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Spanning configurations and matroidal representation stability

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Abstract. Let $V_1, V_2, \ldots$ be a sequence of vector spaces where V_n carries an action of $\mathfrak{S}_n$ for each n. *Representation stability* describes when the sequence V_n has a limit. An important source of stability arises when V_n is the d^{th} homology group (for fixed d) of the configuration space of n distinct points in some topological space X. We replace these configuration spaces with the variety $X_{n,k}$ of *spanning configurations* of n-tuples $(\ell_1, \ldots, \ell_n)$ of lines in $\mathbb{C}^k$ with $\ell_1 + \cdots + \ell_n = \mathbb{C}^k$ as vector spaces. That is, we replace the configuration space condition of *distinctness* with the matroidal condition of *spanning*. We study stability phenomena for the homology groups $H_d(X_{n,k})$ as the parameter (n, k) grows. We also study stability phenomena for a family of multigraded modules related to the Delta Conjecture.

Keywords: symmetric group module, representation stability, subspace configuration

1 Introduction and main result

Suppose that for each $n \geq 1$, we have a representation V_n of the symmetric group $\mathfrak{S}_n$. [1] What does it mean for the sequence $V_1, V_2, V_3, \ldots$ to converge? Representation stability is an answer to this question. We regard $\mathfrak{S}_n$ as the subgroup of permutations in $\mathfrak{S}_{n+1}$ which fix $n+1$, so any $\mathfrak{S}_{n+1}$-module is also an $\mathfrak{S}_n$-module.

Defintion 1.1. Let $(V_n)_{n \geq 1}$ be a sequence of $\mathfrak{S}_n$-modules and let $f_n : V_n \to V_{n+1}$ be a sequence of linear maps. V_n is *(uniformly) representation stable* with respect to f_n if for $n \gg 0$

- the map f_n is injective,
- we have $f_n(w \cdot v) = w \cdot f_n(v)$ for all $w \in \mathfrak{S}_n$ and all $v \in V_n$,

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[1] We only consider finite-dimensional representations over $\mathbb{Q}$.

- the $\mathfrak{S}_{n+1}$-module generated by the image $f_n(V_n) \subseteq V_{n+1}$ is all of V_{n+1}, and
- the transposition $(n+1, n+2) \in \mathfrak{S}_{n+2}$ acts trivially on the image of the composition $(f_{n+1} \circ f_n)(V_n) \subseteq V_{n+2}$.

Let $n \geq 0$. A *partition* λ of n is a weakly decreasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ of positive integers with $\lambda_1 + \lambda_2 + \cdots = n$. We use $\lambda \vdash n$ to mean that λ is a partition of n and write $|\lambda| = n$ for the sum of the parts of λ. There is a one-to-one correspondence between partitions $\lambda \vdash n$ and irreducible representations of $\mathfrak{S}_n$; given $\lambda \vdash n$, let S^λ be the corresponding $\mathfrak{S}_n$-irreducible.

If μ is any partition and $n \geq |\mu| + \mu_1$, the *padded partition* $\mu[n] \vdash n$ is given by $\mu[n] := (n - |\mu|, \mu_1, \mu_2, \ldots)$. Any partition $\lambda \vdash n$ may be written uniquely as $\lambda = \mu[n]$ for some partition μ: if $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ we have $\mu = (\lambda_2, \lambda_3, \ldots)$.

Let $(V_n)_{\geq 1}$ be a sequence of $\mathfrak{S}_n$-modules. Decomposing V_n into irreducibles yields multiplicities $m_{\mu,n} \geq 0$ such that $V_n \cong \bigoplus_\mu m_{\mu,n} S^{\mu[n]}$, where the direct sum is over all partitions μ. Defintion 1.1 has the following combinatorial interpretation.

Theorem 1.2. *(Church-Ellenberg-Farb [2]) Let $(V_n)_{n\geq 1}$ be a sequence of $\mathfrak{S}_n$-modules and define the multiplicities $m_{\mu,n}$ as above. The following are equivalent.*

1. *The sequence $(V_n)_{n\geq 1}$ is representation stable with respect to some maps $f_n : V_n \to V_{n+1}$.*
2. *There exists N such that for any partition μ we have $m_{\mu,n} = m_{\mu,N}$ for all $n \geq N$.*

A famous geometric instance of representation stability comes from configuration spaces. Let X be a topological space and $n \geq 0$. The *configuration space* $\mathrm{Conf}_n X$ is the set of all n-tuples $(x_1, \ldots, x_n)$ of distinct points in X. The set $\mathrm{Conf}_n X$ is topologized via its inclusion into the n-fold product $X \times \cdots \times X$. A point in $\mathrm{Conf}_3 X$ where X is the torus is shown on the left of Figure 1.

