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Vandermondes, Superspace, and Delta Conjecture modules

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Abstract. Superspace is an algebra Ω_n with n commuting generators $x_1, \ldots, x_n$ and n anticommuting generators $\theta_1, \ldots, \theta_n$. We present an extension $\delta_{n,k}$ of the Vandermonde determinant to Ω_n which depends on positive integers $k \leq n$. We use superspace Vandermondes to build representations of the symmetric group S_n. In particular, we construct a doubly graded S_n-module $\mathbb{V}_{n,k}$ whose bigraded Frobenius image $\mathrm{grFrob}(\mathbb{V}_{n,k}; q, t)$ conjecturally equals the symmetric function $\Delta'_{e_{k-1}} e_n$ appearing in the Haglund-Remmel-Wilson Delta Conjecture. We prove the specialization of our conjecture at $t = 0$. We use a differentiation action of Ω_n on itself to build bigraded quotients $\mathbb{W}_{n,k}$ of Ω_n which extend the Delta Conjecture coinvariant rings $R_{n,k}$ defined by Haglund-Rhoades-Shimozono and studied geometrically by Pawlowski-Rhoades. Despite the fact that the Hilbert polynomials of the $R_{n,k}$ are not palindromic, we show that $\mathbb{W}_{n,k}$ exhibits a superspace version of Poincaré Duality.

Keywords: Vandermonde, superspace, S_n-module

1 Introduction

The symmetric group S_n acts on the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$ by subscript permutation. Polynomials in the invariant subring

$$\mathbb{Q}[x_1, \ldots, x_n]^{S_n} := \{ f \in \mathbb{Q}[x_1, \ldots, x_n] \,:\, w.f = f \text{ for all } w \in S_n \} \tag{1.1}$$

are called *symmetric polynomials*. The $\mathbb{Q}$-algebra $\mathbb{Q}[x_1, \ldots, x_n]^{S_n}$ is generated by the n elementary symmetric polynomials $e_1, e_2, \ldots, e_n$.

Let $\mathbb{Q}[x_1, \ldots, x_n]^{S_n}_+$ be the space of symmetric polynomials with vanishing constant term. The *invariant ideal* $I_n \subseteq \mathbb{Q}[x_1, \ldots, x_n]$ is given by

$$I_n := \langle \mathbb{Q}[x_1, \ldots, x_n]^{S_n}_+ \rangle = \langle e_1, e_2, \ldots, e_n \rangle, \tag{1.2}$$

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and the *coinvariant ring* is the corresponding quotient

$$R_n := \mathbb{Q}[x_1, \ldots, x_n]/I_n. \tag{1.3}$$

The quotient R_n is simultaneously a graded ring and a graded S_n-module. The module R_n is among the most important in algebraic combinatorics, with representation theory tied to permutation combinatorics and a geometric realization as the cohomology of the flag variety [1, 3].

The symmetric group S_n acts diagonally on the polynomial ring $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ in $2n$ variables, viz. $w.x_i = x_{w(i)}$ and $w.y_i := y_{w(i)}$ for all $w \in S_n$ and $1 \leq i \leq n$. Garsia and Haiman [5] initiated the study of the the *diagonal coinvariant ring* DR_n defined by modding out by those S_n-invariants with vanishing constant term:

$$DR_n := \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]/\langle \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]_+^{S_n} \rangle. \tag{1.4}$$

Considering x-degree and y-degree separately, the ring DR_n is a doubly graded S_n-module which specializes to R_n when the y-variables are set to zero.

Haiman proved [8] that as ungraded S_n-modules we have $DR_n \cong \mathbb{Q}[\mathrm{Park}_n] \otimes \mathrm{sign}$ where Park_n is the permutation action of S_n on size n parking functions and sign is the 1-dimensional sign representation of S_n. Haiman also proved more refined results on the bigraded S_n-module structure of DR_n; to state these we recall some facts about S_n-modules.

The irreducible representations of S_n over $\mathbb{Q}$ are indexed by partitions of n; if $\lambda \vdash n$ is a partition, let S^λ be the corresponding S_n-irreducible. If V is any finite-dimensional S_n-module, there exist unique multiplicities $c_\lambda \geq 0$ so that $V \cong \bigoplus_{\lambda \vdash n} c_\lambda S^\lambda$. Let Λ denote the ring of symmetric functions over the ground field $\mathbb{Q}(q,t)$ in the infinite variable set $\mathbf{x} = (x_1, x_2, \ldots)$. The *Frobenius image* of V is the symmetric function $\mathrm{Frob}(V) \in \Lambda$ given by $\mathrm{Frob}(V) := \sum_{\lambda \vdash n} c_\lambda s_\lambda$, where s_λ is the Schur function.

