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The dual of the type B permutohedron as a Tchebyshev triangulation

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Abstract. We show that the order complex of intervals of a poset, ordered by inclusion, is a Tchebyshev triangulation of the order complex of the original poset. Besides studying the properties of this transformation, we show that the dual of the type B permutohedron is combinatorially equivalent to the order complex of the poset of intervals of a Boolean algebra (with the minimum and maximum elements removed).

Résumé. Nous montrons que le complexe d'ordre des intervaux d'un ensemble partiellement ordonné, ordonnés par inclusion, forment une triangulation de Tchebyshev du complex d'ordre de l'ensemble partiellement ordonné original. À part d'étudier les propriétés de cette transformation, nous montrons que le polytope dual du permutohèdre du type B est combinatoirement équivalent au complex d'ordre des intervaux d'un algèbre de Boole (sans l'élément minimal et maximal).

Keywords: permutohedron, type B, Tchebyshev triangulation, cd-index

Introduction

Inspired by Postnikov's seminal work [10], we have seen a surge in the study of root polytopes in recent years. A basic object in these investigations is the permutohedron. This talk connects permutohedra with the *Tchebyshev transform* of a poset, introduced by the present author [5, 6] and studied by Ehrenborg and Readdy [3], respectively the (generalized) Tchebyshev triangulations of a simplicial complex, first introduced by the present author in [7] and studied in collaboration with Nevo in [8]. The key idea of a Tchebyshev triangulation may be summarized as follows: we add the midpoint to each edge of a simplicial complex, and perform a sequence of stellar subdivisions, until we obtain a triangulation containing all the newly added vertices. Regardless of the order chosen, the face numbers of the triangulation will be the same, and may be obtained from the face numbers f_j of the original complex by replacing the powers of x with Tchebyshev polynomials of the first kind if we work with the appropriate generating function. The appropriate generating function in this setting is the polynomial $\sum_j f_{j-1}((x-1)/2)^j$.

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It is easy to verify that the face numbers of the type A and type B permutohedra are connected by a similar formula. These permutohedra are simple polytopes and their duals are simplicial polytopes, their boundary complexes are called the type A resp. type B Coxeter complexes. The suspicion arises that the type B Coxeter complex is a Tchebyshev triangulation of the type A Coxeter complex.

The present work contains the verification of this conjecture. The type A Coxeter complex is known to be the order complex of the Boolean algebra, and the type B Coxeter complex turns out to be an order complex as well, namely of the partially ordered set of intervals of the Boolean algebra, ordered by inclusion. We show that the operation of associating the poset of intervals to a partially ordered sets always induces a Tchebyshev triangulation at the level of order complexes. This observation may be helpful in constructing "type B analogues" of other polytopes and partially ordered sets.

This extended abstract is structured as follows. After the Preliminaries, we introduce the poset of intervals in Section 2 and show that the order complex of the poset of intervals is always a Tchebyshev transform of the order complex of the original poset. We also introduce a graded variant of this operation that takes a graded poset into a graded poset. In Section 3 we show that the type B Coxeter complex is the order complex of the graded poset of intervals of the Boolean algebra. In Section 4 we show how to compute the flag f-vector of graded a poset of intervals. The operation is recursive, unfortunately. Finally, in Section 5 we make the first steps towards computing the effect of taking the graded poset of intervals on the cd-index of an Eulerian poset.

1 Preliminaries

1.1 Graded Eulerian posets

A poset is *graded* if it contains a unique minimum element $\widehat{0}$, a unique maximum element $\widehat{1}$ and a rank function ρ satisfying $\rho(\widehat{0}) = 0$ and $\rho(y) = \rho(x) + 1$ for each x and y such that y covers x. The number of chains containing elements of fixed sets of ranks in a graded poset P of rank $n+1$ is encoded by the *flag f-vector* $(f_S(P) \ : \ S \subseteq \{1, \ldots, n\})$. The entry f_S in the flag f-vector is the number of chains $x_1 < x_2 < \cdots < x_{|S|}$ such that their set of ranks $\{\rho(x_i) \ : \ i \in \{1, \ldots, |S|\}\}$ is S. Inspired by Stanley [12] we introduce the *upsilon invariant* of a graded poset P of rank $n+1$ by