Let $H_\bullet(\mathrm{Conf}_n X))$ be the homology of $\mathrm{Conf}_n X$ (singular with rational coefficients). For any $d \geq 0$, the symmetric group $\mathfrak{S}_n$ acts continuously on $\mathrm{Conf}_n X$ by point permutation and so endows the vector space $H_d(\mathrm{Conf}_n X)$ with the structure of an $\mathfrak{S}_n$-module. Many theorems in representation stability state that if X is a 'nice' space and $d > 0$, the sequence $(H_d(\mathrm{Conf}_n X))_{n\geq 1}$ is representation stable (for example, see [1]).

In this paper we prove a new geometric family of **matroidal representation stability** results where the configuration space condition of **distinctness** is replaced by the matroidal condition of **spanning**. The key example is as follows. Given positive integers $k \leq n$, Pawlowski and Rhoades [7] introduced the following space of spanning line configurations:

$$X_{n,k} := \{(\ell_1, \ldots, \ell_n) : \ell_i \subseteq \mathbb{C}^k \text{ a 1-dimensional subspace and } \ell_1 + \cdots + \ell_n = \mathbb{C}^k\}. \quad (1.1)$$

A point in $X_{5,3}$ is shown in the middle of Figure 1. When $k = n$, the space $X_{n,k}$ is homotopy equivalent to the variety $\mathcal{F}\ell_n$ of complete flags in $\mathbb{C}^n$.

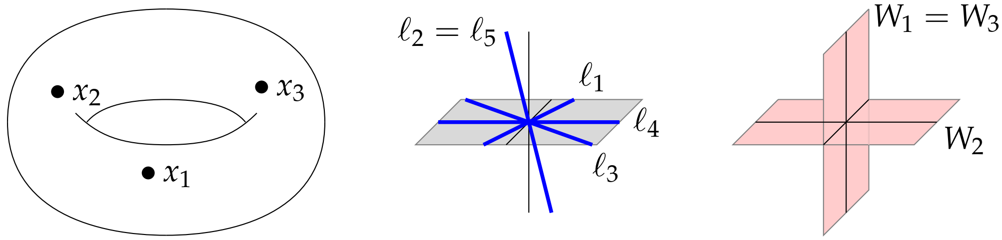

Figure 1: A point configuration, a line configuration, and a 2-plane configuration.

The group $\mathfrak{S}_n$ acts on $X_{n,k}$ by $w.(\ell_1, \ldots, \ell_n) := (\ell_{w(1)}, \ldots, \ell_{w(n)})$ for all $w \in \mathfrak{S}_n$ and $(\ell_1, \ldots, \ell_n) \in X_{n,k}$. This induces an action of $\mathfrak{S}_n$ on the homology group $H_d(X_{n,k})$ for each $d \geq 0$. There are two natural ways to grow a pair (n,k) subject to the condition $k \leq n$:

$$(n,k) \rightsquigarrow (n+1,k) \qquad \text{and} \qquad (n,k) \rightsquigarrow (n+1,k+1).$$

Both of these growth rules leads to a stability result. The following matroidal stability theorem will be proved in **Section 3**.

Theorem 1.3. *Fix $d \geq 0$. The following sequences of modules are representation stable with respect to some linear maps f_n:*

1. $(H_d(X_{n,k}))_{n\geq 1}$ *for $k \geq 0$ fixed, and*
2. $(H_d(X_{n,n-m}))_{n\geq 1}$ *for $m \geq 0$ fixed.*

Here we adopt the convention $X_{n,k} = \varnothing$ for $n < k$ or $k < 0$ so that $H_d(X_{n,k}) = 0$ in this case.

Pawlowski and Rhoades [7] presented the rational cohomology of $X_{n,k}$ as

$$H^\bullet(X_{n,k}) = \mathbb{Q}[x_1, \ldots, x_n] / \langle x_1^k, \ldots, x_n^k, e_n, e_{n-1}, \ldots, e_{n-k+1} \rangle. \tag{1.2}$$

Here e_d is the degree d elementary symmetric polynomial and x_i represents the Chern class $c_1(\ell_i^*) \in H^2(X_{n,k})$ of the line bundle $\ell_i^* \twoheadrightarrow X_{n,k}$. By the Universal Coefficient Theorem, this also gives the homology of $X_{n,k}$. The graded $\mathfrak{S}_n$-isomorphism type of the quotient appearing in (1.2) was calculated by Haglund, Rhoades, and Shimozono [5] in terms of statistics on standard Young tableaux.

The presentation (1.2) of $H^\bullet(X_{n,k})$ in [7] and the calculation of the graded $\mathfrak{S}_n$-module structure of $H^\bullet(X_{n,k})$ in [5] involved a substantial amount of combinatorics, algebra, and geometry. One might think that the proof of the stability result of Theorem 1.3 would rely on these or other similarly difficult arguments, but representation stability exhibits the following leitmotif.

