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On the combinatorics of LLT polynomials in Sp_{2n}

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Abstract. LLT polynomials were originally defined as q-generating functions for tuples of semistandard tableaux and later generalized to arbitrary Lie type. We introduce a combinatorial definition at $q = 1$ for LLT polynomials of type C as a similar generating function over tuples of symplectic tableaux. The definition uses a correspondence between symplectic tableaux and oscillating tableaux that is used to give a proof of a Cauchy identity for Sp_{2n} using Berele insertion, generalizing the combinatorial proof of Schur-Weyl duality for Sp_{2n}.

Résumé. Les polynômes LLT étaient définis à l'origine comme des fonctions génératrices de q pour les tuplets de tableaux semistandard, puis généralisés au type de Lie arbitraire. Nous introduisons une définition combinatoire à $q = 1$ pour les polynômes de type C de type LLT en tant que fonction génératrice similaire sur des tuplets de tableaux symplectiques. La définition utilise une correspondance entre des tableaux symplectiques et des tableaux oscillants, utilisée pour donner la preuve d'une identité de Cauchy pour Sp_{2n} en utilisant l'insertion de Berele, généralisant la preuve combinatoire de la dualité Schur-Weyl pour Sp_{2n}.

Keywords: LLT polynomials, symplectic tableaux, oscillating tableaux, type C

1 Introduction

LLT polynomials were first defined by Lascoux, Leclerc, and Thibon [11] in their study of plethystic substitutions of Hall-Littlewood polynomials. They have since then enjoyed a wide range of applications, from branching rules in the modular representation theory of S_n [10] to crystal base theory of $\mathcal{U}_q(\widehat{\mathfrak{sl}_n})$ Fock spaces [11] to the combinatorics of Macdonald polynomials [4] and diagonal coinvariants [5]. Of chief interest to this author is their last role above, in which LLT polynomials are used to give a monomial expansion of Macdonald polynomials by relating the dinv statistic of fillings of a Young diagram to the inversion statistic of LLT polynomials.

At the time of their origination, LLT polynomials, known as "spin" and "cospin" ribbon Schur functions, were conjectured to be Schur positive when indexed by a tuple of straight shapes. This was later proved [12] using the positivity of certain Kazhdan-Lusztig polynomials, and then extended slightly in [5] to when the indexing tuple consists of n-cores. The case of arbitrary skew shapes was then proved by Grojnowski and

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Haiman [3], and in their work, they expanded the definition of LLT polynomials to all complex reductive Lie groups. Together with the combinatorial expansion of Macdonald polynomials into LLT polynomials, this gave another proof of Macdonald positivity, although still building on a heavily geometric proof.

The problem addressed at hand is to provide a combinatorial description of LLT polynomials in the case of Sp_{2n}, for which we have the following main result:

Definition/Theorem 1.1. Let $G = Sp_{2n}$ and fix a Levi $L = GL_{r_1} \times \cdots \times GL_{r_{\ell-1}} \times Sp_{2r_\ell}$, and weights $\beta = (\beta^{(1)}, \ldots, \beta^{(\ell)}), \gamma = (\gamma^{(1)}, \ldots, \gamma^{(\ell)}) \in X_+(L)$. For $1 \leq j \leq \ell$, choose R_j so that $\widetilde{\gamma}^{(j)} := \gamma^{(j)} + (R_j^{r_j})$ and $\widetilde{\beta}^{(j)} + (R_j^{r_j})$ have all part sizes at least n. For $k >> 0$ sufficiently large, define

$$\boldsymbol{\tau} = (\widetilde{\beta}')^c, \text{ each complement taken in a } ((k+n)^{r_j}) \text{ box}$$

$$\boldsymbol{\sigma} = (\widetilde{\gamma}')^c, \text{ each complement taken in a } (k^{r_j}) \text{ box}$$

The *LLT polynomials of type* $G = Sp_{2n}$ at $q = 1$ are

$$\mathcal{G}^{(k)}_{\beta,\gamma}(z^{\pm 1}_{k+1}, \ldots, z^{\pm 1}_{k+n}; q = 1) = \sum_{T \in Symp(\boldsymbol{\tau}/\boldsymbol{\sigma})} z^T$$

the sum over all skew symplectic tableaux of shape $\boldsymbol{\tau}/\boldsymbol{\sigma}$.

