Séminaire Lotharingien de Combinatoire **84B** (2020)
Article #48, 12 pp.

Proceedings of the 32nd Conference on Formal Power Series and Algebraic Combinatorics (Online)

An insertion algorithm for diagram algebras

Laura Colmenarejo*[1], Rosa Orellana†[2], Franco Saliola‡[3], Anne Schilling§[4], and Mike Zabrocki¶[5]

[1]*Department of Mathematics and Statistics, University of Massachusetts Amherst, MA, USA*
[2]*Department of Mathematics, Dartmouth College, NH, USA.*
[3]*Département de mathématiques, Université du Québec à Montréal, Canada*
[4]*Department of Mathematics, University of California, Davis, CA, USA*
[5]*Department of Mathematics and Statistics, York University, Toronto, Canada*

Abstract. We generalize the Robinson–Schensted–Knuth algorithm to the insertion of two row arrays of multisets. This generalization leads to an algorithm from partition diagrams to pairs of a standard tableau and a standard multiset tableau of the same shape, which has the remarkable property that it is well-behaved with respect to restricting a representation to a subalgebra. This insertion algorithm matches recent representation-theoretic results of Halverson and Jacobson.

Keywords: multisets, diagram algebras, RSK algorithm, enumerative results

1 Introduction

We explore a variant of the Robinson–Schensted–Knuth (RSK) algorithm, where we insert multisets instead of integer entries. If we restrict the multisets to all have size one, the algorithm we are using is the usual RSK algorithm. Applying this insertion to different arrays of multisets gives rise to a purely enumerative result that is a combinatorial manifestation of a double centralizer theorem from representation theory.

The RSK algorithm evolved over the last century from a procedure defined on permutations (in the work of Robinson [19]) to a procedure defined on finite sequences of integers (in the work of Schensted [20]) and finally to a procedure defined on "generalized permutations" by Knuth [14]. In each of these versions, the algorithm establishes a correspondence between the initial input and pairs of combinatorial objects called tableaux subject to certain constraints (see Section 2 for definitions).

In this paper, we adapt the RSK algorithm to the insertion of arrays of multisets and obtain an algorithm from elements of diagram algebras to pairs of standard multiset

*laura.colmenarejo.hernando@gmail.com.
†Rosa.C.Orellana@dartmouth.edu. Supported by NSF grant DMS–1700058.
‡saliola.franco@uqam.ca. Supported by NSERC Discovery Grants.
§anne@math.ucdavis.edu. Supported by NSF grants DMS–1760329 and 1764153.
¶zabrocki@mathstat.yorku.ca. Supported by NSERC Discovery Grants.

tableaux. Note that algorithms that relate partition diagrams and pairs of paths in the Bratteli diagram for the partition algebras have been known since the late 1990s [10, 15]. These paths are referred as "vacillating tableaux" and they are analogues of a path in the Young's lattice, which is the Bratteli diagram for the symmetric groups. Paths in the Young lattice are encoded by standard Young tableaux.

Recently, a new combinatorial interpretation for the dimensions of the irreducible representations for the partition algebra has appeared in the literature [2, 3, 18, 9, 8]. In particular, Benkart and Halverson [2] presented a bijection between vacillating tableaux and "set-partition tableaux" (tableaux whose entries are sets of positive integers). There are two main advantages to working with set-partition tableaux instead of vacillating tableaux. Firstly, they are closer in spirit to the ubiquitous Young tableaux. Secondly, the definition extends naturally to the notion of multiset tableaux (tableaux whose entries are multisets of positive integers, see [17]).

Our insertion algorithm from partition diagrams to pairs of a standard tableau and a standard multiset tableau of the same shape has the remarkable property that it is well-behaved with respect to the subalgebra structure of the partition algebra. One surprising consequence is that we are able to provide explicit combinatorial descriptions of the sets of tableaux that give the dimensions of the irreducible representations associated to the prominent subalgebras of the partition algebras, such as the symmetric group, Brauer, rook, rook-Brauer, Temperley–Lieb, Motzkin, planar rook, and the planar algebras (see Lemma 3.3). This gives rise to analogues of the famous identity $n! = \sum_{\lambda \vdash n} (f^\lambda)^2$ for the symmetric group, where f^λ is the number of standard tableaux of shape λ, to all of the above mentioned algebras (see Corollary 3.4). We prove that the dimensions of the irreducible representations of the various algebras are equal to the number of our combinatorially-defined tableaux by establishing that the branching rules are encoded in the tableaux (see Section 3.3 and Corollary 3.10). Our insertion is different from combining the insertion of Halverson and Lewandowski [10] from partition diagrams to paths in the Bratteli diagram with the bijection of Halverson and Benkart [2] from paths in the Bratteli diagram to set-partition tableaux (see Section 3.3).