$$\Upsilon_P(a, b) = \sum_{S \subseteq \{1, \ldots, n\}} f_S u_S$$

where $u_S = u_1 \cdots u_n$ is a monomial in noncommuting variables a and b such that $u_i = b$ for all $i \in S$ and $u_i = a$ for all $i \notin S$. The term upsilon invariant is *not used elsewhere* in the literature, most sources switch to the *ab-index* $\Psi_P(a, b)$ defined to be equal to $\Upsilon_P(a - b, b)$. A graded poset P is *Eulerian* if every nontrivial interval of P has the same number of elements of even rank as of odd rank. All linear relations satisfied by the flag f-vectors of Eulerian posets were found by Bayer and Billera [1]. A very useful and compact rephrasing of the Bayer–Billera relations was given by Bayer and Klapper in [2]: they proved that satisfying the Bayer–Billera relations is equivalent to stating that the ab-index may be rewritten as a polynomial of $c = a + b$ and $d = ab + ba$. The resulting polynomial in noncommuting variables c and d is called the *cd-index*.

1.2 Tchebyshev triangulations and Tchebyshev transforms

A finite simplicial complex $\triangle$ is a family of subsets of a finite vertex set V. The elements of $\triangle$ are called *faces*, subject to the following rules: a subset of any face is a face and every singleton is a face. The *dimension* of a face is one less than the number of its elements, the dimension $d-1$ of the complex $\triangle$ is the maximum of the dimension of its faces. The number of j-dimensional faces is denoted by $f_j(\triangle)$ and the vector $(f_{-1}, f_0, \ldots, f_{d-1})$ is the *f-vector* of the simplicial complex. We define the *F-polynomial* $F_\triangle(x)$ of a finite simplicial complex $\triangle$ as

$$F_\triangle(x) = \sum_{j=0}^{d} f_{j-1}(\triangle) \cdot \left(\frac{x-1}{2}\right)^j. \tag{1.1}$$

The *join* $\triangle_1 * \triangle_2$ of two simplicial complexes $\triangle_1$ and $\triangle_2$ on disjoint vertex sets is the simplicial complex $\triangle_1 * \triangle_2 = \{\sigma \cup \tau \ : \ \sigma \in \triangle_1, \tau \in \triangle_2\}$. It is easy to show that the F-polynomials satisfy $F_{\triangle_1 * \triangle_2}(x) = F_{\triangle_1}(x) \cdot F_{\triangle_2}(x)$. A special instance of the join operation is the *suspension* operation: the suspension $\triangle * \partial(\triangle^1)$ of a simplicial complex $\triangle$ is the join of $\triangle$ with the boundary complex of the one dimensional simplex. (A $(d-1)$-dimensional simplex is the family of all subsets of a d-element set, its boundary is obtained by removing its only facet from the list of faces.) The *link* of a face σ is the subcomplex $\mathrm{link}_\triangle(\sigma) = \{\tau \in \triangle : \ \sigma \cap \tau = \emptyset, \ \sigma \cup \tau \in K\}$. A special type of simplicial complex we will focus on is the *order complex* $\triangle(P)$ of a finite partially ordered set P: its vertices are the elements of P and its faces are the increasing chains. The order complex of a finite poset is a *flag complex*: its minimal non-faces are all two-element sets (these are the pairs of incomparable elements). Every finite simplicial complex $\triangle$ has a *standard geometric realization* in the vector space with a basis $\{e_v \ : \ v \in V\}$ indexed by the vertices, where each face σ is realized by the convex hull of the basis vectors e_v indexed by the elements of σ.