The theorem in the above statement is that the LLT polynoimals as defined above coincide with the algebro-geometric definition given in [3] at $q = 1$ (see Theorem 4.2 below). The proof uses a combinatorial result interesting in its own right between symplectic tableaux and oscillating tableaux:

Proposition 1.2. *There is a bijection between (horizontal) semistandard oscillating tableaux from* $\varnothing$ *to* λ *in* k *steps, with all parts at most* N*, and symplectic tableaux of shape* λ^c*, the complement taken in an* (N^k) *box.*

This was recently and independently shown in [14]. In particular, this gives a connection between weight multiplicities of irreducible representations of Sp_{2n} (as indexed by symplectic tableaux) and the multiplicities of irreducible representations inside tensor powers of the standard representation (as indexed by oscillating tableaux).

Given that LLT polynomials were used to give a monomial expansion for Macdonald polynomials, it follows that a combinatorial formula for Sp_{2n} LLT polynomials could illuminate a similar expansion for type C Macdonald polynomials. As it stands, Macdonald polynomials are defined for any root system, but with only a combinatorial (and geometric) understanding in type A. What's more, the general type LLT polynomials, as defined below in (2.1), coincide with Hall-Littlewood polynomials when the indexing Levi L is the torus T, and hence a combinatorial formula for LLT polynomials could lead towards a formula for Kostka-Foulkes polynomials akin to Lascoux and Schutzenberger's celebrated and mysterious charge formula in type A. A charge statistic for other

Lie types has been proposed in [13] for Kashiwara-Nakashima tableaux, and in fact the current problem was suggested to the author after first trying to extend the charge formula to type C for King tableaux.

This paper is organized as follows. In section 2 we provide the Lie theory background needed to roughly define general LLT polynomials and we recall the original definition of LLT polynomials for GL_n. In section 3 we define the combinatorial objects that arise in Sp_{2n} and exhibit the underlying combinatorial bijection between symplectic tableaux and oscillating tableaux. In section 4 we expound on the connection between combinatorial LLT polynomials and general type LLT polynomials, and state our main result. We conclude in section 5 with future directions of researChapter

2 Preliminaries

In the current setting we work over the ring $\mathbb{Z}[u^{\pm 1}]$ and we set $q = u^2$. We let G denote a complex reductive Lie group with $\mathfrak{g} = \mathrm{Lie}(G)$ and we fix the Cartan data $(X, X^\vee, \Pi, \Pi^\vee)$ consisting of the weight and coweight lattices $X, X^\vee$ and the sets of simple roots $\alpha_i \in \Pi \subseteq X$ and simple coroots $\alpha_i^\vee \in \Pi^\vee \subseteq X^\vee$. We let X_+, X_{++} denote the set of dominant and regular dominant weights, respectively. The reader should keep in mind the specific cases $G = GL_n$ and $G = Sp_{2n}$, in which the dominant weights are non-increasing integer sequences $(\lambda_1 \geq \cdots \geq \lambda_n)$ and $(\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$, respectively, and regular dominant weights those sequences that are strictly increasing.