For more details on the insertion algorithm for multiset partitions and the proofs of the results presented in this extended abstract, see [5].

2 Basic definitions

Throughout this paper, we work with tableaux whose entries are multisets. Note that any Young tableau—that is, a tableau with integer entries—can be viewed as a multiset tableau by considering each entry to be a multiset of cardinality 1. In this section, we fix notation and define the total order on multisets that we use in order to extend the property of being (semi)standard to multiset tableaux.

A *partition* of $n \in \mathbb{N}$ is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > 0$ whose sum is n. The notation $\lambda \vdash n$ is used to indicate that λ is a partition of n. The *length* of the partition is denoted by $\ell(\lambda) = r$. The *cells* of the partition are the coordinates of the boxes in the diagram; that is, $\text{cells}(\lambda) = \{(i,j) \mid 1 \leqslant i \leqslant \lambda_j, 1 \leqslant j \leqslant \ell(\lambda)\}$. The operation of removing the first row of the partition λ is denoted by $\overline{\lambda} = (\lambda_2, \lambda_3, \ldots, \lambda_r)$.

A *set partition* π of a set S is a collection $\{\pi_1, \ldots, \pi_k\}$ of non-empty subsets of S that are mutually disjoint, i.e., $\pi_i \cap \pi_j = \varnothing$ for all $i \neq j$, and $\bigcup_i \pi_i = S$. The subsets π_i are called the *blocks*. We write $\pi \vdash S$ to mean that π is a set partition of the set S.

Let $(A, \leqslant_A)$ be a totally ordered set, which we refer to as an *(ordered) alphabet*. A *multiset* $S = \{\!\{a_1, a_2, \ldots, a_r\}\!\}$ over A is an unordered collection of elements of A, allowing repeats. The collection of multisets forms an associative monoid with operation

$$\{\!\{a_1, a_2, \ldots, a_r\}\!\} \uplus \{\!\{b_1, b_2, \ldots, b_d\}\!\} = \{\!\{a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_d\}\!\} \ .$$

To simplify notation, we let $\{\!\{a^{m_a}, b^{m_b}, c^{m_c}, \ldots\}\!\}$ denote the multiset that contains m_a occurrences of a, m_b occurrences of b, and so on; for example $\{\!\{1^2, 4^3, 5\}\!\} = \{\!\{1,1,4,4,4,5\}\!\}$.

A *multiset partition* of a multiset S is a multiset of multisets, $\pi = \{\!\{S^{(1)}, S^{(2)}, \ldots, S^{(r)}\}\!\}$, such that $S = S^{(1)} \uplus S^{(2)} \uplus \cdots \uplus S^{(r)}$. We indicate this by the notation $\pi \Vdash S$.

Given two disjoint sets S and S' with elements in an ordered set A, we say $S < S'$ in the *last letter order* if $\max(S) <_A \max(S')$, where $\max(S)$ is the largest element in S. For example, $\{1,3,5\} < \{2,7\}$.

Let λ be a partition, A an ordered alphabet, and a fixed total order $<$ on multisets (such as the graded lexicographic order or the last letter order if the multisets are all disjoint sets). A *semistandard multiset tableau* of *shape* λ is a function T that associates with each cell $(i,j) \in \text{cells}(\lambda)$ a multiset over A such that:

- $T(i,j) \leqslant T(i,j+1)$ whenever (i,j) and $(i,j+1)$ both belong to cells(λ); and
- $T(i,j) < T(i+1,j)$ whenever (i,j) and $(i+1,j)$ both belong to cells(λ).

The *shape* of a multiset tableau T is the partition λ, and the *cells* of T are the cells of its shape. If $T(i,j) = S$, then we say that S *labels* the cell (i,j), and that S is an *entry* of T. When drawing multiset tableaux, the multisets are abbreviated as words without the surrounding multiset delimiters $\{\!\{, \}\!\}$, and empty sets are depicted by blank cells. The *content* of a semistandard multiset tableau T is the (disjoint) union of the entries of T. More precisely, the content of T is the multiset

$$\text{content}(T) = \biguplus_{(i,j) \in \text{cells}(T)} T(i,j).$$

A semistandard multiset tableau is said to be

- a *standard multiset tableau* if its content is the set $[k] := \{1, 2, \ldots, k\}$ for some $k \in \mathbb{N}$;

- a *semistandard Young tableau* if all its entries are multisets of size 1;
- a *standard Young tableau* if it is both standard and a semistandard Young tableau.

