neg\, e_0) \geq \gamma_{si}(\mathcal{H}) - |e_0| + 1.$$

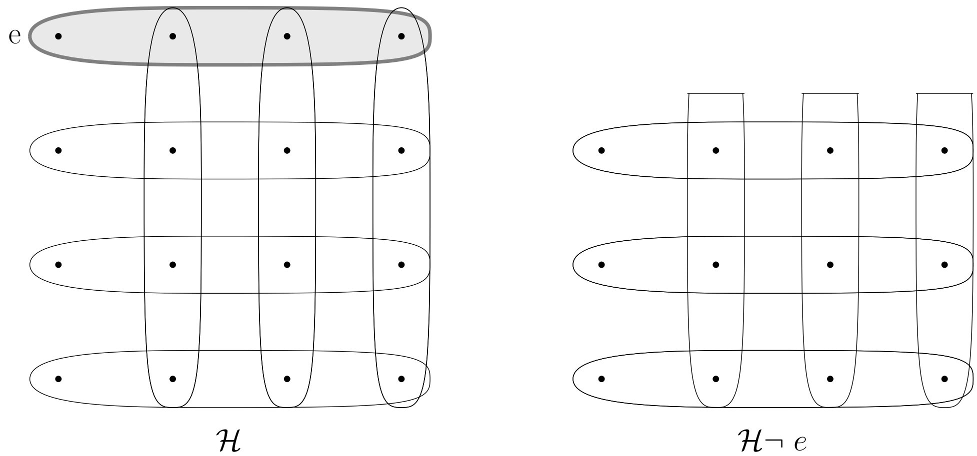


Figure 3: $\mathcal{H}\neg\, e$ is obtained from $\mathcal{H}$ by annihilate the edge e.

3. *$\gamma_E(\mathcal{H}\neg\, e) \geq \gamma_E(\mathcal{H}) - |e| + 1$ for every edge e with $|e| \geq 2$.*

4. *Let e be an edge in $\mathcal{H}$, and let $\mathcal{H}'$ be the hypergraph obtained from $\mathcal{H} - e$ by deleting all isolated vertices. Then*
$$\gamma_E(\mathcal{H}') \geq \gamma_E(\mathcal{H}) - f(e),$$
where $f(e) = 1$ if there is an isolated vertex in $\mathcal{H} - e$ and $f(e) = 0$ otherwise.

4.2 An exact sequence for noncover complexes

The proof of Theorem 2.2 is based on the Mayer–Vietoris exact sequence for noncover complexes. Let K be an abstract simplicial complex and let A and B be complexes such that $K = A \cup B$. Then the following sequence is exact:

$$\cdots \to \tilde{H}_i(A \cap B) \to \tilde{H}_i(A) \oplus \tilde{H}_i(B) \to \tilde{H}_i(K) \to \tilde{H}_{i-1}(A \cap B) \to \cdots . \tag{4.1}$$

In particular, for any integer i_0, if $\tilde{H}_i(A) = \tilde{H}_i(B) = \tilde{H}_{i-1}(A \cap B) = 0$ for all $i \geq i_0$ then $\tilde{H}_i(K) = 0$ for all $i \geq i_0$.

Lemma 4.2. *Let $\mathcal{H}$ be a hypergraph and e be an edge in $\mathcal{H}$. Let e^c be the complement of e, i.e. $e^c = V(\mathcal{H}) \setminus e$. If every edge in $\mathcal{H}$ is inclusion-minimal, then*

$$\mathcal{NC}(\mathcal{H}) = \mathcal{NC}(\mathcal{H} - e) \cup 2^{e^c} \text{ and } \mathcal{NC}(\mathcal{H} - e) \cap 2^{e^c} = \mathcal{NC}(\mathcal{H}\neg e).$$

Suppose a hypergraph $\mathcal{H}$ contains two edges $e \neq f$ such that $f \subset e$. Since $e^c \subset f^c$, deleting e from $\mathcal{H}$ does not affect to the noncover complex. That is, $\mathcal{NC}(\mathcal{H}) = \mathcal{NC}(\mathcal{H} - e)$. Therefore, when we compute the homology of noncover complexes of hypergraphs, we may assume that every edge is inclusion-minimal.