Leitmotif. *It is often easier to show that a sequence $(V_n)_{n\geq 1}$ of $\mathfrak{S}_n$-modules is representation stable then it is to calculate the $\mathfrak{S}_n$-isomorphism types of the V_n.*

Indeed, in **Section 3** we prove Theorem 1.3 using only a geometric property of $X_{n,k}$ coming from linear algebra (a realization as a terminal part of a nonstandard affine paving of the n-fold projective space product $\mathbb{P}^{k-1} \times \cdots \times \mathbb{P}^{k-1}$ discovered in [7]) and not relying on any explicit presentation of the (co)homology of $X_{n,k}$.

We will also illustrate our leitmotif for modules V_n whose isomorphism types are unknown.

- In Section 4 we generalize Theorem 1.3 to spanning configurations of higher-dimensional subspaces; see the right of Figure 1 for a spanning configuration of three 2-places in $\mathbb{C}^3$. The cohomology rings of these moduli spaces were presented by Rhoades [8], but their graded $\mathfrak{S}_n$-module decomposition is unknown. Our proof relies only on a nonstandard affine paving of a product of Grassmannians.

- The space $X_{n,k}$ was introduced in [7] to give geometric context to the Haglund-Remmel–Wilson *Delta Conjecture* [4] in symmetric function theory. Zabrocki [11] and Rhoades-Wilson [9] defined multigraded $\mathfrak{S}_n$-modules and conjectured that their isomorphism types are given by the Delta Conjecture. In Section 5 we give stability results for a family of multigraded $\mathfrak{S}_n$-modules including those studied in [11, 9]. There is not even a conjecture for the multigraded $\mathfrak{S}_n$-isomorphism types of these modules.

2 Background

Let Λ be the ring of symmetric functions in the infinite variable set $\mathbf{x} = (x_1, x_2, \ldots)$ over the ground field $\mathbb{Q}(q,t)$. If V is any $\mathfrak{S}_n$-module, there are unique multiplicities c_λ so that $V \cong \bigoplus_{\lambda \vdash n} c_\lambda S^\lambda$. The *Frobenius image* $\mathrm{Frob}(V) \in \Lambda$ is the symmetric function $\mathrm{Frob}(V) := \sum_{\lambda \vdash n} c_\lambda s_\lambda$, where s_λ is the Schur function.

We will consider (multi)graded $\mathfrak{S}_n$-modules. Suppose $V = \bigoplus_{i \geq 0} V_i$ is a graded $\mathfrak{S}_n$-module. The *graded Frobenius image* of V is $\mathrm{grFrob}(V;q) := \sum_{i \geq 0} q^i \cdot \mathrm{Frob}(V_i)$. More generally, if $V = \bigoplus_{i,j} V_{i,j}$ (or $V = \bigoplus_{i,j,k} V_{i,j,k}$) is a doubly (resp. triply) graded $\mathfrak{S}_n$-module, the multigraded Frobenius image is $\mathrm{grFrob}(V;q,t) := \sum_{i,j} q^i t^j \cdot \mathrm{Frob}(V_{i,j})$ (resp. $\mathrm{grFrob}(V;q,t,z) := \sum_{i,j,k} q^i t^j z^k \cdot \mathrm{Frob}(V_{i,j,k})$).

For any symmetric function $F \in \Lambda$, the *(primed) delta operator* $\Delta'_F : \Lambda \to \Lambda$ is defined as follows. For any partition μ, let $\widetilde{H}_\mu(\mathbf{x};q,t)$ be the modified Macdonald symmetric function. The set $\{\widetilde{H}_\mu(\mathbf{x};q,t) \,:\, \mu \text{ a partition}\}$ is a basis of Λ. The operator Δ'_F is the Macdonald eigenoperator given by

$$\Delta'_F : \widetilde{H}_\mu(\mathbf{x};q,t) \mapsto F(\ldots, q^{i-1}t^{j-1}, \ldots) \times \widetilde{H}_\mu(\mathbf{x};q,t), \tag{2.1}$$

where (i,j) ranges over all cells $\neq (1,1)$ in the Young diagram of μ. For example, if $\mu = (3,2)$ we fill the cells of μ as follows

$\cdot$	q	q^2
t	qt	

so that $\Delta'_F : \widetilde{H}_{(3,2)}(\mathbf{x};q,t) \mapsto F(q,q^2,t,qt) \times \widetilde{H}_{(3,2)}(\mathbf{x};q,t)$.

The *Delta Conjecture* of Haglund, Remmel, and Wilson [4] predicts the monomial expansion of $\Delta'_{e_{k-1}} e_n$ for $k \leq n$. It reads

$$\Delta'_{e_{k-1}} e_n = \mathrm{Rise}_{n,k}(\mathbf{x};q,t) = \mathrm{Val}_{n,k}(\mathbf{x};q,t), \tag{2.2}$$

where Rise and Val are certain formal power series defined using lattice path combinatorics; see [4] for more details.