The *set-partition tableaux* of [2, Definition 3.14] are closely related to our standard multiset tableaux. Finally, for a multiset S, let

- $\mathrm{SSMT}(\lambda, S)$ be the set of *semistandard* multiset tableaux of shape λ and content S;
- $\mathrm{SMT}(\lambda, k)$ be the set of *standard* multiset tableaux of shape λ and content $[k]$;
- $\mathrm{SSYT}(\lambda, S)$ be the set of semistandard Young tableaux of shape λ and content S;
- $\mathrm{SYT}(\lambda)$ be the set of standard Young tableaux of shape λ.

Example 2.1. Let $A = \{1, 2, 3, 4, 5\}$ with the usual order on integers. Then

$$\begin{array}{|c|cc} \hline 114 & & \\ \hline 2 & \multicolumn{1}{|c|}{14} & \\ \hline & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad \begin{array}{|c|cc} \hline 3 & & \\ \hline 2 & \multicolumn{1}{|c|}{3} & \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \qquad \begin{array}{|c|cc} \hline 134 & & \\ \hline 5 & \multicolumn{1}{|c|}{26} & \\ \hline & \multicolumn{1}{|c|}{7} & \multicolumn{1}{c|}{8} \\ \hline \end{array}$$

are three semistandard multiset tableaux of shape $(3, 2, 1)$. The leftmost tableau has content $\{\{1^3, 2^2, 4^2, 5\}\}$. The middle tableau has content $\{\{1^2, 2, 3^2, 4\}\}$ and is also a semistandard Young tableau. The rightmost multiset tableau is standard since its content is $\{1, \ldots, 8\}$.

Let A and B be two ordered alphabets. A *generalized permutation* from A to B is a two-line array of the form $\binom{a_1\ a_2\ \cdots\ a_r}{b_1\ b_2\ \cdots\ b_r}$ satisfying: $a_1, \ldots, a_r \in A$; $b_1, \ldots, b_r \in B$; $a_i \leqslant_A a_{i+1}$ for $1 \leqslant i \leqslant r-1$; and $b_i \leqslant_B b_{i+1}$ whenever $a_i = a_{i+1}$.

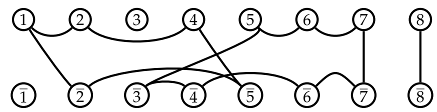
Theorem 2.2 (Knuth)**.** *There is a one-to-one correspondence between generalized permutations from A to B and pairs of tableaux (P, Q) satisfying: P and Q are of the same shape; P is semistandard with entries in B; and Q is semistandard with entries in A.*

3 Insertion algorithm for diagram algebras

For any parameter n and positive integer k, the partition algebra $P_k(n) = \mathrm{span}_{\mathbb{C}}\{\pi \mid \pi \vdash [k] \cup [\overline{k}]\}$ is defined as the complex vector space with basis given by the set partitions on two disjoint sets $[k] \cup [\overline{k}] = \{1, 2, \ldots, k\} \cup \{\overline{1}, \overline{2}, \ldots, \overline{k}\}$. Although we do not define the product here, as we will not use it explicitly, we remark that the dependency of the algebra on n arises when we multiply the set partitions [11].

A *diagram* is a graphical representation of a set partition of the set $[k] \cup [\overline{k}]$: the vertex set of the graph is $[k] \cup [\overline{k}]$ arranged in two horizontal rows, where the top row is labelled by $1, 2, \ldots, k$ and the bottom row are labelled by $\overline{1}, \overline{2}, \ldots, \overline{k}$; and there is a path connecting two vertices if and only if they belong to the same block of the set partition. Note that there is more than one graph that represents a set partition, but this is immaterial to the following. In our examples, we will connect vertices in the same block with a cycle.

Example 3.1. The set partition $\pi = \{\{1,2,4,\overline{2},\overline{5}\},\{3\},\{5,6,7,\overline{3},\overline{4},\overline{6},\overline{7}\},\{8,\overline{8}\},\{\overline{1}\}\}$ is represented by the following diagram:

$\pi =$ 

The partition algebra $P_k(n)$ is semisimple whenever the parameter $n \notin \{0,1,\ldots,2k-2\}$ in which case the irreducible representations are indexed by partitions λ with $0 \leqslant |\lambda| \leqslant k$ [16]. We assume throughout that $n \geqslant 2k$, so that $P_k(n)$ is semisimple and isomorphic to $\mathrm{End}_{S_n}(V^{\otimes k})$.

In [10, 15], the authors introduce RSK-type algorithms between partition algebra diagrams and pairs of paths in the Bratteli diagram of the partition algebras; in [10] these paths are called vacillating tableaux. In [2], the authors define a bijection between vacillating tableaux and standard multiset tableaux. In this section we provide a different bijection from partition algebra diagrams to standard multiset tableaux. This algorithm not only encodes the representation theory of the partition algebra, in the sense that the tableaux of shape λ index an irreducible representation associated with λ, but it also encodes the representation theory of subalgebras of the partition algebra when we restrict the set of diagrams considered. This allows us to obtain enumerative results for representations of various diagram algebras using standard multiset tableaux.