In this extended abstract we will consider (multi)graded S_n-modules. If $V = \bigoplus_{i \geq 0} V_i$ is a graded S_n-module with each graded piece V_i finite-dimensional, the *graded Frobenius image* of V is $\mathrm{grFrob}(V; q) := \sum_{i \geq 0} q^i \cdot \mathrm{Frob}(V_i)$. Even more generally, if $V = \bigoplus_{i,j \geq 0} V_{i,j}$ or $V = \bigoplus_{i,j,k \geq 0} V_{i,j,k}$ is a doubly or triply graded S_n-module, we have the associated bigraded and trigraded Frobenius images

$$\mathrm{grFrob}(V; q, t) := \sum_{i,j \geq 0} q^i t^j \cdot \mathrm{Frob}(V_{i,j}) \quad \text{or} \quad \mathrm{grFrob}(V; q, t, z) := \sum_{i,j,k \geq 0} q^i t^j z^k \cdot \mathrm{Frob}(V_{i,j,k}),$$

respectively.

Haiman proved [8] that $\mathrm{grFrob}(DR_n; q, t) = \nabla e_n$, where e_n is the degree n elementary symmetric function and ∇ is the Bergeron-Garsia *nabla operator*. Therefore, describing the bigraded S_n-isomorphism type of DR_n is equivalent to finding the Schur expansion of ∇e_n, but there is not even a conjecture in this direction. The monomial expansion of ∇e_n is given by the *Shuffle Theorem* [2].

The *Delta Conjecture* is a conjectural extension of the Shuffle Theorem due to Haglund, Remmel, and Wilson [6]. It depends on two positive integers $k \leq n$ and reads

$$\Delta'_{e_{k-1}} e_n = \mathrm{Rise}_{n,k}(\mathbf{x}; q, t) = \mathrm{Val}_{n,k}(\mathbf{x}; q, t). \tag{1.5}$$

Here $\Delta'_{e_{k-1}}$ is a certain symmetric function operator and Rise and Val are formal power series defined using the combinatorics of lattice paths; see [6] for details. When $k = n$, the Delta Conjecture reduces to the Shuffle Theorem.

The Delta Conjecture is open as of this writing, but combining the work of [4, 7, 11, 15] it is known at $q = 0$. More precisely, we have

$$\Delta'_{e_{k-1}} e_n \mid_{t=0} = \mathrm{Rise}_{n,k}(\mathbf{x}; q, 0) = \mathrm{Rise}_{n,k}(\mathbf{x}; 0, q) = \mathrm{Val}_{n,k}(\mathbf{x}; q, 0) = \mathrm{Val}_{n,k}(\mathbf{x}; 0, q). \tag{1.6}$$

In this paper we define a doubly graded S_n-module $\mathbb{V}_{n,k}$ for any positive integers $k \leq n$ and conjecture that $\mathrm{grFrob}(\mathbb{V}_{n,k}; q, t) = \Delta'_{e_{k-1}} e_n$ (see Conjeture 2.6). That is, we conjecture that $\mathbb{V}_{n,k}$ is a module for the Delta Conjecture. We prove this conjecture at $t = 0$. In order to describe $\mathbb{V}_{n,k}$, we introduce new combinatorial objects called *superspace Vandermondes*.

Superspace of rank n is the unital associative $\mathbb{Q}$-algebra Ω_n generated by $2n$ symbols $x_1, \ldots, x_n, \theta_1, \ldots, \theta_n$ subject to the relations

$$x_i x_j = x_j x_i \qquad x_i \theta_j = \theta_j x_i \qquad \theta_i \theta_j = -\theta_j \theta_i$$

for all $1 \leq i, j \leq n$.[1] Setting the θ-variables to zero recovers the classical polynomial ring $\mathbb{Q}[x_1, \ldots, x_n]$. By considering x-degree and θ-degree separately, Ω_n is a doubly graded algebra. The ring Ω_n carries a diagonal action of S_n given by $w.x_i := x_{w(i)}$ and $w.\theta_i := \theta_{w(i)}$ for $w \in S_n$ and $1 \leq i \leq n$.

Defintion 1.1. Let $k \leq n$ be positive integers. The *superspace Vandermonde* $\delta_{n,k}$ is the following element of Ω_n:

$$\delta_{n,k} := \varepsilon_n.(x_1^{k-1} x_2^{k-1} \cdots x_{n-k+1}^{k-1} x_{n-k+2}^{k-2} \cdots x_{n-1}^1 x_n^0 \theta_1 \theta_2 \cdots \theta_{n-k}). \tag{1.7}$$

Here $\varepsilon_n := \sum_{w \in S_n} \mathrm{sign}(w) \cdot w \in \mathbb{Q}[S_n]$ is the antisymmetrizing element in the symmetric group algebra.

For example, when $n = 3$ and $k = 2$ we have

$$\delta_{3,2} = \varepsilon_3.(x_1 x_2 \theta_1) = x_1 x_2 \theta_1 - x_1 x_2 \theta_2 - x_1 x_3 \theta_1 + x_1 x_3 \theta_3 + x_2 x_3 \theta_2 - x_2 x_3 \theta_3.$$

[1] The 'super' in superspace comes from supersymmetry in physics: the x-variables index bosons and the θ-variables index fermions. Extending coefficients to the reals, Ω_n is the ring of polynomial-valued differential forms on Euclidean n-space – this is why we write Ω.