Definition 1.1. We define a *Tchebyshev triangulation* $T(\triangle)$ of a finite simplicial complex $\triangle$ as follows. We number the edges $e_1, e_2, \ldots, e_{f_1(\triangle)}$ in some order, and we associate to each edge $e_i = \{u_i, v_i\}$ a midpoint w_i. We associate a sequence $\triangle_0 := \triangle, \triangle_1, \triangle_2 \ldots, \triangle_{f_1(\triangle)}$ of simplicial complexes to this numbering of edges, as follows. For each $i \geq 1$, the complex $\triangle_i$ is obtained from $\triangle_{i-1}$ by replacing the edge e_i and the faces contained therein with the one-dimensional simplicial complex L_i, consisting of the vertex set $\{u_i, v_i, w_i\}$ and edge set $\{\{u_i, w_i\}, \{w_i, v_i\}\}$, and by replacing the family of faces $\{e_i \cup \tau \ : \ \tau \in \operatorname{link}_{\Delta_{i-1}}(e_i)\}$ containing e_i with the family of faces $\{\sigma' \cup \tau \ : \ \sigma' \in L_i\}$. In other words, we subdivide the edge e_i into a path of length 2 by adding the midpoint w_i and we also subdivide all faces containing e_i, by performing a stellar subdivision.

It has been shown in [8] in a more general setting that a Tchebyshev triangulation of $\triangle$ as defined above is indeed a triangulation of $\triangle$ in the following sense: if we consider the standard geometric realization of $\triangle$ and associate to each midpoint w the midpoint of the line segment realizing the corresponding edge $\{u, v\}$ then the convex hulls of the vertex sets representing the faces of $T(\triangle)$ represent a triangulation of the geometric realization of $\triangle$. Furthermore, a direct consequence of [8, Theorem 3.3] is the following theorem.

Theorem 1.2 (Hetyei and Nevo)**.** *All Tchebyshev triangulations of a simplicial complex have the same f-vector.*

The following result has been shown in [7, Proposition 3.3] for a specific Tchebyshev triangulation. By the preceding theorem it holds for all Tchebyshev triangulations and motivates the choice of the terminology. The *Tchebyshev transform of the first kind of polynomials* used in the next result is the linear map $T : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$ sending x^n into the Tchebyshev polynomial of the first kind $T_n(x)$.

Theorem 1.3. *For any finite simplicial complex $\triangle$, the F-polynomial of any Tchebyshev triangulation $T(\triangle)$ is the Tchebyshev transform of the first kind of the F-polynomial of $\triangle$:*

$$F_{T(\triangle)}(x) = T(F_{\triangle}(x)).$$

The notion of the Tchebyshev triangulation of a simplicial complex was motivated by a poset operation, first considered in [5] and formally introduced in [6] .

Definition 1.4. Given a locally finite poset P, its *Tchebyshev transform of the first kind* $T(P)$ is the poset whose elements are the intervals $[x, y] \subset P$ satisfying $x \neq y$, ordered by the following relation: $[x_1, y_1] \leq [x_2, y_2]$ if either $y_1 \leq x_2$ or both $x_1 = x_2$ and $y_1 \leq y_2$ hold.

A geometric interpretation of this operation may be found in [6, Theorem 1.10]. The graded variant of this poset operation is defined in [3]. Given a graded poset P with minimum element $\widehat{0}$ and maximum element $\widehat{1}$, we introduce a new minimum element $\widehat{-1} < \widehat{0}$ and a new maximum element $\widehat{2}$. The *graded Tchebyshev transform of the first kind* of a graded poset P is then the interval $[(\widehat{-1}, \widehat{0}), (\widehat{1}, \widehat{2})]$ in $T(P \cup \{\widehat{-1}, \widehat{2}\})$. By abuse of notation we also denote the graded Tchebyshev transform of a graded poset P by $T(P)$. It is easy to show that $T(P)$ is also a graded poset, whose rank is one more than that of P. The following result may be found in [7, Theorem 1.5].

Theorem 1.5. *Let P be a graded poset and $T(P)$ its graded Tchebyshev transform. Then the order complex $\triangle(T(P) \setminus \{(\widehat{-1}, \widehat{0}), (\widehat{1}, \widehat{2})\})$ is a Tchebyshev triangulation of the suspension of $\triangle(P \setminus \{\widehat{0}, \widehat{1}\})$.*

It has been shown by Ehrenborg and Readdy [3] that there is a linear transformation assigning to the flag f-vectors of each graded poset P the flag f-vector of $T(P)$. For Eulerian posets, they also compute the effect on the cd-index of this Tchebyshev transform.