3.1 The correspondence

A block is said to be *isolated* if it is of size 1. A block in a set partition π is called *propagating* if it contains vertices in both $[k]$ and $[\overline{k}]$ and it is *non-propagating* otherwise. For example, $\{1,2,4,\overline{2},\overline{5}\}$ is a propagating block and $\{\overline{3}\}$ is isolated. All isolated blocks are non-propagating. The number of propagating blocks in π is called the *propagating number* and we denote it by $\mathrm{pr}(\pi)$. For example, the set partition π in Example 3.1 has $\mathrm{pr}(\pi) = 3$. Let $\pi = \{\pi_1, \pi_2, \ldots, \pi_r\}$ be a set partition of $[k] \cup [\overline{k}]$. We associate with π a pair (T, S) of standard multiset tableaux as follows. To begin,

- let $\pi_{j_1}, \pi_{j_2}, \ldots, \pi_{j_p}$ denote the propagating blocks of π ordered so that $\pi^+_{j_1} < \cdots < \pi^+_{j_p}$ in the last letter order, where $\pi^+_j = \pi_j \cap [k]$;
- let $\sigma_{i_1}, \ldots, \sigma_{i_a} \subseteq [k]$ denote the non-propagating blocks contained in $[k]$ and ordered so that $\sigma_{i_1} < \cdots < \sigma_{i_a}$ in the last letter order;
- let $\tau_{i_1}, \ldots, \tau_{i_b} \subseteq [\overline{k}]$ denote the non-propagating blocks contained in $[\overline{k}]$ and ordered so that $\tau_{i_1} < \cdots < \tau_{i_b}$ in the last letter order.

Let (P,Q) denote the pair of standard multiset tableaux obtained by applying the RSK algorithm to the generalized permutation

$$\begin{pmatrix} \pi_{j_1}^+ & \pi_{j_2}^+ & \cdots & \pi_{j_p}^+ \\ \pi_{j_1}^- & \pi_{j_2}^- & \cdots & \pi_{j_p}^- \end{pmatrix},$$

where $\pi_j^+ = \pi_j \cap [k]$ and $\pi_j^- = \pi_j \cap [\bar{k}]$. Let T be the tableau obtained from P by adjoining a row containing $n-p-b$ empty cells followed by cells labelled $\tau_{i_1},\ldots,\tau_{i_b}$. Let S be the tableau obtained from Q by adjoining a row containing $n-p-a$ empty cells followed by cells labelled $\sigma_{i_1},\ldots,\sigma_{i_a}$.

Theorem 3.2. *Let $n \geqslant 2k$. The set partitions of $[k]\cup[\bar{k}]$ are in bijection with pairs (T,S) of standard multiset tableaux satisfying: T and S are of the same shape λ, where λ is a partition of n; T has content $[\bar{k}]$; and S has content $[k]$.*

3.2 Restriction to subalgebras

There are other bijections between partition algebra diagrams and pairs of standard multiset tableaux, but an important aspect of the algorithm in this paper is that it is compatible with the (representation theory) restriction to many prominent subalgebras of $P_k(n)$. More precisely, we will see that this single procedure captures the combinatorics of the representation theory of all these subalgebras. For instance, for an integer r with $0 \leqslant r \leqslant k$ the subspace spanned by the set partitions with propagating number at most r is a subalgebra of $P_k(n)$ and the irreducible representations of this subalgebra are indexed by partitions of size less than or equal to r. Then, we have

$$\#\left\{\pi \vdash [k]\cup[\bar{k}] \;\middle|\; \mathrm{pr}(\pi) \leqslant r\right\} = \sum_{\substack{\lambda\vdash n \\ |\bar{\lambda}|\leqslant r}} \#\mathrm{SMT}(\lambda,k)^2 \,. \tag{3.1}$$

3.2.1 Definition of the subalgebras

We introduce some terminology that will make it easier to define the subalgebras. See Figure 1 for examples of the types of diagrams that we define below. A set partition π is called *planar* if it can be represented as a graph without edge crossings inside the rectangle formed by its vertices. A set partition is called a *matching* if all its blocks are of size at most 2. We call a set partition a *perfect matching* if all its blocks are of size 2. Note that the number of perfect matchings of $2n$ elements is equal to $(2n-1)!! = (2n-1)(2n-3)\cdots(1)$. A perfect matching, where each block contains an element in $[k]$ and an element in $[\bar{k}]$ is a *permutation*. A set partition is a *partial permutation* if all its blocks have size one or two and every block of size two is propagating.