The superpolynomial $\delta_{n,k}$ is always a nonzero element of Ω_n, thanks to the θ-variables. When $k = n$, the superspace Vandermonde $\delta_{n,k}$ reduces to the classical Vandermonde determinant $\varepsilon_n.(x_1^{n-1}x_2^{n-2}\cdots x_{n-1}^1 x_n^0)$.

The $\delta_{n,k}$ are seeds we use to grow modules. By starting with $\delta_{n,k}$ and closing under various differential operators and linearity we will construct:

- A singly graded subspace $V_{n,k}$ of Ω_n which satisfies $\mathrm{grFrob}(V_{n,k};q) = \Delta'_{e_{k-1}}e_n \mid_{t=0}$ (see **Section 2**).
- A doubly graded extension $\mathbb{V}_{n,k}$ of $V_{n,k}$ with $\mathrm{grFrob}(\mathbb{V}_{n,k};q,t)$ conjecturally given by $\Delta'_{e_{k-1}}e_n$ (see **Section 2**).
- A doubly graded S_n-stable quotient $\mathbb{W}_{n,k}$ of Ω_n which extends $V_{n,k}$ and exhibits a number of symmetries including a superspace variant of Poincaré Duality (see **Section 4**). $\mathbb{W}_{n,k}$ extends the cohomology of the space of spanning line configurations studied by Pawlowski and Rhoades [10].

This paper is not the first to propose connections between the Delta Conjecture and superspace. The Fields Institute Combinatorics Group in general, and Mike Zabrocki in particular, conjectured [16] that representation-theoretic models for the Delta Conjecture can be obtained by looking at coinvariant-type quotients defined using superspace Ω_n and an extension $\Omega_n[y_1,\ldots,y_n]$ of superspace involving n new commuting variables $y_1,\ldots,y_n$. We discuss the connection between our work and their conjectures in **Section 3**. In a nutshell, we are able to prove that our proposed Delta model $\mathbb{V}_{n,k}$ is valid at $t = 0$, but the corresponding case of their conjecture remains open.

2 The S_n-modules $V_{n,k}$ and $\mathbb{V}_{n,k}$ and the Delta Conjecture

For $1 \leq i \leq n$, the partial derivative operator $\partial/\partial x_i$ acts naturally on the polynomial ring $\mathbb{Q}[x_1,\ldots,x_n]$. Superspace admits the tensor product decomposition

$$\Omega_n = \mathbb{Q}[x_1,\ldots,x_n] \otimes \wedge\{\theta_1,\ldots,\theta_n\} \tag{2.1}$$

where $\wedge\{\theta_1,\ldots,\theta_n\}$ is the exterior algebra on the generators $\theta_1,\ldots,\theta_n$. The operator $\partial/\partial x_i$ therefore acts on Ω_n by acting on the first tensor factor.

Our first new S_n-module is defined as follows. Starting with the superspace Vandermonde $\delta_{n,k}$, we close under the operators $\partial/\partial x_1,\ldots,\partial/\partial x_n$ and linearity.

Defintion 2.1. Let $k \leq n$ be positive integers. The vector space $V_{n,k}$ is the smallest $\mathbb{Q}$-linear subspace of Ω_n which

- contains the superspace Vandermonde $\delta_{n,k}$, and

- is closed under the n partial derivatives $\partial/\partial x_1, \ldots, \partial/\partial x_n$.

The subspace $V_{n,k} \subseteq \Omega_n$ is closed under the action of S_n. Furthermore, $V_{n,k}$ a doubly graded subspace of Ω_n. If we ignore the θ-grading (which is constant of degree $n-k$) and focus on the x-grading, we see that $V_{n,k}$ is a singly-graded S_n-module.

To describe the Schur expansion of $\mathrm{grFrob}(V_{n,k}; q)$, we need some notation. Let T be a standard Young tableau with n boxes. A number $1 \leq i \leq n-1$ is a *descent* of T if i appears in a row above $i+1$. The *descent number* $\mathrm{des}(T)$ is the number of descents and the *major index* $\mathrm{maj}(T)$ is the sum of the descents in T. We write $\mathrm{shape}(T) \vdash n$ for the partition of n obtained by erasing the numbers in T. We also use the standard q-numbers, q-factorials, and q-binomials:

$$[n]_q := 1 + q + \cdots + q^{n-1} \quad [n]!_q := [n]_q [n-1]_q \cdots [1]_q \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q [n-k]!_q}. \tag{2.2}$$

The following theorem, and other results in this extended abstract, are proven in [12].

Theorem 2.2. *Let $k \leq n$ be positive integers. The graded Frobenius image of $V_{n,k}$ is given by either of the expressions*

$$\mathrm{grFrob}(V_{n,k}; q) = \sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}(T) + \binom{n-k}{2} - (n-k)\cdot \mathrm{des}(T)} \begin{bmatrix} \mathrm{des}(T) \\ n-k \end{bmatrix}_q s_{\mathrm{shape}(T)} \tag{2.3}$$

$$= \Delta'_{e_{k-1}} e_n \mid_{t=0} \tag{2.4}$$

where the sum is over all standard Young tableaux T with n boxes.