1.3 Permutohedra of type A and B

Permutohedra of type A and B have a vast literature, the results cited here may be found in [4] and in [14]. The type A permutohedron $\mathrm{Perm}(A_{n-1})$ is the convex hull of the $n!$ vertices $(\pi(1), \ldots, \pi(n)) \in \mathbb{R}^n$, where π is any permutation of the set $[1, n] := \{1, 2, \ldots, n\}$. The type B permutohedron $\mathrm{Perm}(B_n)$ is the convex hull of all points of the form $(\pm\pi(1), \pm\pi(2) \ldots, \pm\pi(n)) \in \mathbb{R}^n$. Combinatorially equivalent polytopes may be obtained by taking the A_{n-1}-orbit, respectively B_n orbit, of any sufficiently generic point in an $(n-1)$-dimensional (respectively n-dimensional) space, and the convex hull of the points in the orbit. [4, Section 2].

The type A and B permutohedra are simple polytopes, their duals are simplicial polytopes. The boundary complexes of these duals are order complexes of graded posets (with their minimum and maximum elements removed): in the type A case we have the order complex of $P([1, n]) - \{\emptyset, [1, n]\}$, where $P([1, n])$ is the Boolean algebra of rank n, in the type B case we have the face lattice of the n-dimensional crosspolytope [14, Lecture 1]. The standard n-dimensional crosspolytope is the convex hull of the vertices $\{\pm e_i : i \in [1, n]\}$, where $\{e_1, e_2, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$. Each nontrivial face of the crosspolytope is the convex hull of a set of vertices of the form $\{e_i, i \in K^+\} \cup \{-e_i, i \in K^-\}$, where K^+ and K^- is are disjoint subsets of $[1, n]$ and their union is not empty.

Corollary 1.6. *Each facet of* $\mathrm{Perm}(B_n)$ *is uniquely labeled with a pair of sets (K^+, K^-) where K^+ and K^- is are subsets of $[1, n]$, satisfying $K^+ \subseteq [1, n] - K^-$ and K^+ and K^- cannot be both empty. For a set of valid labels $\{(K_1^+, K_1^-), (K_2^+, K_2^-), \ldots, (K_m^+, K_m^-)\}$ the intersection of the corresponding set of facets is a nonempty face of* $\mathrm{Perm}(B_n)$ *if and only if*

$$K_1^+ \subseteq K_2^+ \subseteq \cdots \subseteq K_m^+ \subseteq [1, n] - K_m^- \subseteq [1, n] - K_{m-1}^- \subseteq \cdots \subseteq [1, n] - K_1^- \quad \textit{holds.}$$

The triangle of f-vectors of the type B Coxeter complexes is given in sequence A145901 in [11].

2 The poset of intervals as a Tchebyshev transform

Definition 2.1. An *interval* $[u,v]$ in a partially ordered set P is the set of all elements $w \in P$ satisfying $u \le w \le v$. For a finite partially ordered set P we define the *poset $I(P)$ of the intervals of P* as the set of all intervals $[u,v] \subseteq P$, ordered by inclusion.

We may identify the singleton intervals $[u,u]$ in $I(P)$ with the elements of P. Figure 1 shows a partially ordered set and its order complex. The poset of its intervals and the order complex thereof may be seen in Figure 2.

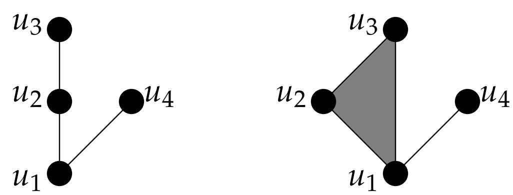

Figure 1: A partially ordered set P and its order complex $\triangle(P)$

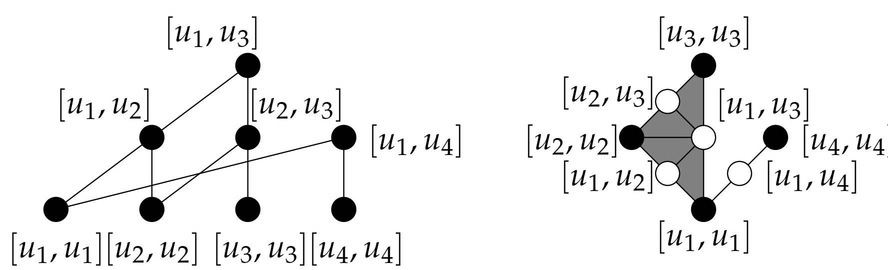

Figure 2: The poset $I(P)$ of intervals of P and its order complex

The following result is a generalization of [9, Remark 10], and an equivalent restatement of [15, Theorem 4.1].