We now review category-theoretic material related to representation stability. The notion of an FI-module will allow us to prove that a sequence $(V_n)_{n\geq 1}$ is representation stable by embedding it in another sequence $(W_n)_{n\geq 1}$ for which representation stability is known.

For $n \geq 1$, write $[n] := \{1,2,\ldots,n\}$ Let FI be the category consisting of

- the single object $[n]$ for each positive integer n, and
- morphisms given by injective maps $f : [n] \hookrightarrow [m]$.

Let Vect be the category of finite-dimensional $\mathbb{Q}$-vector spaces with morphisms given by arbitrary linear maps.

An FI-*module* is a covariant functor $V : \mathsf{FI} \to \mathrm{Vect}$. We write $V(n)$ for the image of the object $[n]$ in FI under V. More explicitly, an FI-module consists of a finite-dimensional $\mathbb{Q}$-vector space $V(n)$ for each $n \geq 1$ and a linear map $V(f) : V(n) \to V(m)$ associated to any injection $f : [n] \hookrightarrow [m]$ such that

- if $\mathrm{id}_{[n]} : [n] \to [n]$ is the identity, then $V(\mathrm{id}_{[n]}) : V(n) \to V(n)$ is the identity, and
- if $f : [n] \hookrightarrow [m]$ and $g : [m] \hookrightarrow [p]$, then $V(g \circ f) = V(g) \circ V(f)$.

If V is an FI-module, the vector space $V(n)$ is naturally a $\mathfrak{S}_n$-module for each $n \geq 1$.

As an example of an FI-module, fix $d \geq 0$ and let $\mathbb{Q}[x_1,\ldots,x_n]_d$ be the space of polynomials in $x_1,\ldots,x_n$ which are homogeneous of degree d. The assignment $[n] \mapsto \mathbb{Q}[x_1,\ldots,x_n]_d$ is an FI-module where $f : [n] \hookrightarrow [m]$ is sent to the map $\mathbb{Q}[x_1,\ldots,x_n]_d \to \mathbb{Q}[x_1,\ldots,x_m]_d$ defined on variables by $x_i \mapsto x_{f(i)}$.

If $V, W : \mathsf{FI} \to \mathrm{Vect}$ are FI-modules, we say that W is a *submodule* of V if $W(n) \subseteq V(n)$ for all n and for any injection $f : [n] \hookrightarrow [m]$ the following diagram commutes

$$\begin{array}{ccc} V(n) & \xrightarrow{V(f)} & V(m) \\ \uparrow & & \uparrow \\ W(n) & \xrightarrow[W(f)]{} & W(m) \end{array}$$

where the vertical arrows are inclusions. If W is a submodule of V, we have a quotient FI-module $V/W : [n] \mapsto V(n)/W(n)$.

An FI-module V is *finitely-generated* if there is a finite subset $S \subseteq \bigsqcup_{n \geq 1} V(n)$ such that no proper submodule $W \subsetneq V$ contains every element of S. The FI-module $\mathbb{Q}[x_1, \ldots, x_n]_d$ described above is finitely-generated. In fact, it is generated by the set of monomials

$$S = \bigsqcup_{n \leq d} \{x_1^{a_1} \cdots x_n^{a_n} \, : \, a_1 + \cdots + a_n = d\} \subseteq \bigsqcup_{n \leq d} \mathbb{Q}[x_1, \ldots, x_n]_d.$$

We also define the category coFI to be the opposite category to FI. That is, the objects of coFI are the same as those in FI but the arrows are reversed. A coFI-module is a covariant functor $V : \mathrm{coFI} \to \mathrm{Vect}$. Submodules, quotient modules, and finite generation are defined as in the setting of FI-modules. We state two key results about FI and coFI.

Theorem 2.1.

1. *(Snowden [10]) Any submodule or quotient module of a finitely-generated* FI*-module or* coFI*-module is finitely-generated.*

2. *(Church-Ellenberg-Farb [2]) If V is a finitely-generated* FI*-module or* coFI*-module then the sequence $(V(n))_{n \geq 1}$ of $\mathfrak{S}_n$-modules exhibits representation stability with respect to the maps $V([n] \hookrightarrow [n+1]) : V(n) \to V(n+1)$ induced by containment for* FI *or the duals of the maps $V([n] \hookrightarrow [n+1]) : V(n+1) \to V(n)$ for* coFI*.*

3 Geometric proof of Theorem 1.3

To use the category FI to prove Theorem 1.3, we need the geometric notion of an affine paving. Let X be a complex algebraic variety. A chain

$$\varnothing = Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_m = X \tag{3.1}$$

of Zariski closed subsets is an *affine paving* of X if each difference $Z_i - Z_{i-1}$ is isomorphic to a disjoint union $\bigsqcup_j A_{ij}$ of affine spaces (of possibly different dimensions).