Equation (1.6) allows us to replace the $\Delta'_{e_{k-1}} e_n \mid_{t=0}$ in Theorem 2.2 with any of the symmetric functions $\mathrm{Rise}_{n,k}(\mathbf{x}; q, 0), \mathrm{Rise}_{n,k}(\mathbf{x}; 0, q), \mathrm{Val}_{n,k}(\mathbf{x}; q, 0)$, or $\mathrm{Val}_{n,k}(\mathbf{x}; 0, q)$. Thanks to Theorem 2.2, it is easy to describe the ungraded S_n-isomorphism type of $V_{n,k}$.

Corollary 2.3. *Let $k \leq n$ be positive integers and consider the permutation action of S_n on the family $\mathcal{OP}_{n,k}$ of k-block ordered set partitions $(B_1, B_2, \ldots, B_k)$ of $\{1, 2, \ldots, n\}$. As ungraded S_n-modules we have*

$$V_{n,k} \cong \mathbb{Q}[\mathcal{OP}_{n,k}] \otimes \mathrm{sign} \tag{2.5}$$

where sign *is the 1-dimensional sign representation of S_n.*

The *(signless) Stirling number of the second kind* $\mathrm{Stir}(n,k)$ counts (unordered) k-block set partitions of $\{1, 2, \ldots, n\}$. Corollary 2.3 implies $\dim V_{n,k} = k! \cdot \mathrm{Stir}(n,k)$. The graded dimension of $V_{n,k}$ is given by a suitable q-analog of this formula.

Recall that the *Hilbert series* of a graded vector space $V = \bigoplus_{i \geq 0} V_i$ is the formal power series $\mathrm{Hilb}(V; q) := \sum_{i \geq 0} q^i \cdot \dim V_i$. The *$q$-Stirling number* $\mathrm{Stir}_q(n,k)$ is defined by the recursion

$$\mathrm{Stir}_q(n,k) = \mathrm{Stir}_q(n-1, k-1) + [k]_q \cdot \mathrm{Stir}_q(n-1, k) \tag{2.6}$$

together with the initial conditions $\mathrm{Stir}_q(0,0) = 1$ and $\mathrm{Stir}_q(0,k) = 0$ for any $k > 0$.

Corollary 2.4. *The Hilbert series of* $V_{n,k}$ *is* $\mathrm{Hilb}(V_{n,k};q) = [k]!_q \cdot \mathrm{Stir}_q(n,k)$.

In order to describe our proposed model for the Delta Conjecture, we need more variables. Let $y_1, \ldots, y_n$ be n new commuting variables and consider the extension $\Omega_n[y_1, \ldots, y_n]$ of superspace defined formally by the tensor product

$$\Omega_n[y_1, \ldots, y_n] := \mathbb{Q}[x_1, \ldots, x_n] \otimes \mathbb{Q}[y_1, \ldots, y_n] \otimes \wedge\{\theta_1, \ldots, \theta_n\}. \tag{2.7}$$

This is a *triply* graded S_n-module with action $w.x_i := x_{w(i)}, w.y_i := y_{w(i)}, w.\theta_i := \theta_{w(i)}$. This ring admits an action of partial derivatives $\partial/\partial x_i$ and $\partial/\partial y_i$ in both the x-variables and y-variables.

Defintion 2.5. For $k \leq n$, let $\mathbb{V}_{n,k}$ be the smallest $\mathbb{Q}$-linear subspace of $\Omega_n[y_1, \ldots, y_n]$ which

- contains the superspace Vandermonde $\delta_{n,k}$ (in the x-variables and θ-variables alone),
- is closed under the *polarization operator* $\sum_{s=1}^{n} y_s(\partial/\partial x_s)^j$ for each $j \geq 1$, and
- is closed under the $2n$ partial derivatives $\partial/\partial x_1, \ldots, \partial/\partial x_n, \partial/\partial y_1, \ldots, \partial/\partial y_n$.

The S_n-module $\mathbb{V}_{n,k}$ is concentrated in θ-degree $n-k$. By considering x-degree and y-degree, the space $\mathbb{V}_{n,k}$ attains the structure of a doubly graded S_n-module.

Conjecture 2.6. *Let* $k \leq n$ *be positive integers. The doubly graded Frobenius image of* $\mathbb{V}_{n,k}$ *is given by*

$$\mathrm{grFrob}(\mathbb{V}_{n,k};q,t) = \Delta'_{e_{k-1}} e_n. \tag{2.8}$$

Conjeture 2.6 is true at $t = 0$ by Theorem 2.2. Conjeture 2.6 is true when $k = n$ by the work of Haiman [8]. Conjeture 2.6 has been checked on computer for $n \leq 4$ (and at $n = 5$ on the level of bigraded Hilbert series). Since every increase $n \to n+1$ adds two new commuting variables and one new anticommuting variable, studying Conjeture 2.6 involves considerable computational challenges as n grows.