Theorem 2.2. *For any finite partially ordered set P the order complex $\triangle(I(P))$ of its poset of intervals is isomorphic to a Tchebyshev triangulation of $\triangle(P)$ as follows. For each $u \in P$ we identify the vertex $[u,u] \in \triangle(I(P))$ with the vertex $u \in \triangle(P)$ and for each nonsingleton interval $[u,v] \in I(P)$ we identify the vertex $[u,v] \in \triangle(I(P))$ with the midpoint of the edge $\{[u,u],[v,v]\}$. We number the midpoints $[u_1,v_1],[u_2,v_2],\ldots$ in such an order that $i<j$ holds whenever the interval $[u_i,v_i]$ contains the interval $[u_j,v_j]$.*

Definition 2.3. For a graded poset P we define its *graded poset of intervals $\widehat{I}(P)$* as the poset of all intervals of P, *including the empty set*, ordered by inclusion.

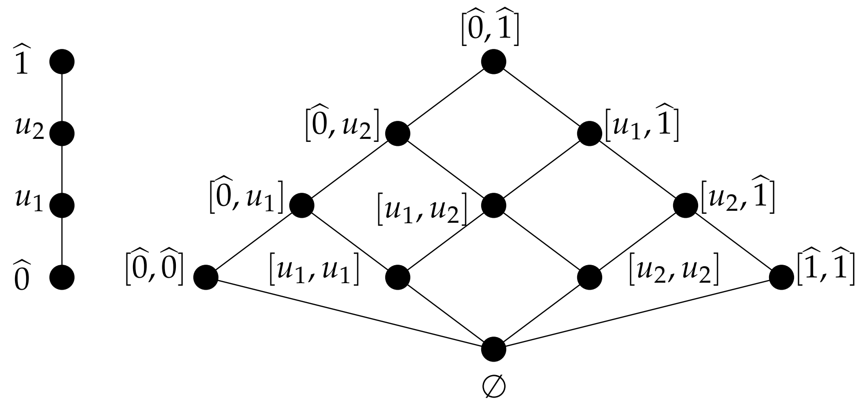

Figure 3: The graded poset of intervals of a chain

Remark 2.4. *Figure 3 represents the graded poset of intervals of a chain of rank* 3*. Comparing it with [6, Figure 2] where the Tchebyshev transform of a chain of rank* 3 *is represented, we see the two transforms are different.*

Proposition 2.5. *If P is a graded poset of rank n with rank function ρ then $\widehat{I}(P)$ is a graded poset of rank $n+1$, in which the rank of a nonempty interval $[u,v]$ is $\rho(v)-\rho(u)+1$.*

In analogy to Theorem 1.5 we have the following result.

Proposition 2.6. *Let P be a graded poset and $\widehat{I}(P)$ its graded poset of intervals. Then the order complex $\triangle(\widehat{I}(P)-\{\emptyset,[\widehat{0},\widehat{1}]\})$ is a Tchebyshev triangulation of the suspension of $\triangle(P-\{\widehat{0},\widehat{1}\})$.*

3 The type B Coxeter complex as a Tchebyshev triangulation

After introducing $X:=K^{+}$ and $Y:=[1,n]-K^{-}$, we may rephrase Corollary 1.6 as follows.