For example, let $\mathbb{P}^{k-1}$ denote the $(k-1)$-dimensional complex projective space of lines through the origin in $\mathbb{C}^k$. The variety $\mathbb{P}^{k-1}$ admits the following affine paving (in projective coordinates)

$$\varnothing \subset [\star : 0 : \cdots : 0] \subset [\star : \star : \cdots : 0] \subset \cdots \subset [\star : \star : \cdots : \star] = \mathbb{P}^{k-1}. \tag{3.2}$$

We need only one fact about affine pavings. Suppose X is a variety and $U \subseteq X$. The inclusion $\iota : U \to X$ induces a map on homology

$$\iota_* : H_\bullet(U) \longrightarrow H_\bullet(X). \tag{3.3}$$

Although the nature of the map ι_* is generally inscrutable:

Suppose $\varnothing = Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_m = X$ is an affine paving and $U = X - Z_i$ for some i. Then the induced map ι_* on homology is injective.

Proof. We prove Theorem 1.3 (2); the proof of Theorem 1.3 (1) is similar, but easier. The strategy is to give the homology groups in question the structure of an FI-module which embeds inside a finitely-generated FI-module, and then apply Theorem 2.1. We start by describing our embedding.

The n-fold product $(\mathbb{P}^{k-1})^n$ consists of all n-tuples $(\ell_1, \ldots, \ell_n)$ of 1-dimensional subspaces of $\mathbb{C}^k$. We have an inclusion $\iota : X_{n,k} \hookrightarrow (\mathbb{P}^{k-1})^n$.

While one can take products of the subvarieties in (3.2) to get a product paving of $(\mathbb{P}^{k-1})^n$, this paving interacts poorly with the inclusion ι. Pawlowski and Rhoades [7] exhibit a *different* affine paving $\varnothing = Z_0 \subseteq Z_1 \subseteq \cdots \subseteq (\mathbb{P}^{k-1})^n$ with $X_{n,k} = (\mathbb{P}^{k-1})^n - Z_i$ for some i. Therefore, the map

$$\iota_* : H_d(X_{n,k}) \hookrightarrow H_d((\mathbb{P}^{k-1})^n) \tag{3.4}$$

is an injection for all k and n.

Given $f : [n] \hookrightarrow [p]$ define $\nu_f : (\mathbb{P}^{n-m-1})^n \to (\mathbb{P}^{p-m-1})^p$ by $\nu_f : (\ell_1, \ldots, \ell_n) \mapsto (\ell'_1, \ldots, \ell'_p)$ where the ℓ'_j are defined as follows. Write the complement of the image of f as

$$\{1, 2, \ldots, p\} - \{f(1), f(2), \ldots, f(n)\} := \{c_1 < c_2 < \cdots < c_{p-n}\}.$$

Now set

$$\ell'_j := \begin{cases} \ell_i & \text{if } f(i) = j, \\ \langle e_{n+i} \rangle & \text{if } c_i = j. \end{cases} \tag{3.5}$$

In the first branch we consider ℓ_i as a line in $\mathbb{C}^{p-m}$ by embedding $\mathbb{C}^{n-m} \hookrightarrow \mathbb{C}^{p-m}$ into the first $n - m$ coordinates and in the second branch $\langle e_{n+i} \rangle \subseteq \mathbb{C}^{p-m}$ is the line spanned by the $(n+i)^{th}$ standard basis vector. We have $\ell'_1 + \cdots + \ell'_p = \mathbb{C}^{p-m}$ whenever $\ell_1 + \cdots + \ell_n = \mathbb{C}^{n-m}$ so that $\nu_f(X_{n,n-m}) \subseteq X_{p,p-m}$.

If $f : [n] \hookrightarrow [p]$ and $g : [p] \hookrightarrow [r]$ are two injections, we do **not** necessarily have the equality of maps $\nu_{g \circ f} = \nu_g \circ \nu_f$. For example, suppose $f : [2] \hookrightarrow [4]$ and $g : [4] \hookrightarrow [6]$ are given by

$$f(1) = 3,\ f(2) = 1 \quad \text{and} \quad g(1) = 2,\ g(2) = 6,\ g(3) = 5,\ g(4) = 3.$$