3 The Fields and Zabrocki Conjectures

In this section we describe alternative conjectural representation-theoretic models for the Delta Conjecture arising from quotients of Ω_n and $\Omega_n[y_1, \ldots, y_n]$. Recall that the symmetric group S_n acts diagonally on superspace Ω_n. Solomon proved [13] that the ring $(\Omega_n)^{S_n} \subseteq \Omega_n$ of S_n-invariants is a free $\mathbb{Q}[x_1, \ldots, x_n]^{S_n}$-module on the generating set $\{de_{i_1} \cdots de_{i_r} : 1 \leq i_1 < \cdots < i_r \leq n\}$ where $d := \sum_{j=1}^{n} \theta_j \cdot (\partial/\partial x_j)$ is the total derivative operator.

Let $\langle (\Omega_n)_+^{S_n} \rangle \subseteq \Omega_n$ be the two-sided ideal of Ω_n generated by S_n-invariants with vanishing constant term. By considering x-degree and θ-degree, the quotient $\Omega_n / \langle (\Omega_n)_+^{S_n} \rangle$ is a doubly graded S_n-module. We view this quotient as a 'superspace coinvariant ring'. The following conjecture about its doubly graded Frobenius image was made by the Combinatorics Group at the Fields Institute.

Fields Conjecture. *(see [16]) Let n be a positive integer. The doubly graded Frobenius image of $\Omega_n / \langle (\Omega_n)_+^{S_n} \rangle$ is given by*

$$\mathrm{grFrob}(\Omega_n / \langle (\Omega_n)_+^{S_n} \rangle; q, z) = \sum_{k=1}^{n} z^{n-k} \cdot \Delta'_{e_{k-1}} e_n \mid_{t=0}, \tag{3.1}$$

where q tracks x-degree and z tracks θ-degree.

If the Fields Conjecture is true, the bigraded Hilbert series of $\Omega_n / \langle (\Omega_n)_+^{S_n} \rangle$ would be given by

$$\mathrm{Hilb}(\Omega_n / \langle (\Omega_n)_+^{S_n} \rangle; q, z) = \sum_{k=1}^{n} z^{n-k} \cdot [k]!_q \cdot \mathrm{Stir}_q(n, k) \tag{3.2}$$

where q tracks x-degree and z tracks θ-degree. The Fields Combinatorics Group proved (personal communication) the inequality

$$\mathrm{Hilb}(\Omega_n / \langle (\Omega_n)_+^{S_n} \rangle; q, z) \geq \sum_{k=1}^{n} z^{n-k} \cdot [k]!_q \cdot \mathrm{Stir}_q(n, k) \tag{3.3}$$

where $f(q,z) \geq g(q,z)$ means that the difference $f(q,z) - g(q,z)$ is a polynomial in q, z with nonnegative coefficients.

Recall that the *alternating subspace* of an S_n-module V is given by

$$\{v \in V \,:\, w.v = \mathrm{sign}(w) \cdot v \text{ for all } w \in S_n\}.$$

Let A_n be the alternating subspace of $\Omega_n / \langle (\Omega_n)_+^{S_n} \rangle$. The alternant space A_n is a doubly graded vector space. The Fields Conjecture would imply that

$$\mathrm{Hilb}(A_n; q, z) = \sum_{k=1}^{n} z^{n-k} \cdot q^{\binom{k}{2}} \cdot \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q. \tag{3.4}$$

Equation (3.4) has been verified by Swanson and Wallach [14], giving further evidence for the Fields Conjecture.

If the Fields Conjecture is true, we would have an isomorphism of ungraded S_n-modules $\Omega_n / \langle (\Omega_n)_+^{S_n} \rangle \cong \bigoplus_{k=1}^{n} (\mathbb{Q}[\mathcal{OP}_{n,k}] \otimes \mathrm{sign})$. At present, it is unknown whether either of these S_n-modules injects into the other.

The symmetric functions appearing in the Fields Conjecture and Theorem 2.2 are closely related. We propose the following 'bridge conjecture' whose truth would yield the Fields Conjecture. Let φ be the composite linear map

$$\varphi : V_{n,1} \oplus \cdots \oplus V_{n,n} \hookrightarrow \Omega_n \twoheadrightarrow \Omega_n / \langle (\Omega_n)_+^{S_n} \rangle \tag{3.5}$$

obtained by including the direct sum $V_{n,1} \oplus \cdots \oplus V_{n,n}$ into superspace and then projecting onto the superspace coinvariant ring.

Conjecture 3.1. *The linear map φ is bijective.*

Mike Zabrocki studied the triply diagonal action of S_n on the ring $\Omega_n[y_1, \ldots, y_n]$ and the associated space $\Omega_n[y_1, \ldots, y_n]_+^{S_n}$ of S_n-invariants with vanishing constant term. He checked the following conjecture by computer for $n \leq 6$.

Zabrocki Conjecture. *([16]) Let n be a positive integer. We have*

$$\mathrm{grFrob}(\Omega_n[y_1, \ldots, y_n] / \langle \Omega_n[y_1, \ldots, y_n]_+^{S_n} \rangle; q, t, z) = \sum_{k=1}^{n} z^{n-k} \cdot \Delta'_{e_{k-1}} e_n \tag{3.6}$$

where q tracks x-degree, t tracks y-degree, and z tracks θ-degree.