Corollary 3.1. *We may label each facet of the type B permutohedron* $\mathrm{Perm}(B_n)$ *with a nonempty interval $[X,Y]$ of the Boolean algebra $P([1,n])$ that is different from $P([1,n])=[\emptyset,[1,n]]$. The set $\{[X_1,Y_1],[X_2,Y_2],\ldots,[X_m,Y_m]\}$ labels a collection of facets with a nonempty intersection if and only if the intervals form an increasing chain in $\widehat{I}(P([1,n]))-\{\emptyset,[\emptyset,[1,n]]\}$.*

The representation of each face of $\mathrm{Perm}(B_n)$ as an intersection of facets is unique, hence we obtain the following result.

Proposition 3.2. *The dual of* $\mathrm{Perm}(B_n)$ *is a simplicial polytope whose boundary complex is combinatorially equivalent to the order complex $\triangle(\widehat{I}(P([1,n]))-\{\emptyset,[\emptyset,[1,n]]\})$.*

Corollary 3.3. *The dual of* $\mathrm{Perm}(B_n)$ *is a simplicial polytope whose boundary complex is combinatorially equivalent to a Tchebyshev triangulation of the suspension of* $\triangle(P([1,n]) - \{\emptyset, [1,n]\})$*, and hence to a Tchebyshev triangulation of the suspension of the boundary complex of the dual of the permutohedron* $\mathrm{Perm}(A_{n-1})$*.*

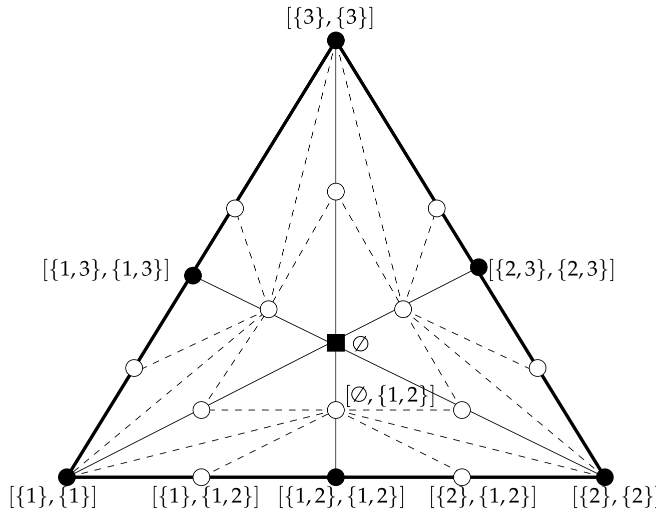

Figure 4: Half of the dual of $\mathrm{Perm}(B_3)$

Figure 4 represents "half" of the dual of $\mathrm{Perm}(B_3)$. The boundary of the triangle whose vertices are labeled with singleton intervals $[\{i\},\{i\}]$ is shown in bold. The vertices of the barycentric subdivision of the boundary are marked with black circles. These correspond to singleton intervals of the form $[X,X]$, where X is a subset of $[1,3]$. (In general, X is a subset of $[1,n]$.) The suspending vertex $\emptyset$ is marked with a black square. The other suspending vertex $[1,3]$ (in general: $[1,n]$) is not shown in the picture. One would need to make another picture showing the boundary of the triangle with the suspending vertex, and "glue" the two pictures along the boundary of the triangle. The midpoints of the edges are marked with white circles. These are labeled with intervals $[X,Y]$ such that X is properly contained in Y. The edges arising when we take the appropriate Tchebyshev triangulation are indicated with dashed lines. This part of the picture is different on the "other side" of the dual of $\mathrm{Perm}(B_3)$.

4 Computing the flag f-vector of the graded poset of intervals

In this section we show that for any graded poset P, the flag f-vector of its graded poset of intervals $\widehat{I}(P)$ may be obtained from the flag f-vector of P by a linear transformation.

By "chain" in this section we always mean a chain containing the unique minimum element and the unique maximum element.

Definition 4.1. Given a chain $\emptyset \subset [u_1, v_1] \subset [u_2, v_2] \subset \cdots \subset [u_k, v_k] \subset [u_{k+1}, v_{k+1}] = [\widehat{0}, \widehat{1}]$ in the graded poset of intervals $\widehat{I}(P)$ of a graded poset P, we call the set

$$\{u_1, v_1, u_2, v_2, \ldots, u_{k+1}, v_{k+1}\}$$

the *support* of the chain.