Let L be a Levi subgroup of G with Weyl group W_J for a parabolic subset J. Fix a dominant weight $\eta \in X_+$ in the fundamental alcove on a level k of $\mathfrak{g}$ so that $\mathrm{Stab}^{\widetilde{W}}(\eta) = W_J$. Recall here that on level k, $\widetilde{W}$ acts on η in the usual way for $w \in W_f$, but as translations by multiples of k for $\lambda \in X$. More concretely, for GL_n and Sp_{2n}, the fundamental alcoves consist of the partitions λ with $\lambda_1 \leq k$ and $\lambda_1 \leq k/2$, respectively. For $G = GL_n$ and $L = GL_{r_1} \times \cdots \times GL_{r_\ell}$, we can choose k, η such that

$$k > \eta_1 = \cdots = \eta_{r_1} > \eta_{r_1+1} = \cdots = \eta_{r_1+r_2} > \cdots > 0 = \cdots = 0$$

Now, for a regular dominant weight $\beta \in X_{++}(L)$, we can find $w \in W_f$ such that

$$\mu := w(\eta + k\beta) \in X_{++} \cap \widetilde{W} \cdot \eta$$

and the resulting μ is unique if we require $w \in W^J$. Surprisingly enough, in GL_n this is simply the usual combinatorics of k-cores and k-quotients. More specifically, if we let ρ, ρ_L denote the Weyl vectors for G and L, so that $\mu - \rho \in X_+$ and $\beta - \rho_L \in X_+(L)$, then provided μ, β have no negative entries, $\mu - \rho$ is precisely the partition whose k-core is η and whose k-quotient is $\beta - \rho_L$. The uninitiated reader can refer to [15, Chapter I.1] for a thorough primer on the combinatorics of k-cores, k-quotients, and abaci.

We can now essentially skip to a (cursory) definition of general LLT polynomials. The interested reader is encouraged to refer to [3] for the complete definition. To summarize, there is a $\mathbb{Q}(q)$–algebra with elements χ_λ indexed by irreducible characters of G. This algebra will act on a module with basis elements $|\gamma\rangle$ that are indexed by $\gamma \in X_{++}(L)$ and whose construction involves η, k, μ as above. For $\chi = \sum a_\lambda x^\lambda$, the action is given by

$$\chi|\gamma\rangle = \sum a_\lambda |\gamma + k\lambda\rangle$$

The general LLT polynomials will be generating functions for matrix coefficients of this action. More specifically, we define the polynomials $Q^\lambda_{\beta,\gamma}(u)$ by

$$\chi^*_\lambda \mid \gamma\rangle = \sum_{\beta \in X_{++}(L)} Q^\lambda_{\beta,\gamma}(u) \mid \beta\rangle$$

We note that there is an implicit dependence on k, η everywhere, namely changing k and η will change the basis elements $|\gamma\rangle$ and also how χ_λ acts on these basis elements; the polynomials $Q^\lambda_{\beta,\gamma}(u)$ however will be unchanged.

Definition 2.1. Let L be a Levi of G with Weyl group W_J and fix $\beta, \gamma \in X_{++}(L)$. The associated *general LLT polynomial* is

$$\mathcal{L}^G_{L,\beta,\gamma}(x;q) = u^m \sum_\lambda Q^\lambda_{\beta,\gamma}(u)\chi_\lambda(x) \tag{2.1}$$

for some intricately defined power m.

The polynomials $Q^\lambda_{\beta,\gamma}(u)$ are akin to Kazhdan-Lusztig polynomials, in that they are matrix coefficients for a change of basis, and their positivity is shown using their interpretation as decomposition multiplicities of certain non-irreducible perverse sheaves on the flag variety. Just as is the case for the usual Kazhdan-Lusztig polynomials, it's not a priori so easy to compute or find a combinatorial formula for these matrix coefficients, and as such, it remains to see how this definition coincides with the original combinatorial definition. For now we recall the combinatorial definition, reformulated as in [5] for our purposes.