The Zabrocki Conjecture is related to Conjeture 2.6 in the same way as the Fields Conjecture is related to Theorem 2.2. Since Theorem 2.2 is proven whereas the Fields Conjecture remains open, superspace Vandermondes might prove an easier road to Delta Conjecture modules than quotient rings.

4 The ring $\mathbb{W}_{n,k}$ and Super Poincaré Duality

So far we have built S_n-modules $V_{n,k}$ and $\mathbb{V}_{n,k}$ by starting with the superspace Vandermonde $\delta_{n,k}$ and closing under partial derivatives in the commuting variables x_i, y_i (and potentially polarization operators). The modules $V_{n,k}$ and $\mathbb{V}_{n,k}$ have the defect of not being closed under multiplication and not admitting a natural ring structure. In this section we build a new bigraded S_n-module $\mathbb{W}_{n,k}$ from $\delta_{n,k}$. The module $\mathbb{W}_{n,k}$ is naturally a bigraded quotient of Ω_n. The module $\mathbb{W}_{n,k}$ turns out to extend both $V_{n,k}$ and the cohomology ring $H^\bullet(X_{n,k}; \mathbb{Q})$ of a variety $X_{n,k}$ of line configurations studied by Pawlowski and Rhoades. In order to define $\mathbb{W}_{n,k}$, we need operators $\partial/\partial\theta_i$ on Ω_n which differentiate with respect to anticommuting variables.

For $1 \leq i \leq n$, let $\partial/\partial\theta_i : \Omega_n \to \Omega_n$ be the $\mathbb{Q}[x_1, \ldots, x_n]$-module endomorphism characterized by

$$\partial/\partial\theta_i : \theta_{j_1} \cdots \theta_{j_r} \mapsto \begin{cases} (-1)^{s-1} \theta_{j_1} \cdots \widehat{\theta_{j_s}} \cdots \theta_{j_r} & \text{if } j_s = i \\ 0 & \text{if } i \neq j_1, \ldots, j_r \end{cases} \tag{4.1}$$

where $1 \leq j_1, \ldots, j_r \leq n$ are distinct indices and $\widehat{\cdot}$ means omission. The sign $(-1)^{s-1}$ is necessary to ensure that $\partial/\partial\theta_i$ is well-defined.

Defintion 4.1. For positive integers $k \leq n$, let $\mathbb{W}_{n,k}$ be the smallest linear subspace of Ω_n which

- contains the superspace Vandermonde $\delta_{n,k}$, and
- is closed under the $2n$ operators $\partial/\partial x_1, \ldots, \partial/\partial x_n, \partial/\partial\theta_1, \ldots, \partial/\partial\theta_n$.

The vector space $\mathbb{W}_{n,k}$ is a bigraded S_n-module. We use an action of superspace on itself to show that $\mathbb{W}_{n,k}$ is naturally a bigraded quotient of Ω_n.

The operators $\partial/\partial\theta_i$ and $\partial/\partial x_i$ on Ω_n satisfy the relations

$$(\partial/\partial x_i)(\partial/\partial x_j) = (\partial/\partial x_j)(\partial/\partial x_i) \quad (\partial/\partial x_i)(\partial/\partial\theta_j) = (\partial/\partial\theta_j)(\partial/\partial x_i) \quad (\partial/\partial\theta_i)(\partial/\partial\theta_j) = -(\partial/\partial\theta_j)(\partial/\partial\theta_i)$$

for all $1 \leq i, j \leq n$. These are the defining relations of Ω_n, so for any superpolynomial $f = f(x_1, \ldots, x_n, \theta_1, \ldots, \theta_n)$ we have an unambiguous operator ∂f on Ω_n obtained by replacing each x_i in f with $\partial/\partial x_i$ and each θ_i in f by $\partial/\partial\theta_i$. This gives rise to an action $\odot : \Omega_n \times \Omega_n \to \Omega_n$ of superspace on itself by the rule

$$f \odot g := \partial f(g). \tag{4.2}$$

Proposition 4.2. *Let* $\mathrm{ann}(\delta_{n,k}) := \{f \in \Omega_n \,:\, f \odot \delta_{n,k} = 0\}$ *be the annihilator in* Ω_n *of the superspace Vandermonde* $\delta_{n,k}$*. Then* $\mathrm{ann}(\delta_{n,k})$ *is a two-sided ideal in* Ω_n *which is* S_n*-stable and bigraded. The canonical composition*

$$\mathbb{W}_{n,k} \hookrightarrow \Omega_n \twoheadrightarrow \Omega_n/\mathrm{ann}(\delta_{n,k}) \tag{4.3}$$

is an isomorphism of bigraded S_n*-modules.*

Thanks to Proposition 4.2, there is a natural multiplication operation on $\mathbb{W}_{n,k}$, so that the anticommuting differentiation operators $\partial/\partial\theta_i$ give rise to a ring structure which $V_{n,k}$ and $\mathbb{V}_{n,k}$ lack.