When a hypergraph $\mathcal{H}$ contains exactly one edge which is the whole vertex set, i.e. $\mathcal{H} = \{V(\mathcal{H})\}$, then $\mathcal{NC}(\mathcal{H})$ is an empty complex, thus has non-vanishing homology only in dimension -1. In this case, $\eta(\mathcal{I}(\mathcal{H})) = |V(\mathcal{H})| - 1$. Otherwise, suppose $\mathcal{H} \neq \{V(\mathcal{H})\}$. If we set $K = \mathcal{NC}(\mathcal{H})$, $A = \mathcal{NC}(\mathcal{H} - e)$, and $B = 2^{e^c}$, then Lemma 4.2 and the sequence (4.1) gives us an exact sequence

$$\begin{aligned} &\cdots \to \tilde{H}_i(\mathcal{NC}(\mathcal{H} \neg\, e)) \to \tilde{H}_i(\mathcal{NC}(\mathcal{H} - e)) \to \tilde{H}_i(\mathcal{NC}(\mathcal{H})) \\ &\to \tilde{H}_{i-1}(\mathcal{NC}(\mathcal{H} \neg\, e)) \to \tilde{H}_{i-1}(\mathcal{NC}(\mathcal{H} - e)) \to \tilde{H}_{i-1}(\mathcal{NC}(\mathcal{H})) \to \cdots . \end{aligned} \tag{4.2}$$

By applying Lemma 4.1 to the sequence (4.2), we obtain a hypergraph analogue of (2.1). By Theorem 1.1, this implies Theorem 2.2.

Theorem 4.3. *Let $\mathcal{H}$ be a hypergraph. Then*

$$\tilde{H}_i(\mathcal{NC}(\mathcal{H})) = 0 \text{ for all } i \geq |V(\mathcal{H})| - \max\{\left\lceil \frac{\tilde{\gamma}(\mathcal{H})}{2} \right\rceil, \gamma_{si}(\mathcal{H}), \gamma_E(\mathcal{H})\} - 1.$$

5 Applications

In this section, we present applications of our results.

5.1 Tight paths and tight cycles

A repeated application of the sequence (4.2) is sometimes useful when we compute the homology of the independence complexes of hypergraphs. In this section, we introduce two examples that are generalizations of paths and cycles.

Let n and k be positive integers and V be a set of size n. A k-uniform hypergraph on $V = \{v_1, \ldots, v_n\}$ is called the *(k-uniform) tight path*, denoted by $P_{n,k}$, if there exists a linear ordering $<$, say $v_1 < v_2 < \cdots < v_n$, on V such that

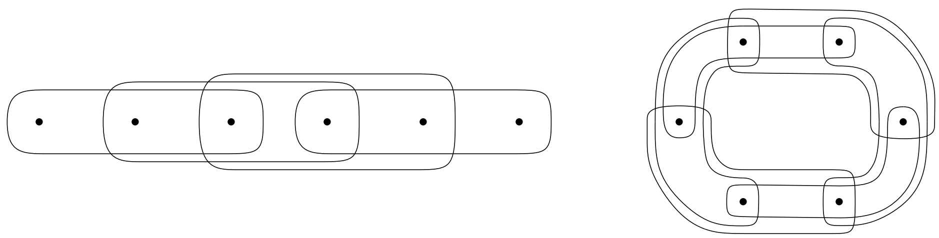
$$P_{n,k} := \{\{v_{i+1}, \ldots, v_{i+k}\} : 0 \leq i \leq n-k\}.$$

When $n < k$, then there is no edge.