Viewing a Young diagram as a subset of $\mathbb{Z} \times \mathbb{Z}$, we define a *skew shape with contents* to be an equivalence class of a skew Young diagram up to content-preserving translations. Recall that the content of a cell (i,j) is defined as $c((i,j)) = j - i$ Given a tuple $\boldsymbol{\beta}/\boldsymbol{\gamma} = (\beta^{(1)}/\gamma^{(1)}, \dots, \beta^{(k)}/\gamma^{(k)})$ of skew shapes with contents, a semistandard Young tableau T of shape $\boldsymbol{\beta}/\boldsymbol{\gamma}$ is a semistandard Young tableau on each $\beta^{(j)}/\gamma^{(j)}$. An *inversion* of $T = (T_1, \dots, T_k)$ is a pair of cells x, y with $x \in \beta^{(i)}/\gamma^{(i)}, \beta^{(j)}/\gamma^{(j)}$ such that $T(x) > T(y)$ and either

- $c(x) = c(y)$ and $i < j$ or

- $c(x)+1=c(y)$ and $i>j$

Definition 2.2. Let $\boldsymbol{\beta}/\boldsymbol{\gamma}$ be a tuple of skew shapes with contents. The *combinatorial* LLT polynomials $\mathcal{G}_{\boldsymbol{\beta}/\boldsymbol{\gamma}}(x;q)$ are given by

$$\mathcal{G}_{\boldsymbol{\beta}/\boldsymbol{\gamma}}(x;q) = \sum_{T \in SSYT(\boldsymbol{\beta}/\boldsymbol{\gamma})} q^{\operatorname{inv} T} x^T \tag{2.2}$$

where $\operatorname{inv} T$ is the number of inversions of T.

It was shown in [3, Corollary 6.19] that for $G = GL_n$ and the choices of L, k, η above,

$$\mathcal{G}_{\boldsymbol{\beta}/\boldsymbol{\gamma}}(X;q) = q^m \mathcal{L}^G_{L,\beta+\rho_L,\gamma+\rho_L}(X;q)_{\mathrm{pol}} \tag{2.3}$$

where m is some explicit power and pol denotes truncation to polynomial characters. We take this truncation because a general LLT polynomial as defined in (2.1) is technically a formal sum, as there are infinitely many dominant weights λ. We defer an overview of the proof of (2.3) to section 4.

3 Combinatorics

Our main interest is in the combinatorics at play for $G = Sp_{2n}$, and so we introduce those objects here.

Definition 3.1. Let λ, μ be straight shapes. An *n-oscillating tableau* of shape λ/μ is a sequence

$$\mu = \nu^0, \nu^1, \nu^2, \ldots, \lambda$$

of partitions such that for each i,

(i) ν^i differs from ν^{i-1} by a single box.

(ii) $\ell(\nu^i) \leq n$.

In the literature [20] this is also known as an *n-symplectic up-down tableau*. When the length restriction is implicit or not imposed, we will drop the n and simply refer to this as an oscillating tableau or an up-down tableau.

Definition 3.2. Let λ, μ be straight shapes. An *N-horizontal (N-vertical) semistandard oscillating tableau* of shape λ/μ is a sequence

$$\mu = \alpha^0 = \beta^0 \subseteq \alpha^1 \supseteq \beta^1 \subseteq \alpha^2 \supseteq \beta^2 \subseteq \cdots \supseteq \lambda$$

of partitions such that for each i,

(i) α^i/β^{i-1} and α^i/β^i is a horizontal (vertical) strip.

(ii) α^i, β^i have all row (column) lengths $\leq N$.

For brevity, we will denote N-hSSOT as the set of N-horizontal semistandard oscillating tableau, and likewise for N-vSSOT. Again, we may often drop the N to avoid clutter or if the condition is not imposed. The *weight* of a horizontal or vertical semistandard is a composition ν, where $\nu_i = |\alpha^i/\beta^{i-1}| + |\alpha^i/\beta^i|$.

A basis for the irreducible representations of GL_n are historically indexed by semistandard tableaux, whose generating functions are Schur polynomials. In Sp_{2n}, *symplectic tableaux* were proposed independently by Kashiwara–Nakashima [6] and King [7]. Their definitions are quite different, the former more compatible with crystal operations, and the latter more compatible with weight multiplicities and restriction to subgroups. An intricate bijection between the two tableaux was given by Sheats [18]. We opt to use King's tableaux in this paper.