What do the bigraded S_n-modules $\mathbb{W}_{n,k}$ look like? We display $\mathrm{grFrob}(\mathbb{W}_{4,2}; q, z)$ in matrix format, with rows labeling θ-degree and columns labeling x-degree.

$$\mathrm{grFrob}(\mathbb{W}_{4,2}; q, z) = \begin{pmatrix} s_4 & s_4 + s_{31} & s_4 + s_{31} + s_{22} & s_{31} \\ s_{31} & 2s_{31} + s_{22} + s_{211} & s_{31} + s_{22} + 2s_{211} & s_{211} \\ s_{211} & s_{22} + s_{211} + s_{1111} & s_{211} + s_{1111} & s_{1111} \end{pmatrix} \tag{4.4}$$

The matrices $\mathrm{grFrob}(\mathbb{W}_{n,k}; q, z)$ enjoy the following properties. Let $U_n = S^{(n-1,1)}$ be the $(n-1)$-dimensional reflection representation of S_n.

Theorem 4.3. *There hold the following facts about the bigraded* S_n*-module* $\mathbb{W}_{n,k}$*.*

1. (Special k) *We have* $\mathbb{W}_{n,n} \cong R_n$ *(coinvariant ring) and* $\mathbb{W}_{n,1} \cong \wedge U_n$ *(exterior algebra).*
2. (Bottom x-degree) *The x-degree* 0 *piece of* $\mathbb{W}_{n,k}$ *is isomorphic to* $\bigoplus_{j=0}^{n-k} \wedge^j U_n$.
3. (Top x-degree) *The top x-degree of* $\mathbb{W}_{n,k}$ *is* $\binom{k}{2} + (n-k) \cdot (k-1)$*; this piece of* $\mathbb{W}_{n,k}$ *is isomorphic to* $\bigoplus_{j=0}^{n-k} \wedge^j U_n \otimes \mathrm{sign}$.
4. (Top θ-degree) *The top* $(= n-k)$ *θ-degree piece of* $\mathbb{W}_{n,k}$ *is isomorphic to* $V_{n,k}$.
5. (Bottom θ-degree) *Let* $I_{n,k} \subseteq \mathbb{Q}[x_1, \ldots, x_n]$ *be* $I_{n,k} := \langle x_1^k, \ldots, x_n^k, e_n, e_{n-1}, \ldots, e_{n-k+1} \rangle$ *and let* $R_{n,k} := \mathbb{Q}[x_1, \ldots, x_n]/I_{n,k}$*. The θ-degree* 0 *piece of* $\mathbb{W}_{n,k}$ *is isomorphic to* $R_{n,k}$.

The quotient rings $R_{n,k}$ in Item 5 of Theorem 4.3 were introduced by Haglund, Rhoades, and Shimozono [7]. They proved that

$$\mathrm{grFrob}(R_{n,k}; q) = (\mathrm{rev}_q \circ \omega) \Delta'_{e_{k-1}} e_n \mid_{t=0}, \tag{4.5}$$

where ω is the symmetric function involution which trades s_λ and $s_{\lambda'}$ and rev_q reverses the coefficient sequences of polynomials in q. The ring $R_{n,k}$ was the first model for a coinvariant ring attached to the Delta Conjecture.

The rings $R_{n,k}$ have a geometric interpretation. A *line* in the k-dimensional complex vector space $\mathbb{C}^k$ is a 1-dimensional linear subspace. Pawlowski and Rhoades defined [10] the variety $X_{n,k}$ of spanning configurations of n lines in $\mathbb{C}^k$:

$$X_{n,k} := \{(\ell_1, \ldots, \ell_n) \,:\, \ell_i \subseteq \mathbb{C}^k \text{ a line and } \ell_1 + \cdots + \ell_n = \mathbb{C}^k\}. \tag{4.6}$$

The space $X_{n,k}$ and its cohomology ring $H^\bullet(X_{n,k}; \mathbb{Q})$ admit S_n-actions by line permutation. Pawlowski and Rhoades presented [10] the cohomology $H^\bullet(X_{n,k}; \mathbb{Q})$ as

$$H^\bullet(X_{n,k}; \mathbb{Q}) = \mathbb{Q}[x_1, \ldots, x_n]/I_{n,k} = R_{n,k}. \tag{4.7}$$

We may therefore interpret the θ-degree 0 piece of $\mathbb{W}_{n,k}$ as the cohomology of $X_{n,k}$.

The 'twist' $(\mathrm{rev}_q \circ \omega)$ involved in Equation (4.5) can be visualized in the matrix representing $\mathrm{grFrob}(\mathbb{W}_{4,2}; q, z)$ in (4.4). Namely, the top row can be obtained from the bottom row by reversal together with applying the operator ω. The reader may notice that the middle row of $\mathrm{grFrob}(\mathbb{W}_{4,2}; q, z)$ is invariant under reversal followed by ω. This observation generalizes as follows.