The *(k-uniform) tight cycle* $C_{n,k}$ is defined as a k-uniform hypergraph on $\mathbb{Z}_n$ with $n \geq k+1$ such that

$$C_{n,k} := \{\{i, \ldots, i+k-1\} : 0 \leq i \leq n-1\}.$$

For example, $P_{n,2}$ and $C_{n,2}$ are a path and a cycle, respectively. See Figure 4 for illustrations of the 3-uniform case.



$P_{6,3}$ $C_{6,3}$

Figure 4: 3-uniform tight path $P_{6,3}$ and tight cycle $C_{6,3}$.

In [14], it was shown that for every integer $i \geq 0$,

$$\begin{aligned} \tilde{\beta}_i(\mathcal{I}(P_{n,2})) &= \begin{cases} 1 & \text{if } n = 3i+2, 3i+3, \\ 0 & \text{otherwise.} \end{cases}, \text{ and} \\ \tilde{\beta}_i(\mathcal{I}(C_{n,2})) &= \begin{cases} 2 & \text{if } n = 3i+3 \\ 1 & \text{if } n = 3i+2, 3i+4, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{5.1}$$

As generalization of (5.1), we compute the reduced Betti numbers for noncover complexes of $P_{n,k}$ and $C_{n,k}$.

Theorem 5.1. *Let k, n be positive integers and let q be a non-negative integer. Then*

$$\tilde{\beta}_i(\mathcal{I}(P_{n,k})) = \begin{cases} 1 & \text{if } i = q(k-1)+k-2, n = q(k+1)+k \\ & \text{or } i = q(k-1)+k-2, n = (q+1)(k+1), \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5.2. *Let k, n be positive integers with $n > k$ and let q be a non-negative integer. Then*

$$\tilde{\beta}_i(\mathcal{I}(C_{n,k})) = \begin{cases} k & \text{if } i = q(k-1)+k-2, n = (q+1)(k+1), \\ 1 & \text{if } i = q(k-1)+k+t-3, \\ & \quad n = (q+1)(k+1)+t \text{ for } t \in [k], \\ 0 & \text{otherwise.} \end{cases}$$

5.2 General position complexes

In this section, we present an application of Theorem 2.2 to the homological connectivity of "general position complexes".

Let P be a set of points in $\mathbb{R}^d$ and let $G(P)$ denote the simplicial complex consisting of those subsets of P which are in general position. Furthermore, let $\varphi(P)$ denote the largest subset of P in general position, that is, $\varphi(P) = \dim(G(P)) + 1$. In [10], it was shown that if $\varphi(P) > d\binom{2k-2}{d}$ then $\eta(G(P)) \geq k$. We give an alternative proof of it, by showing the following matroidal generalization.

Theorem 5.3. *Let M be a matroid of rank r on X. For any finite subset Y of X, define a hypergraph*

$$\mathcal{H}_Y = \{S \subseteq Y : |S| \leq r, S \text{ is a circuit of } M\}.$$

If $\mathcal{H}_Y$ has an independent set of size greater than $(r-1)\binom{2k-2}{r-1}$, then $\eta(I(\mathcal{H}_Y)) \geq k$.

By Theorem 2.2, it is sufficient to show that $\tilde{\gamma}(\mathcal{H}_Y) > 2k-2$.

5.3 Rainbow covers

As an application of Theorem 3.1, we can obtain the following result for "rainbow covers". Let l and m be positive integers with $l \leq m$. Given m covers $X_1, \ldots, X_m$ in a hypergraph $\mathcal{H}$, a *rainbow cover* of size l is a cover $X = \{x_{i_1}, \ldots, x_{i_l}\}$ of l distinct vertices of $\mathcal{H}$ such that $x_{i_j} \in X_{i_j}$ for each $j \in \{1, \ldots, l\}$.