Definition 3.3. A *symplectic tableau* T of shape λ is a filling of the Ferrers diagram of λ with the letters $1 < \overline{1} < 2 < \cdots < n < \overline{n}$ such that

1. T is semistandard with respect to the above ordering
2. The entries $\overline{i}$ must be in row $\leq i$.

For convenience, we will denote the entries with their ordering above as the set $[\pm n]$. The utility of these objects is that the irreducible character χ_λ of Sp_{2n} becomes a generating function for symplectic tableaux of shape λ, with each $i, \overline{i}$ contributing a weight x_i, x_i^{-1}, respectively.

Now, it is obvious and yet miraculous that one can view standard Young tableaux as a special case of semistandard tableaux. Representation theoretically, this is giving a connection between weight multiplicities of irreducible representations of GL_n and multiplicities of irreducible constituents in tensor powers of the standard representation. There is an analogous Schur-Weyl duality for Sp_{2n}, with the symmetric group replaced by a Brauer algebra, whose irreducible representations are indexed by oscillating tableaux. However, the above definition of a symplectic tableaux has no such obvious reformulation to connect to oscillating tableaux (nor does the Kashiwara/Nakashima definition). A priori, an oscillating tableau seems to be a fundamentally different object than a symplectic tableau. One result we present, stated below, is that there is in fact an analogous specialization of symplectic tableaux to oscillating tableaux.

Theorem 3.4. *Fix partitions λ, μ and positive integers N, n, k such that $N \geq \ell(\mu), \ell(\lambda)$ and $k \geq \mu_1$. There is a bijection*

$$\Psi_{k,n,N} : \left\{ \begin{array}{c} N\text{-vSSOT of} \\ \text{shape } \lambda/\mu \text{ and } n \text{ steps} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{skew symplectic tableaux of} \\ \text{shape } \tau/\sigma \text{ and entries in} \\ \{\pm(k+1), \ldots, \pm(k+n)\} \end{array} \right\} \tag{3.1}$$

where $\tau = (\lambda')^c$ *is the complement transpose of* λ *in a* $((n+k)^N)$ *box and* $\sigma = (\mu')^c$ *is the complement transpose of* μ *in a* (k^N) *box. The inverse map sends a skew symplectic tableau of shape* τ/σ *to an* N*-vSSOT of shape* $(\tau')^c/(\sigma')^c$*, the first complement taken in an* $((n+k)^N)$ *box and the latter taken in a* (k^N) *box, for any* $N \geq \tau_1$.

Skew symplectic tableaux were defined by Koike–Terada [8] and follow similar restrictions as for King tableaux. Taking $\mu = \varnothing$ and $k = 0$ above gives

Corollary 3.5. *There is a bijection*

$$\Psi_{n,N} : \left\{ \begin{array}{c} N\text{-vSSOT of} \\ \text{shape } \lambda \text{ and } n \text{ steps} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{symplectic tableaux of} \\ \text{shape } \mu \text{ and entries in } [\pm n] \end{array} \right\} \tag{3.2}$$

where $\mu = (\lambda')^c$ *is the complement transpose of* λ *in an* (n^N) *box.*

Applying the transpose gives a dual bijection for N-hSSOT and restricting to oscillating tableaux gives the following.

Corollary 3.6. *There is a bijection between* n*-oscillating tableaux from* $\varnothing$ *to* λ *in* d *steps and symplectic tableaux of shape* λ^c *and weight* $((d-1)^n)$*, the complement taken in a* (d^n) *box.*

The complement shape is perhaps not surprising when we compare to the situation in GL_n. Indeed, the identity of Schur polynomials

$$(x_1 \cdots x_k)^n s_\lambda(x_1^{-1}, \ldots, x_k^{-1}) = s_{\lambda^c}(x_1, \ldots, x_k) \tag{3.3}$$

where the complement is taken in a (n^k) box, implies a bijection between semistandard Young tableaux of shape λ^c and those of shape λ, with the weight μ mapping to $(n - \mu_k, \ldots, n - \mu_1)$. This bijection is given in [19, Exercise 7.41] and we adapt it to our current case with oscillating tableaux and symplectic tableaux. We outline the bijection in the N-hSSOT case by an example (Figure 1).