Then

$$(\ell_1, \ell_2) \overset{\nu_f}{\longmapsto} (\ell_2, \langle e_3 \rangle, \ell_1, \langle e_4 \rangle) \overset{\nu_g}{\longmapsto} (\langle e_5 \rangle, \ell_2, \langle e_4 \rangle, \langle e_6 \rangle, \ell_1, \langle e_3 \rangle)$$

whereas

$$(\ell_1, \ell_2) \overset{\nu_{g \circ f}}{\longmapsto} (\langle e_3 \rangle, \ell_2, \langle e_4 \rangle, \langle e_5 \rangle, \ell_1, \langle e_6 \rangle).$$

Despite the inequality $\nu_{g \circ f} \neq \nu_g \circ \nu_f$, we have the following

Claim: *We have a homotopy of maps $\nu_{g \circ f} \simeq \nu_g \circ \nu_f$.*

To prove the Claim, consider the translation action of $GL_{r-m}(\mathbb{C})$ on $(\mathbb{P}^{r-m-1})^r$ given by $A \cdot (\ell_1, \ldots, \ell_r) := (A\ell_1, \ldots, A\ell_r)$. For fixed injections $f : [n] \hookrightarrow [p]$ and $g : [p] \hookrightarrow [r]$, there exists a matrix $P \in GL_{r-m}(\mathbb{C})$ such that

$$P \cdot \nu_{g\circ f}(\ell_1, \ldots, \ell_n) = (\nu_g \circ \nu_f)(\ell_1, \ldots, \ell_n) \tag{3.6}$$

for all $(\ell_1, \ldots, \ell_n) \in (\mathbb{P}^{n-m-1})^n$. The matrix P simply permutes the last $r - n$ standard basis vectors in a fashion depending on f and g.

Since $GL_{r-m}(\mathbb{C})$ is path-connected, there is a continuous map $\gamma : [0, 1] \to GL_{r-m}(\mathbb{C})$ with $\gamma(0) = I$ (the identity matrix) and $\gamma(1) = P$. The requisite homotopy equivalence $[0, 1] \times (\mathbb{P}^{n-m-1})^n \to (\mathbb{P}^{r-m-1})^r$ is given by $t \times (\ell_1, \ldots, \ell_n) \mapsto \gamma(t) \cdot \nu_{g\circ f}(\ell_1, \ldots, \ell_n)$. This proves the Claim.

Our Claim implies $(\nu_{g\circ f})_* = (\nu_g)_* \circ (\nu_f)_*$ as functions on $H_d((\mathbb{P}^{n-m-1})^n)$ so the assignment $[n] \mapsto H_d((\mathbb{P}^{n-m-1})^n)$ is an FI-module. The injectivity of the map ι_* in (3.4) means that $[n] \mapsto H_d(X_{n,n-m})$ is a submodule. The FI-module $[n] \mapsto H_d((\mathbb{P}^{n-m-1})^n)$ is finitely-generated, so of Theorem 1.3 (2) follows from Theorem 2.1. □

4 Higher dimensional subspaces

In this section we extend Theorem 1.3 from lines to higher-dimensional subspaces. Let $\mathrm{Gr}(r, k)$ be the Grassmannian of r-dimensional subspaces of $\mathbb{C}^k$ and consider the n-fold product $\mathrm{Gr}(r, k)^n = \mathrm{Gr}(r, k) \times \cdots \times \mathrm{Gr}(r, k)$ of this Grassmannian with itself. We have the variety

$$X_{r,n,k} := \{(V_1, \ldots, V_n) \in \mathrm{Gr}(r, k)^n \,:\, V_1 + \cdots + V_n = \mathbb{C}^k\} \tag{4.1}$$

of spanning subspace configurations. The cohomology of $X_{r,n,k}$ may be presented as follows.

Theorem 4.1. *(Rhoades [8]) Let $N = r \cdot n$ and consider a list $\mathbf{x}_N := (x_1, \ldots, x_N)$ of N variables. For $1 \leq i \leq n$ denote the i^{th} batch of r variables by $\mathbf{x}_N^{(i)} := (x_{(r-1)i+1}, x_{(r-1)i+2}, \ldots, x_{ri})$. We have*

$$H^\bullet(X_{r,n,k}) = (\mathbb{Q}[x_1, x_2, \ldots, x_N]/I)^{\mathfrak{S}_r\times\cdots\times\mathfrak{S}_r} \tag{4.2}$$

where the n-fold symmetric group product $\mathfrak{S}_r \times \cdots \times \mathfrak{S}_r$ permutes variables within batches, the superscript indicates taking invariants, and $I \subseteq \mathbb{Q}[x_1, \ldots, x_N]$ is generated by

- *the top k elementary symmetric polynomials $e_N, e_{N-1}, \ldots, e_{N-k+1}$ in the full variable set $\mathbf{x}_N$ and*
- *for $1 \leq i \leq n$ the complete homogeneous symmetric polynomials $h_k, h_{k-1}, \ldots, h_{k-r+1}$ in the variable set $\mathbf{x}_N^{(i)}$.*

The variables in $\mathbf{x}_N^{(i)}$ *represent the Chern roots of the vector bundle* $V_i^* \twoheadrightarrow X_{r,n,k}$.