Obviously the support of a chain in $\widehat{I}(P)$ is a chain in P containing the minimum element $\widehat{0}$ and the maximum element $\widehat{1}$.

The next statement expresses the number of chains in $\widehat{I}(P)$ having the same support in terms of the *Pell numbers* $P(n)$. These numbers are given by the initial conditions $P(1) = 1$ and $P(2) = 2$ and by the recurrence $P(n) = 2 \cdot P(n-1) + P(n-2)$ for $n \geq 3$. A detailed bibliography on the Pell numbers may be found at sequence A000129 of [11].

Proposition 4.2. *Let P be a graded poset and let $c : \widehat{0} = z_0 < z_1 < \cdots < z_{m-1} < z_m = \widehat{1}$ be a chain in it. Then the number of chains $\emptyset \subset [u_1, v_1] \subset [u_2, v_2] \subset \cdots \subset [u_k, v_k] \subset [u_{k+1}, v_{k+1}] = [\widehat{0}, \widehat{1}]$ whose support is c is the sum $P(m) + P(m+1)$ of two adjacent Pell numbers.*

The proof is by induction on m. The numbers $P(m) + P(m+1)$ are listed as sequence A001333 in [11]. It is transparent in the (omitted) proof of Proposition 4.2 that the contributions of chains of $\widehat{I}(P)$ with a fixed support to $\Upsilon_{\widehat{I}(P)}(a, b)$ depends only on the contribution of their support to $\Upsilon_P(a, b)$. This observation motivates the following definition.

Definition 4.3. Given an ab-word w of degree n, we define $\iota(w)$ as the contribution of all chains of $\widehat{I}(P)$ with a fixed support to $\Upsilon_{\widehat{I}(P)}(a, b)$, whose support is the same chain of P, contributing the word w to $\Upsilon_P(a, b)$.

Theorem 4.4. *The operator ι may be recursively computed using the following formulas.*

1. *$\iota(a^n) = (a + 2b)a^n$ holds for $n \geq 0$. In particular, for the empty word ε we have $\iota(\varepsilon) = (a + 2b)$.*
2. *$\iota(a^i b a^j) = (a + 2b)(a^i b a^j + a^j b a^i) + b a^{i+j+1}$ holds for $i, j \geq 0$.*
3. *$\iota(a^i b w b a^j) = \iota(a^i b w) b a^j + \iota(w b a^j) b a^i + \iota(w) b a^{i+j+1}$ holds for $i, j \geq 0$ and any ab-word w.*

Proof. We only show the third statement, to save space. Consider a chain $c : \widehat{0} < z_1 < z_2 < \cdots < z_k < z_{k+1} = \widehat{1}$ that contributes $a^i b w b a^j$ to the ab-index of a graded poset P of rank $n+1$. In such a chain the rank of z_1 is $i+1$ and the rank of of z_k is $n-j$. The largest element below $[\widehat{0}, \widehat{1}]$ of any chain in $\widehat{I}(P)$ with support c is either $[\widehat{0}, z_k]$ (of rank $n-j+1$) or $[z_1, \widehat{1}]$ (of rank $n+1-i$) or $[z_1, z_k]$ (of rank $n-i-j+1$). The three terms correspond to the contributions of the chains of these three types. $\square$

Corollary 4.5. *There is a linear map $I_n : \mathbb{R}^{2^n} \to \mathbb{R}^{2^{n+1}}$ sending the flag f-vector of each graded poset P of rank $n+1$ into the flag f-vector of its graded poset of intervals $\widehat{I}(P)$. This linear map may be obtained by encoding flag f-vectors with the corresponding upsilon invariants, and extending the map ι by linearity.*

Example 4.6. *Using Theorem 4.4 we obtain $\iota(a) = a^2 + 2ba$, $\iota(b) = 4b^2 + 2ab + ba$, $\iota(a^2) = a^3 + 2ba^2$, $\iota(ab) = \iota(ba) = a^2b + aba + 2bab + 2b^2a + ba^2$ and $\iota(b^2) = 8b^3 + 4ab^2 + 2bab + aba + 2b^2a$.*