We first associate to a horizontal semistandard oscillating tableau a tableau with set-valued entries[1] inside the (N^n) rectangle. More specifically, to each cell in the $N \times n$ rectangle, we will assign a subset of entries in $\{1, \bar{1}, \ldots, n, \bar{n}\}$, viz., if in the i^{th} step of the hSSOT a cell was added or removed, then we add i or $\bar{i}$, respectively, to that cell's label.

Denote T the resulting tableau with set-valued entries, consisting of the cells in (N^n) labelled with a (possibly empty) set. Let $\nu^1, \ldots, \nu^N$ be the (possibly zero) columns of T, left to right. Let $\widetilde{\nu}^i$ be the column whose entries are

$$\{1, \ldots, n\} - \{i \mid i \in \nu^i\} \cup \{\bar{i} \mid \bar{i} \in \nu^i\}$$

arranged in increasing order. Let $\widetilde{T}$ be the tableau with columns $\widetilde{\nu}^N, \ldots, \widetilde{\nu}^1$, left to right. The proof finishes by showing that $\widetilde{T}$ is a symplectic tableau of the desired shape.

[1] We don't refer to this as a *set-valued tableau* because that already has two different definitions that do not seem to apply in this context.

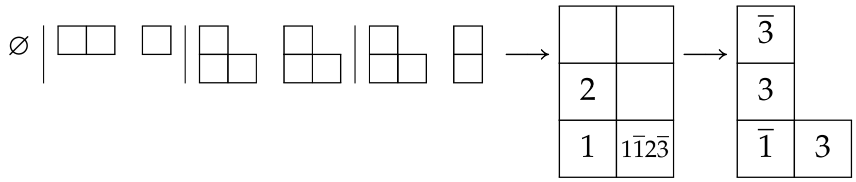

Figure 1: An example of the bijection from a hSSOT to a symplectic tableau. The left sequence of partitions is the 2-hSSOT $\varnothing \subseteq (2) \supseteq (1) \subseteq (2,1) \supseteq (2,1) \subseteq (2,1) \supseteq (1,1)$. The middle diagram is the tableaux T with set-valued entries we associate to the 2-hSSOT and the rightmost diagram is the resulting symplectic tableaux $\widetilde{T}$ with entries in $[\pm 3]$.

Remark 3.7. We note that $\Psi_{n,N}$ and its dual are not quite weight preserving bijections. Given an N-hSSOT $\vec{o}$ with n steps, let T be the intermediate tableau with set-valued entries we associate to $\vec{o}$. Recall that the weight of $\vec{o}$ is the composition $\mu = (\mu_1, \ldots, \mu_n)$ where

$$\mu_i = |\alpha^i/\beta^{i-1}| + |\alpha^i/\beta^i| = \#i\text{'s in T} + \#\bar{i}\text{'s in T}$$

The weight of the resulting symplectic tableau $\widetilde{T}$ will be $\nu = (\nu_1, \ldots, \nu_n)$ where

$$\nu_i = \#i\text{'s in } \widetilde{T} - \#\bar{i}\text{'s in } \widetilde{T} = (N - \#i\text{'s in T}) - (\#\bar{i}\text{'s in T}) = N - \mu_i = (\mu^c)_{n-i}$$

While ν is not always a partition, we can apply the symplectic Bender-Knuth involution to $\widetilde{T}$ to get a symplectic tableau with partition weight.