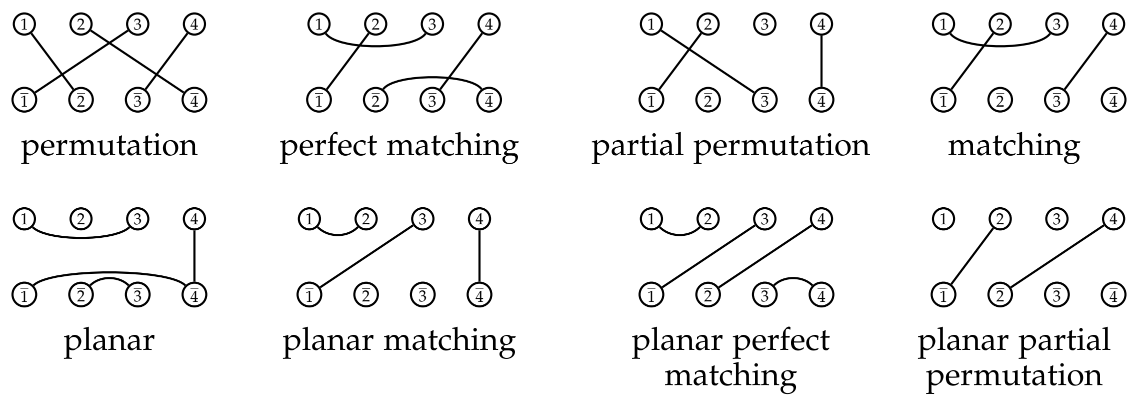


Figure 1: Examples of types of set partition diagrams.

The first two columns in Table 1 summarizes the definitions of the subalgebras that we work with. In [9], the authors construct the irreducible representations of these subalgebras using standard multiset tableaux (which they call *set-partition tableaux*) and compute their characters. Their results provide a detailed study of the representation theory of these subalgebras from which we extract the information in Table 2.

3.2.2 Restricting the correspondence to the subalgebras

We characterize the standard multiset tableaux produced by the correspondence of Section 3.1 when restricted to the diagrams spanning one of the subalgebras A_k in Table 1. We denote this set by $\mathrm{SMT}_{A_k}(\lambda)$.

A standard multiset tableau is *matching* if the first row contains sets of size less than or equal to 2 and all other rows contain only sets of size 1. In Lemma 3.3, we show these are the multiset tableaux that correspond to matching diagrams by our insertion algorithm. Two sets S and S' are *non-crossing* if there do not exist elements $a, b \in S$ and $c, d \in S'$ such that either $a < c < b < d$ or $c < a < d < b$. We say that $c \in [k]$ is *between* a set S if there exist $a, b \in S$ such that $a < c < b$. We call a standard multiset tableau *planar* if it has two rows, if the sets in the first row are pairwise non-crossing, and if no element belonging to one of the sets in the second row is between any set in the tableau (apart from the set containing the element). In Lemma 3.3, we show these are the multiset tableaux that correspond to planar diagrams by our insertion algorithm.

Lemma 3.3. *Let k be any positive integer, λ a partition of an integer n with $n \geqslant 2k$, and A_k one of the subalgebras of $P_k(n)$ defined in Table 1. If we apply the insertion procedure of Theorem 3.2 to the diagrams spanning A_k, then the resulting standard multiset tableaux are characterized by the properties listed in Table 1.*

In addition, we obtain the following corollary of Theorem 3.2 and Lemma 3.3.

Corollary 3.4. *If $n \geqslant 2k$, then for each subalgebra A_k of the partition algebra $P_k(n)$, we have*

$$\dim(A_k) = \sum_{\lambda \vdash n} \left(\#\,\mathrm{SMT}_{A_k}(\lambda)\right)^2.$$

The explicit formulas are summarized in Table 2.

3.3 From standard multiset tableaux to Bratteli diagrams

Let A_k denote one of the subalgebras from Lemma 3.3. We establish a bijection between the standard multiset tableaux for A_k and the paths in the Bratteli diagram for $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$.

A *Bratteli diagram* associated to a tower of algebras $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ is an infinite $\mathbb{N}$-graded graph defined as follows. The vertices at level $k \in \mathbb{N}$ are in bijection with the isomorphism classes of the irreducible representations of A_k; if the irreducible representations are parameterized by some index set, then we label the vertices by the elements of the index set. Note that it is possible that vertices at different levels carry the same label (this happens for some of the index sets listed in Table 2), but the associated representations are different. The edges in the graph connect vertices of level k with vertices at level $k+1$: the number of edges from the vertex associated with an irreducible A_k-representation V to the vertex associated with an irreducible A_{k+1}-representation V' is the multiplicity of V in the restriction of V' to A_k.

In all the examples we consider, there is exactly one irreducible A_0-representation and it is of dimension 1. It follows from an induction argument that the dimension of an irreducible representation V is equal to the number of paths in the Bratteli diagram from the unique level-0 vertex to the vertex associated with V.