The action of $\mathfrak{S}_n$ on the cohomology $H^\bullet(X_{r,n,k})$ corresponds under the presentation of Theorem 4.1 to permuting the variable batches $\mathbf{x}_N^{(1)}, \ldots, \mathbf{x}_N^{(n)}$. As an ungraded $\mathfrak{S}_n$-module, it follows from [8] that $H^\bullet(X_{r,n,k})$ is isomorphic to the column-permuting action of $\mathfrak{S}_n$ on the set of 0,1-matrices of dimension $k \times n$ which have all column sums equal to r and no zero rows. When $r = 2, n = 4$, and $k = 3$ one such matrix is

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

The following stability result concerns the graded structure of $H^\bullet(X_{r,n,k})$.

Theorem 4.2. *Fix a homological degree* $d \geq 0$ *and a subspace dimension* r*. The following sequences of modules are representation stable with respect to some sequence of maps* f_n*.*

1. $(H_d(X_{r,n,k}))_{n \geq 0}$ *for* $k \geq 0$ *fixed, and*
2. $(H_d(X_{r,n,n-m}))_{n \geq 0}$ *for* $m \geq 0$ *fixed.*

The proof of Theorem 4.2 is similar to that of Theorem 1.3. One exhibits a non-standard affine paving of the product $\mathrm{Gr}(r,k)^n$ of Grassmannians which has $X_{r,n,k}$ as a terminal portion. We omit the details in this extended abstract, but remark that the following problem remains open, despite the explicit presentation in Theorem 4.1.

Problem 4.3. Calculate the graded $\mathfrak{S}_n$-isomorphism type of $H_\bullet(X_{r,n,k})$.

This illustrates our introductory leitmotif: it can be easier to prove that a sequence of modules exhibits representation stability than it is to calculate their isomorphism types.

5 Modules from Coinvariants and Vandermondes

In this section we consider a family of multigraded $\mathfrak{S}_n$-modules which arise as subspaces or quotients of rings generated by two n-column matrices of variables: one matrix of commuting variables and one matrix of anticommuting variables. We obtain stability results for modules considered by Orellana-Zabrocki [6], Zabrocki [11], and Rhoades-Wilson [9] which have (sometimes conjectural) ties to the Delta Conjecture [4]. Most of the modules we consider in this section do not have known decompositions into irreducibles. In spite of this (and in keeping with our leitmotif), it is possible to show that they enjoy stability properties.

For $n, m, p \geq 0$, consider an $m \times n$ matrix $(x_j^{(i)})_{1 \leq i \leq m, 1 \leq j \leq n}$ of (commuting) variables and a $p \times n$ matrix $(\theta_j^{(i)})_{1 \leq i \leq p, 1 \leq j \leq n}$ of (anticommuting) variables. Let $S(n, m, p)$ be the unital associative $\mathbb{Q}$-algebra generated by these $mn + pn$ variables subject to the relations

$$x_j^{(i)} x_{j'}^{(i')} = x_{j'}^{(i')} x_j^{(i)} \qquad x_j^{(i)} \theta_{j'}^{(i')} = \theta_{j'}^{(i')} x_j^{(i)} \qquad \theta_j^{(i)} \theta_{j'}^{(i')} = -\theta_{j'}^{(i')} \theta_j^{(i)}$$

The algebra $S(n, m, p)$ has a multigrading obtained by considering each row of the two variable matrices separately. For $\alpha = (\alpha_1, \ldots, \alpha_m) \in (\mathbb{Z}_{\geq 0})^m$ and $\beta = (\beta_1, \ldots, \beta_p) \in (\mathbb{Z}_{\geq 0})^p$ write $S(n, m, p)_{\alpha,\beta}$ for the piece of $S(n, m, p)$ of homogeneous multidegree (α, β).

The group $\mathfrak{S}_n$ acts on $S(n, m, p)$ by the rule $w.x_j^{(i)} := x_{w(j)}^{(i)}$ and $w.\theta_j^{(i)} := \theta_{w(j)}^{(i)}$. Orellana and Zabrocki [6] gave a combinatorial interpretation of the $\mathfrak{S}_n$-isomorphism type of $S(n, m, p)_{\alpha,\beta}$. We consider this object as n varies.