5 The graded poset of intervals of an Eulerian poset

Theorem 5.1. *If a graded poset P is Eulerian then the same holds for the graded poset of its intervals $\widehat{I}(P)$.*

Proof. It is well known consequence of Phillip Hall's theorem (see [13, Propositition 3.8.5]) that a graded poset is Eulerian if and only if the reduced characteristic of the order complex of each open interval (u, v) is $(-1)^{\rho(v)-\rho(u)}$ where ρ is the rank function. Since taking the graded poset of intervals results in taking a triangulation of the suspension of each such order complex, the reduced Euler characteristic remains unchanged. □

As a consequence of Theorem 5.1, the linear map I_n takes the flag f-vector of any graded Eulerian poset of rank $n+1$ into the flag f-vector of a graded Eulerian poset of rank $n+2$. It has been shown by Bayer and Billera [1] that for each n, one may make a list of F_{n+1} graded Eulerian partially ordered sets of rank $n+1$ whose flag f vectors are linearly independent, where F_{n+1} is the $(n+1)$st Fibonacci number ($F_1 = 1$, $F_2 = 2$). The upsilon invariants of such a basis span the vector space of upsilon invariants of all Eulerian posets of rank $n+1$, and the images under ι of these basis vectors have the property that the resulting upsilon invariants are also polynomials of $c = a + 2b$ and $d = ab + ba + 2b^2$. The same observation also holds for all linear combinations, hence we obtain the following result.

Theorem 5.2. *Extending the operator ι to linear combinations of ab-words by linearity, results in a linear operator that takes each polynomial of $c = a + 2b$ and $d = ab + ba + 2b^2$ into a polynomial of $c = a + 2b$ and $d = ab + ba + 2b^2$. This operator takes the cd-index of an Eulerian poset P into the cd-index of its graded poset of intervals $\widehat{I}(P)$.*

By abuse of notation, we will use the same symbol ι to denote the induced operator on cd words.

Example 5.3. *Using the formulas listed in Example 4.6 we obtain $\iota(c) = c^2 + 2d$, $\iota(d) = 2(cd + dc)$ and $\iota(c^2) = c^3 + 2cd + 4dc$.*

We conclude this section with explicitly computing $\iota(c^n)$ for all n. Note that c^n is the cd-index of the "ladder" poset L_n of rank $n+1$. This poset has exactly 2 elements: $-i$ and i for each rank i satisfying $0 < i < n+1$, and any pair of elements at different ranks are comparable. This formula was first found by Jojić [9].

Theorem 5.4. *Assume that the finite vector $(k_0, \ldots, k_r)$ of nonnegative integers satisfies $2r + k_0 + k_2 + \cdots + k_r = n$. Then the coefficient of $c^{k_0}dc^{k_1}d \cdots c_{k_r}dc^{k_r}$ in $\iota(c^n)$ is $2^r(k_1+1)(k_2+1)\cdots(k_r+1)$.*

The proof uses an *R-labeling* on the dual $\widehat{I}(L_n)^*$ of the graded poset of intervals $\widehat{I}(L_n)$ of the ladder poset L_n and proceeds by induction.

Remark 5.5. *Surprisingly this formula is the dual of the one obtained for the other Tchebyshev transform see [5, Theorem 7.1] and [3, Corollary 6.6] (see also [5, Table 1] and compare it with Example 5.3), although the two poset operations are very different.*

6 Concluding remarks

Taking the graded poset of intervals seems to be a fairly straightforward operation, worthy of further study. Some explicit but cumbersome formulas for cd-indices were found by Jojić [9], in the talk we will see simplified proofs of his formulas and results on the analogue of the Tchebyshev transform of the second kind related to the interval transform of a poset. Generalizations of permutohedra abound, and performing an analogous sequence of stellar subdivisions on their duals, respectively taking the graded poset of intervals for an associated poset may result in interesting geometric constructions, producing perhaps new type B analogues. Finally, applying the Tchebyshev transform studied in [5], [6] and [3] to a Boolean algebra creates a poset whose order complex has the same face numbers as the dual of a type B permutohedron. It may be interesting to find out whether the resulting polytope also has a nice geometric representation.

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