By *branching rule*, we mean any combinatorial description of the edge multiplicities in the Bratteli diagram in terms of the index sets of the irreducible representations.

Remark 3.5. A proof that the planar algebra $PP_k(n)$ is isomorphic to the Temperley–Lieb algebra $TL_{2k}(n)$ can be found in [11, Section 1]. Consequently, the branching rule for $PP_k(n)$ is obtained by a repeated application of the branching rule for $TL_{2k}(n)$.

Paths in the Bratteli diagram are often called updown tableaux or oscillating tableaux in the literature; see [10] and the references therein.

Proposition 3.6. *Let k be a positive integer and λ a partition of $n \geqslant 2k$. There is a bijection*

$$\phi\colon\ \mathrm{SMT}(\lambda,k) \to \left\{ (S,\tau) \;\middle|\; S \in \mathrm{SMT}(\mu,k-1),\ \tau \to \mu,\ \tau \to \lambda \right\}.$$

This correspondence is particularly useful because it respects the properties characterizing the tableaux in $\mathrm{SMT}_{A_k}(\lambda)$ (see Table 1).

Example 3.7. Consider the following tableau in $\mathrm{SMT}((n-3,2,1),9)$,

$$T = \begin{array}{|c|c|c|c|c|c|c|c|}\cline{1-1} 7 \\ \cline{1-2} 5 & 8 \\ \hline \; & \; & \; & \cdots & \; & 12 & 46 & 39 \\ \hline \end{array},$$

in particular, it is an element of $\mathrm{SMT}_{B_9(n)}((n-3,2,1))$. To compute $\phi(T) = (S,\tau)$, we remove the cell labelled $\{3,9\}$ and insert $\{3\}$ in the second row, obtaining

$$S = \begin{array}{|c|c|c|c|c|c|c|}\cline{1-1} 7 \\ \cline{1-1} 5 \\ \cline{1-2} 3 & 8 \\ \hline \; & \; & \; & \cdots & \; & 12 & 46 \\ \hline \end{array}$$

where $S \in \mathrm{SMT}_{B_8(n)}((n-4,2,1,1))$ and $\tau = (n-4,2,1)$. Proposition 3.6 also states that T can be recovered from S and the partitions $\tau = (n-4,2,1)$ and $\lambda = (n-3,2,1)$.

Now we are ready for the main result of this section, which states that the standard multiset tableaux in $\mathrm{SMT}_{A_k}(\lambda)$ encode the branching rule for the subalgebra A_k.

Theorem 3.8. *Let A_k be a subalgebras of $P_k(n)$ defined in Table 1 with $n \geqslant 2k$, and $\lambda, \mu \vdash n$.*

1. *If $T \in \mathrm{SMT}_{A_k}(\lambda)$ and $\phi(T) = (S,\tau)$, then $S \in \mathrm{SMT}_{A_{k-1}}(\mu)$.*
2. *For each $S \in \mathrm{SMT}_{A_{k-1}}(\mu)$, the number of $T \in \mathrm{SMT}_{A_k}(\lambda)$ such that $\phi(T) = (S,\tau)$ for some partition τ is equal to the number of edges from $\overline{\mu}$ to $\overline{\lambda}$ in the Bratteli diagram for the tower of algebras $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$.*

The map ϕ from Proposition 3.6 allows us to establish a bijection between standard multiset tableaux and *vacillating tableaux*. A *vacillating tableau* is a sequence partitions satisfying the condition $\lambda^{(r)} \vdash n$ and $\lambda^{(r+\frac{1}{2})} \vdash n-1$ with $\lambda^{(r)} \leftarrow \lambda^{(r+\frac{1}{2})}$ and $\lambda^{(r+\frac{1}{2})} \rightarrow \lambda^{(r+1)}$ for $0 \leqslant r < k$ [10, 2]. A different bijection appears in [2]. The bijection we provide here is compatible with the families of tableaux for each of the subalgebras and the Bratteli diagrams for those subalgebras.

Proposition 3.9. *For each family of subalgebras A_k in Table 1 and for each λ a partition of $n \geqslant 2k$, there is a bijection between $\mathrm{SMT}_{A_k}(\lambda)$ and the set of vacillating tableaux of the form $\left((n) = \lambda^{(0)}, \lambda^{(\frac{1}{2})}, \lambda^{(1)}, \lambda^{(1\frac{1}{2})}, \ldots, \lambda^{(k-\frac{1}{2})}, \lambda^{(k)} = \lambda\right)$, where $\overline{\lambda^{(0)}} \Rightarrow \overline{\lambda^{(1)}} \Rightarrow \cdots \Rightarrow \overline{\lambda^{(k)}}$ is a path in the Bratteli diagram for the tower of algebras $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$.*

What we have presented in this section completes the connection between the results in [9] and those in [10]. The insertion presented in Theorem 3.2 is a correspondence between diagrams and pairs of standard multiset tableaux that motivates the tableaux that arise in the paper [9]. Theorem 3.8 then provides a correspondence between standard multiset tableaux and paths in the Bratteli diagram.