Theorem 4.4. *The matrix representing* $\mathrm{grFrob}(\mathbb{W}_{n,k}; q, z)$ *is invariant under* 180° *rotation followed by the application of ω to each entry.*

Recall that a sequence of numbers $(a_0, a_1, \ldots, a_d)$ is *palindromic* if $a_i = a_{d-i}$ for all i and *unimodal* if $a_0 \leq a_1 \leq \cdots \leq a_r \geq a_{r+1} \geq \cdots \geq a_d$ for some r. A famous example of a polynomial in $\mathbb{Q}[q]$ with a palindromic and unimodal coefficient sequence is the

q-factorial $[n]!_q = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$. While these facts about $[n]!_q$ follow from showing that if $f(q)$ and $g(q)$ have palindromic unimodal coefficient sequences, so does $f(q)\cdot g(q)$, there is a deeper derivation coming from geometry.

A finite-dimensional graded $\mathbb{Q}$-algebra $A = \bigoplus_{i=0}^d A_i$ is a *Poincaré Duality Algebra* if $A_d \cong \mathbb{Q}$ is 1-dimensional and if for all $0 \leq i \leq d$ the map $A_i \otimes A_{d-i} \to A_d \cong \mathbb{Q}$ is a perfect pairing. This forces $\dim A_i = \dim A_{d-i}$.

Let $\mathcal{F}\ell_n$ be the variety of complete flags in $\mathbb{C}^n$. Borel proved [1] that the cohomology of $\mathcal{F}\ell_n$ has presentation $H^\bullet(\mathcal{F}\ell_n;\mathbb{Q}) = R_n$ given by the coinvariant ring. Since $\mathcal{F}\ell_n$ is a compact complex manifold, the ring $H^\bullet(\mathcal{F}\ell_n;\mathbb{Q})$ is a Poincaré Duality Algebra and the palindromicity of its Hilbert polynomial $[n]!_q$ follows.

The complex variety $X_{n,k}$ is smooth, but usually not compact. Indeed, the cohomology ring $H^\bullet(X_{n,k};\mathbb{Q}) = R_{n,k}$ does not usually have a palindromic Hilbert series, e.g. $\mathrm{Hilb}(R_{3,2};q) = 1+3q+2q^2$. However, the extension $\mathbb{W}_{n,k} \supseteq R_{n,k}$ exhibits a superspace version of Poincaré Duality.

Let $A = \bigoplus_{i=0}^d \bigoplus_{j=0}^e A_{i,j}$ be a finite-dimensional bigraded $\mathbb{Q}$-algebra. We say that A is a *Super Poincaré Duality Algebra* if $A_{d,e} \cong \mathbb{Q}$ and $A_{i,j} \otimes A_{d-i,e-j} \to A_{d,e}$ is a perfect pairing for all $0 \leq i \leq d$ and $0 \leq j \leq e$.

Theorem 4.5. *The bigraded algebra $\mathbb{W}_{n,k}$ is a Super Poincaré Duality Algebra.*

Does Theorem 4.5 have geometric meaning? Is there a 'superspace version' of cohomology which yields $\mathbb{W}_{n,k}$ when applied to $X_{n,k}$?

The unimodality of $[n]!_q$ also has geometric meaning. A Poincaré Duality Algebra $A = \bigoplus_{i=0}^d A_i$ satisfies the *Hard Lefschetz Property* if there exists an element $\ell \in A_1$ (called a *Lefschetz element*) such that for any $i \leq d/2$ the map $A_i \xrightarrow{\times \ell^{d-2i}} A_{d-i}$ of multiplication by ℓ^{d-2i} is bijective.

Since $\mathcal{F}\ell_n$ is a compact complex manifold and $H^\bullet(\mathcal{F}\ell_n;\mathbb{Q}) = R_n$, the ring R_n satisfies the Hard Lefschetz Property. Maneo, Numata, and Wachi proved [9] that a linear form $\ell = c_1x_1 + \cdots + c_nx_n$ is a Lefschetz element if and only if $c_1,\ldots,c_n \in \mathbb{Q}$ are distinct.

As a closing example, we display the bigraded Hilbert series $\mathrm{Hilb}(\mathbb{W}_{4,2};q,z)$ as a matrix where rows index θ-degree and columns index x-degree.

$$\mathrm{Hilb}(\mathbb{W}_{4,2};q,z) = \begin{pmatrix} 1 & 4 & 6 & 3 \\ 3 & 11 & 11 & 3 \\ 3 & 6 & 4 & 1 \end{pmatrix} \tag{4.8}$$

Either Theorem 4.4 or Theorem 4.5 imply that the matrix $\mathrm{Hilb}(\mathbb{W}_{n,k};q,z)$ is always invariant under 180° rotation.

Conjecture 4.6. *Each row and column in the matrix representing* $\mathrm{Hilb}(\mathbb{W}_{n,k};q,z)$ *is unimodal.*

Conjeture 4.6 would be best proven by showing that $\mathbb{W}_{n,k}$ satisfies an as-yet-undefined 'Super Hard Lefschetz Property'.

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