Theorem 5.4. *Let $\mathcal{H}$ be a hypergraph with no isolated vertices. Then each of the following holds:*

1. *Suppose that every edge in $\mathcal{H}$ has size at most 3. Then for every $|V(\mathcal{H})| - \left\lceil \frac{\tilde{\gamma}(\mathcal{H})}{2} \right\rceil$ covers of $\mathcal{H}$, there exists a rainbow cover.*
2. *Suppose that every edge in $\mathcal{H}$ has size at most 2. Then for every $|V(\mathcal{H})| - \gamma_{si}(\mathcal{H})$ covers of $\mathcal{H}$, there exists a rainbow cover.*
3. *For every $|V(\mathcal{H})| - \gamma_E(\mathcal{H})$ covers of $\mathcal{H}$, there exists a rainbow cover.*

Theorem 5.4 follows from the topological colorful Helly theorem. Here we state the special case of a famous result by Kalai and Meshulam [11].

Theorem 5.5 (Topological colorful Helly theorem)**.** *Let K be a d-Leray simpicial complex with a vertex partition $V(K) = V_1 \cup \cdots \cup V_m$ with $m \geq d+1$. If $\sigma \in K$ for every $\sigma \subset V(K)$ with $|\sigma \cap V_i| = 1$, then there exists $I \subset \{1, \ldots, m\}$ of size at least $m-d$ such that $\bigcup_{i \in I} V_i \in K$.*

6 Remarks

Bounding Leray numbers of noncover complexes in terms of domination numbers of (hyper)graphs also has been studied from an algebraic viewpoint. (See [8, 9].) See [12] to understand the relation between algebra and topology of an abstract simplicial complex in this context. It is worth to mention here a result in [9], which deals with a different type of independence domination numbers of hypergraphs.

Let $\mathcal{H}$ be a hypergraph on V. We say $W \subset V$ *weakly dominates* $A \subset V$ if for each $v \in A$, either v is a loop in $\mathcal{H}$ or there exists a vertex $w \neq v$ in W such that w and v belong to some edge of $\mathcal{H}$. Let

$$\gamma(\mathcal{H}; A) := \min\{|W| : W \subset V \setminus A,\ W \text{ weakly dominates } A\},$$

and $t(\mathcal{H}) := \max\{\gamma(\mathcal{H}; A) : A \in \mathcal{I}(\mathcal{H})\}$. The following is a reformulation of [9, Theorem 5.2].

Theorem 6.1. *Let $\mathcal{H}$ be a hypergraph on V with no isolated vertices. Then $L(\mathcal{NC}(\mathcal{H})) \leq |V| - t(\mathcal{H}) - 1$.*

Consequently, we obtain $\eta(\mathcal{I}(\mathcal{H})) \geq t(\mathcal{H})$. Also, Theorem 6.1 gives an analogue of Theorem 5.4: every $|V| - t(\mathcal{H})$ covers in $\mathcal{H}$ assigns a rainbow cover. Note that $t(\mathcal{H}) = \gamma_{si}(\mathcal{H})$ when $\mathcal{H}$ is a graph.

The two independence domination parameters $t(\mathcal{H})$ and $\gamma_{si}(\mathcal{H})$ are not comparable in general. In particular, we can construct examples so that one of the parameters is arbitrarily large while the other remains constant.

1. Let $\mathcal{H}$ be a complete k-uniform hypergraph $\binom{[n]}{k}$ on $n \geq k$ vertices. Then we have $\gamma_{si}(\mathcal{H}) = k - 1$ and $t(\mathcal{H}) = 1$.

2. Let k and n be positive integers such that $k \geq 3$ and $n \geq 2$. We construct a k-uniform hypergraph $\mathcal{A}_{n,k}$ with vertex set $V_{n,k}$ such that $|V_{n,k}| = \binom{(k-1)n}{k-1} + (k-1)n$ as follows.