Since the dimensions of the irreducibles are equal to the number of paths in the Bratteli diagram, it follows that the number of tableaux of a given shape is equal to the dimension of the irreducible representation. This establishes the following result, which can also be proven by enumerating the tableaux in Lemma 3.3 by a purely combinatorial argument and verifying that the values agree with the formulas in Table 2.

Corollary 3.10. *Let $n \geqslant 2k$ and $\lambda \vdash n$. For each of the algebras A_k described in Table 2, let $V_{A_k}^{\overline{\lambda}}$ be the irreducible A_k-representation indexed by $\overline{\lambda}$. Then*

$$\dim\left(V_{A_k}^{\overline{\lambda}}\right) = \#\mathrm{SMT}_{A_k}(\lambda).$$

Remark 3.11. Benkart and Halverson [2] give a bijection between standard multiset tableaux and vacillating tableaux that is different from the correspondence that we have just described. Their bijection does not behave well under restriction to all of the subalgebras A_k and the corresponding standard multiset tableaux in $\mathrm{SMT}_{A_k}(\lambda)$.

Acknowledgements

The authors would like to thank BIRS and AIM for the opportunity to collaborate in Banff and San José, which greatly facilitated the work on this project. We thank Tom Halverson for sharing an early version of [8] with us.

References

[1] G. Benkart and T. Halverson. "Motzkin algebras". *European J. Combin.* **36** (2014), pp. 473–502. Link.

[2] G. Benkart and T. Halverson. "Partition Algebras and the Invariant Theory of the Symmetric Group". 2017. arXiv:1709.07751.

[3] G. Benkart, T. Halverson, and N. Harman. "Dimensions of irreducible modules for partition algebras and tensor power multiplicities for symmetric and alternating groups". *J. Algebraic Combin.* **46**.1 (2017), pp. 77–108. Link.

[4] R. Brauer. "On algebras which are connected with the semisimple continuous groups". *Ann. of Math. (2)* **38**.4 (1937), pp. 857–872. Link.

[5] L. Colmenarejo, R. Orellana, F. Saliola, A. Schilling, and M. Zabrocki. "An insertion algorithm on multiset partitions with applications to diagram algebras". *J. Algebra* **557**.1 (2020), pp. 97–128. Link.

[6] Z. Daugherty and R. Orellana. "The quasi-partition algebra". *J. Algebra* **404** (2014), pp. 124–151.

[7] D. Flath, T. Halverson, and K. Herbig. "The planar rook algebra and Pascal's triangle". *Enseign. Math. (2)* **55**.1-2 (2009), pp. 77–92. Link.

[8] T. Halverson. "Set-partition tableaux, symmetric group multiplicities, and partition algebra modules". *Sémin. Loth. Combin.* (2019). to appear.

[9] T. Halverson and T. N. Jacobson. "Set-partition tableaux and representations of diagram algebras". 2018. arXiv:1808.08118.

[10] T. Halverson and T. Lewandowski. "RSK insertion for set partitions and diagram algebras". *Electron. J. Combin.* **11**.2 (2004/06), Research Paper 24, 24. Link.

[11] T. Halverson and A. Ram. "Partition algebras". *European J. Combin.* **26**.6 (2005), pp. 869–921. Link.

[12] V. F. R. Jones. "Index for subfactors". *Invent. Math.* **72**.1 (1983), pp. 1–25. Link.

[13] V. F. R. Jones. "The Potts model and the symmetric group". *Subfactors (Kyuzeso, 1993)*. World Sci. Publ., River Edge, NJ, 1994, pp. 259–267.

[14] D. E. Knuth. "Permutations, matrices, and generalized Young tableaux". *Pacific J. Math.* **34** (1970), pp. 709–727. Link.

[15] P. P. Martin and G. Rollet. "The Potts model representation and a Robinson-Schensted correspondence for the partition algebra". *Compositio Math.* **112**.2 (1998), pp. 237–254. Link.

[16] P. Martin and H. Saleur. "On an algebraic approach to higher-dimensional statistical mechanics". *Comm. Math. Phys.* **158**.1 (1993), pp. 155–190. Link.

[17] R. Orellana and M. Zabrocki. "A combinatorial model for the decomposition of multivariate polynomials rings as an S_n-module". 2019. arXiv:1906.01125.

[18] R. Orellana and M. Zabrocki. "Symmetric group characters as symmetric functions". 2016. arXiv:1605.06672.