We can use these bijections to give a more general version of Berele insertion (a thorough exposition on Berele insertion can be found in [20]). Let's recall first a variation of RSK known as the (dual) Burge correspondence [2]. We will say a 2-lined array

$$\begin{pmatrix} a_1 \, a_2 \, \ldots \, a_r \\ b_1 \, b_2 \, \ldots \, b_r \end{pmatrix}$$

is arranged in *antilexicographic order* if $a_i \geq a_{i+1}$ and $a_i = a_{i+1} \implies b_i < b_{i+1}$. In one guise, the dual Burge correspondence is a bijection between 2-lined arrays in antilexicographic order and pairs of SSYT (P, Q) with conjugate shapes via row bumping $b_r b_{r-1} \ldots b_1$ to form P and placing $a_r a_{r-1} \ldots a_1$ in the newly added cell of the conjugate shape to form Q. We give an analogue in type C:

Proposition 3.8. *Let* $\begin{pmatrix} a_1 \, a_2 \, \ldots \, a_r \\ b_1 \, b_2 \, \ldots \, b_r \end{pmatrix}$ *be a 2-lined array arranged in antilexicographic order, with the top entries* $a_i \in [m]$ *and the bottom entries* $b_j \in [\pm n]$*. The following procedure gives a bijection to pairs of symplectic tableaux* $(\widehat{P}, \widehat{Q})$ *with conjugate complement shapes:*

- *Row Berele bump* $b_r b_{r-1} \ldots b_1$ *to form* $\widehat{P}$*.*

- *Keep track of the intermediate shapes as a vSSOT of weight $(a_r, a_{r-1}, \ldots, a_1)$, and then apply Corollary 3.5 to form $\widehat{Q}$.*

As a corollary, we get the following Cauchy-like identity

Corollary 3.9.

$$\prod_{i=1}^{n}\prod_{j=1}^{m}(x_i + x_i^{-1} + y_j + y_j^{-1}) = \sum_{\lambda \subseteq (m^n)} sp_\lambda(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) sp_{(\lambda')^c}(y_1^{\pm 1}, \ldots, y_m^{\pm 1}) \tag{3.4}$$

This appears in [17], where it is referred to as a "Morris Identity". To the best of the author's knowledge, this does not seem to be related to other identities commonly known as a Morris identity. A stronger form was shown by Mimachi [16], where he proved the identity for Koornwinder polynomials. This is a curious identity, as a similar identity holds for Schur functions, dating back to Littlewood [19, Exercise 7.42]. Taking the coefficient of $y_1^{n-1} \cdots y_m^{n-1}$ in (3.4) and applying Corollary 3.5 recovers the combinatorial manifestation of Sp_{2n} Schur-Weyl duality.

Corollary 3.10 (Berele [1]).

$$(x_1 + x_1^{-1} + \ldots + x_n + x_n^{-1})^m = \sum_{\lambda, \ell(\lambda) \leq n} sp_\lambda(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) \widetilde{f}_m^\lambda(n) \tag{3.5}$$

where $\widetilde{f}_m^\lambda(n)$ is the number of n-oscillating tableaux of shape λ and m steps.

In GL_n, the representation theoretic statements of $GL_m - GL_n$ duality and Schur-Weyl duality are bundled into one cohesive combinatorial algorithm. The previous corollaries exhibit the same such bundling with Berele insertion for the analogous dualities in Sp_{2n}. In particular, Corollary 3.9 should be a statement about $Sp_{2n} - \mathfrak{sp}_{2m}$ duality, although at present the author has not worked through the details.

4 Connections; Main Result

The question still remains as to why (2.3) holds. The proof involves explicitly computing the polynomials $Q^\lambda_{\beta,\gamma}(u)$. However, acting by χ_λ on a basis element $|\gamma\rangle$ outright is complicated and so instead one acts by e_λ, wraps up the coefficients as a generating function with monomial symmetric functions, and then uses a Cauchy identity to relate this back to a generating function with Schur functions. As $|\beta\rangle$ and $|\gamma\rangle$ are meant to represent quotients of partitions, we visualize them as abaci. With this, in GL_n each generator $x_i \in \mathbb{Z}[u^{\pm 1}]X$ acts by moving the i^{th} bead on the abacus one unit to the right (or down depending on how one prefers to draw their abacus).