   Let $W_{n,k} \subset V_{n,k}$ be a subset of size $(k-1)n$. Consider a bijection $\phi : \binom{W_{n,k}}{k-1} \to V_{n,k} \setminus W_{n,k}$. Now we define the edges of $\mathcal{A}_{n,k}$ as

   $$\mathcal{A}_{n,k} := \left\{ \{\phi(X)\} \cup X : X \subset \binom{W_{n,k}}{k-1} \right\} \cup \binom{V_{n,k} \setminus W_{n,k}}{k}.$$

   Since $W_{n,k}$ is an independent set and $\gamma_{weak}(\mathcal{A}_{n,k}, W_{n,k}) = n$, we have $t(\mathcal{A}_{n,k}) \geq n$. Observe that any strongly independent set of $\mathcal{A}_{n,k}$ contains at most one vertex from $W_{n,k}$ and at most one vertex from $V_{n,k} \setminus W_{n,k}$. Take $u \in W_{n,k}$ and $v \in V_{n,k} \setminus W_{n,k}$ such that u and v are not contained in the same edge of $\mathcal{A}_{n,k}$. Then the strongly independent set $\{u, v\}$ can be strongly dominated by k vertices. First observe that a $(k-1)$-set $\phi^{-1}(v)$ in $W_{n,k}$ strongly dominates v. Then take any $(k-2)$-subset U in $\phi^{-1}(v)$ and let $u' = \phi(U \cup \{u\})$. Clearly $U \cup \{u'\}$ strongly dominates u. This shows $\gamma_{si}(\mathcal{A}_{n,k}) \leq k$.

Acknowledgements

We thank Ron Aharoni and Andreas Holmsen for their insightful comments. Both authors were supported by ISF grant no. 2023464 and BSF grant no. 2006099.

References

[1] B. D. Acharya. "Domination in hypergraphs". *AKCE J. Graph Combin.* **4** (2007), pp. 117–126.

[2] R. Aharoni, M. Chudnovsky, and A. Kotlov. "Triangulated spheres and colored cliques". *Discrete Comput. Geom.* **28**.2 (2002), pp. 223–229. Link.

[3] R. Aharoni and P. Haxell. "Hall's theorem for hypergraphs". *J. Graph Th.* **35**.2 (2000), pp. 83–88. Link.

[4] A. Björner and M. Tancer. "Note: Combinatorial Alexander duality – A short and elementary proof". *Discrete Comput. Geom.* **42** (2009), pp. 586–593. Link.

[5] C. Bujtás, M. Henning, Z. Tuza, and A. Yeo. "Total transversals and total domination in uniform hypergraphs". *Elctron. J. Combin.* **21**.2 (2014), #P2.24. Link.

[6] I. Choi, J. Kim, and B. Park. "Collapsibility of non-cover complexes of graphs". *Elctron. J. Combin.* **27**.1 (2020), #P1.20. Link.

[7] M. Chudnovsky. "Systems of disjoint representatives". PhD thesis, unpublished. Haifa, 2000.

[8] H. Dao and J. Schweig. "Projective dimension, graph domination parameters, and independence complex homology". *J. Combin. Theory, Ser. A* **120**.2 (2013), pp. 453–469. Link.

[9] H. Dao and J. Schweig. "Further applications of clutter domination parameters to projective dimension". *J. Algebra* **432** (2015), pp. 1–11. Link.

[10] A. Holmsen, L. Martinez-Sandoval, and L. Montejano. "A geometric Hall-type theorem". *Proc. Amer. Math. Soc.* **144** (2016), pp. 503–511. Link.

[11] G. Kalai and R. Meshulam. "A topological colorful Helly theorem". *Adv. Math.* **191**.2 (2005), pp. 305–311. Link.

[12] G. Kalai and R. Meshulam. "Intersections of Leray complexes and regularity of monomial ideals". *J. Combin. Theory, Ser. A* **113**.7 (2006), pp. 1586–1592. Link.

[13] R. Meshulam. "The clique complex and hypergraph matching". *Combinatorica* **21**.1 (2001), pp. 89–94. Link.

[14] R. Meshulam. "Domination numbers and homology". *J. Combin. Theory, Ser. A* **102**.2 (2003), pp. 321–330. Link.