[19] G. d. B. Robinson. "On the Representations of the Symmetric Group". *Amer. J. Math.* **60**.3 (1938), pp. 745–760. Link.

[20] C. Schensted. "Longest increasing and decreasing subsequences". *Canad. J. Math.* **13** (1961), pp. 179–191. Link.

[21] L. Solomon. "Representations of the rook monoid". *J. Algebra* **256**.2 (2002), pp. 309–342. Link.

[22] H. Wenzl. "On the structure of Brauer's centralizer algebras". *Ann. of Math. (2)* **128**.1 (1988), pp. 173–193. Link.

[23] B. W. Westbury. "The representation theory of the Temperley-Lieb algebras". *Math. Z.* **219**.4 (1995), pp. 539–565. Link.

Table 1: Properties characterizing the standard multiset tableaux in $\mathrm{SMT}_{A_k}(\lambda)$.

Subalgebra A_k of $P_k(n)$	diagrams spanning A_k	properties characterizing SMT_{A_k}	
		sizes of entries in first row	other properties
Partition algebra $P_k(n)$	all diagrams	—	—
Quasi-partition algebra $QP_k(n)$	no isolated blocks	not 1	—
Group algebra $\mathbb{C}S_k$	permutations	0	matching
Brauer algebra $B_k(n)$	perfect matchings	0, 2	matching
Rook algebra $R_k(n)$	partial permutations	0, 1	matching
Rook-Brauer algebra $RB_k(n)$	matchings	0, 1, 2	matching
Planar algebra $PP_k(n)$	planar diagrams	—	planar
Temperley-Lieb algebra $TL_k(n)$	planar perfect matchings	0, 2	matching & planar
Motzking algebra $M_k(n)$	planar matchings	0, 1, 2	matching & planar
Planar rook algebra $PR_k(n)$	planar partial permutations	0, 1	matching & planar

Table 2: Index sets and dimensions of irreducible representations for A_k in Table 1.

Subalgebra A_k	Dimension of A_k	Index set for irreducibles	Dimension of irreducible $V_{A_k}^{\lambda}$
$P_k(n)$	B_{2k}	$\{\lambda \mid \lambda \vdash m,\ 0 \leqslant m \leqslant k\}$	$f^\lambda \sum_{i=\lvert\lambda\rvert}^{k} \binom{k}{i} \left\{ {i \atop \lvert\lambda\rvert} \right\} B_{k-i}$
$QP_k(n)$ [6, 3]	$1 + \sum_{j=1}^{2k} (-1)^{j-1} B_{2k-j}$	$\{\lambda \mid \lambda \vdash m, 0 \leqslant m \leqslant k\}$	$f^\lambda \sum_{\ell=0}^{k} \sum_{i=\lvert\lambda\rvert}^{\ell} (-1)^{k-\ell} \binom{k}{\ell} \binom{\ell}{i} \left\{ {i \atop \lvert\lambda\rvert} \right\} B_{\ell-i}$
$\mathbb{C}S_k$	$k!$	$\{\lambda \mid \lambda \vdash k\}$	f^λ
$B_k(n)$ [4, 22]	$(2k-1)!!$	$\{\lambda \mid \lambda \vdash k-2r, 0 \leqslant 2r \leqslant k\}$	$f^\lambda \binom{k}{\lvert\lambda\rvert} (k - \lvert\lambda\rvert - 1)!!$
$R_k(n)$ [21]	$\sum_{i=0}^{k} \binom{k}{i}^2 i!$	$\{\lambda \mid \lambda \vdash m, 0 \leqslant m \leqslant k\}$	$f^\lambda \binom{k}{\lvert\lambda\rvert}$
$TL_k(n)$ [12, 23]	$\frac{1}{k+1} \binom{2k}{k}$	$\{(k-2r) \mid 0 \leqslant 2r \leqslant k\}$	$\binom{k}{(k-m)/2} - \binom{k}{(k-m)/2-1}$
$M_k(n)$ [1]	$\sum_{i=0}^{k} \frac{1}{i+1} \binom{2i}{i} \binom{2k}{2i}$	$\{(m) \mid 0 \leqslant m \leqslant k\}$	$\sum_{i=0}^{\lfloor \frac{k-m}{2} \rfloor} \binom{k}{m+2i} \left(\binom{m+2i}{i} - \binom{m+2i}{i-1} \right)$
$PR_k(n)$ [7]	$\binom{2k}{k}$	$\{(m) \mid 0 \leqslant m \leqslant k\}$	$\binom{k}{m}$
$PP_k(n)$ [13]	$\frac{1}{2k+1} \binom{4k}{2k}$	$\{(m) \mid 0 \leqslant m \leqslant k\}$	$\binom{2k}{k-m} - \binom{2k}{k-m-1}$