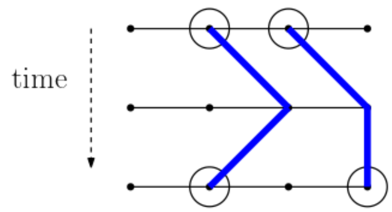

Figure 2: A sequence of actions of $x_i^{\pm 1}$ on $|\gamma\rangle$, viewed as a non-intersecting path

The action $e_\lambda \cdot |\gamma\rangle$ can then be viewed as a sequence of bead moves on each rung, which in turn can be visualized as a time evolution, in which each sequence of moves becomes a tuple of non-intersecting paths (see Figure 2).

In type A, it is well known that a tuple of non-intersecting paths is in bijection with semistandard Young tableaux, as is used in a combinatorial proof of the Jacobi-Trudi identity using the Lindström-Gessel-Viennot lemma. In general type, the coefficient of an abacus $|\beta\rangle$ in $e_\lambda \cdot |\gamma\rangle$ is gotten by applying *straightening rules*, which can be found in full in [3, Prop. 6.3]. In our case of Sp_{2n}, the symplectic characters involve terms x_i and x_i^{-1}, and so the non-intersecting paths can move right and left, seemingly complicating the combinatorics. However, after a *folding procedure* we can simplify the action at $q = 1$ so that the analogous tableaux reformulation replaces semistandard Young tableaux with semistandard oscillating tableaux, namely we have

Proposition 4.1. *There is a sign-reversing involution on the set of non-intersecting paths for which the fixed points are in bijection with vertical semistandard oscillating tableaux.*

This and further combinatorics of non-intersecting paths and oscillating tableaux can be found in [9]. Combining Proposition 4.1 and Corollary 3.5 gives a combinatorial tool to count the coefficient $\langle\beta \mid e_\lambda|\gamma\rangle$, given schematically by

$$\langle\beta \mid e_\lambda|\gamma\rangle \leftrightarrow \#\{\text{non-intersecting paths}\} \leftrightarrow \#\{\text{vertical SSOT}\} \leftrightarrow \#\{\text{symplectic tableaux}\}$$

Choosing everything appropriately, we arrive at

Theorem 4.2. *Let $G = Sp_{2n}$ and fix a Levi L and weights $\beta, \gamma \in X_+(L)$. As in Definition/Theorem 1.1, define $\mathcal{G}^{(k)}_{\beta,\gamma}(z_{k+1}^{\pm 1}, \ldots, z_{k+n}^{\pm 1})$. Then,*

$$\mathcal{G}^{(k)}_{\beta,\gamma}(z_{k+1}^{\pm 1}, \ldots, z_{k+n}^{\pm 1})\Big|_{z_{k+i} \mapsto x_i} = \mathcal{L}^G_{L,\beta+\rho_L,\gamma+\rho_L}(X_n; 1)\Big|_{pol}$$

where $|_{pol}$ denotes truncation to weights that fit in an (n^n) box and then swapping all the coefficients of χ_λ with $\chi_{(\lambda')^c}$.

5 Future Work

The obvious next step would be to extend our proposed tableaux definition of LLT polynomials in type C to arbitrary q. However, the straightening relations for when an irreducible character acts on a basis element $|\gamma\rangle$ becomes more complicated when $q \neq 1$ and it still remains to overcome this. Another direction is that in GL_n, Theorem 4.2 can be restated without the polynomial truncation by writing general LLT polynomials as an inverse limit of combinatorial LLT polynomials. This follows from a natural stability of Schur polynomials to infinitely many variables, which does not hold for symplectic characters. It would be desirable to have such a restatement for Sp_{2n} so as to remove the somewhat arbitrary polynomial truncation in this case.

A final natural progression would be to provide a combinatorial definition of general LLT polynomials for other Lie types. In the odd orthogonal group, one current obstacle is a lack of a Cauchy identity as in Corollary 3.9 for orthogonal tableaux. Even more, the bijection in Corollary 3.5 does not seem to carry over to any similar bijection between orthogonal tableaux and the analogues of oscillating tableaux.

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