Proposition 5.1. *Let $m, p \geq 0$ and let $\alpha \in (\mathbb{Z}_{\geq 0})^m$ and $\beta \in (\mathbb{Z}_{\geq 0})^p$ be multidegrees. The sequence $(S(n, m, p)_{\alpha,\beta})_{n \geq 1}$ is representation stable with respect to the inclusion maps*

$$f_n : S(n, m, p)_{\alpha,\beta} \hookrightarrow S(n + 1, m, p)_{\alpha,\beta}.$$

Let $S(n, m, p)_+^{\mathfrak{S}_n} \subseteq S(n, m, p)$ be the space of $\mathfrak{S}_n$-invariants with vanishing constant term and let $I(n, m, p) \subseteq S(n, m, p)$ be the ideal generated by $S(n, m, p)_+^{\mathfrak{S}_n}$. We consider the quotient

$$R(n, m, p) := S(n, m, p)/I(n, m, p). \tag{5.1}$$

Write $R(n, m, p)_{\alpha,\beta}$ for the piece of homogeneous multidegree (α, β).

The $\mathfrak{S}_n$-modules $R(n, m, p)$ have received significant attention in algebraic combinatorics. $R(n, 1, 0)$ is the classical coinvariant ring attached to the symmetric group which presents the cohomology of the flag variety $\mathcal{F}\ell_n$ (or the space $X_{n,n}$). $R(n, 2, 0)$ is the *diagonal coinvariant ring* studied by Garsia and Haiman [3]. The trigraded $\mathfrak{S}_n$-module $R(n, 2, 1)$ was studied by Zabrocki [11] in the context of the Delta Conjecture. Zabrocki conjectured that

$$\mathrm{grFrob}(R(n, 2, 1); q, t, z) = \sum_{k=1}^{n} z^{n-k} \cdot \Delta'_{e_{k-1}} e_n. \tag{5.2}$$

Theorem 5.2. *Let $m, p \geq 0$ and let $\alpha \in (\mathbb{Z}_{\geq 0})^m$ and $\beta \in (\mathbb{Z}_{\geq 0})^p$ be multidegrees. The sequence $(R(n, m, p)_{\alpha,\beta})_{n \geq 1}$ is representation stable with respect to some sequence of maps f_n.*

Proof. The assignment $[n] \mapsto I(n, m, p)_{\alpha,\beta}$ is a submodule of the coFI-module $[n] \mapsto S(n, m, p)_{\alpha,\beta}$. Now apply Proposition 5.1 and Theorem 2.1. ☐

Another Delta Conjecture model was proposed by Rhoades and Wilson [9]. Assuming $m, p \geq 1$, the *superspace Vandermonde* $\delta_{n,k} \in S(n, m, p)$ is the element

$$\delta_{n,k} := \varepsilon_n \cdot (x_1^{k-1} \cdots x_{n-k}^{k-1} \, x_{n-k+1}^{k-1} x_{n-k+2}^{k-2} \cdots x_{n-1}^{1} \, x_n^{0} \cdot \theta_1 \cdots \theta_{n-k}), \tag{5.3}$$

where the x's and θ's are drawn from the 'first' commuting and anticommuting tensor factors and $\varepsilon_n = \sum_{w \in \mathfrak{S}_n} \mathrm{sign}(w) \cdot w \in \mathbb{Q}[\mathfrak{S}_n]$ is the antisymmetrizing group algebra element.

To describe the representations in [9] we need polarization operators. Let $y_1, \ldots, y_n$ and $z_1, \ldots, z_n$ be two rows of commuting generators of $S(n, m, p)$ (renamed y and z for clarity). For $j \geq 1$ the *commuting polarization operator* from y to z of order j is the operator on $S(n, m, p)$ defined by

$$\rho^{(j)}_{y \to z} := z_1(\partial/\partial y_1)^j + \cdots + z_n(\partial/\partial y_n)^j. \tag{5.4}$$

This operator lowers y-degree by j and raises z-degree by 1.

To define an anticommuting version of the $\rho^{(j)}_{y \to z}$ we need to differentiate with respect to anticommuting variables. Let $\xi_1, \ldots, \xi_n$ and $\tau_1, \ldots, \tau_n$ be two rows of anticommuting generators of $S(n, m, p)$ (renamed ξ and τ for clarity). For $1 \leq i \leq n$, the operator $\partial/\partial \xi_i$ acts on $S(n, m, p)$ by commuting with multiplication by any commuting variable and the rule

$$\partial/\partial \xi_i : \zeta_{j_1} \cdots \zeta_{j_r} \mapsto \begin{cases} (-1)^{s-1} \zeta_{j_1} \cdots \widehat{\xi_{j_s}} \cdots \zeta_{j_r} & \text{if } \zeta_{j_s} = \xi_i, \\ 0 & \text{if } \xi_i \text{ does not appear in } \zeta_{j_1}, \ldots, \zeta_{j_r}, \end{cases} \tag{5.5}$$

where $\zeta_{j_1}, \ldots, \zeta_{j_r}$ are distinct anticommuting variables and $\widehat{\cdot}$ means omission. The *anticommuting polarization operator* from ξ to τ is the operator on $S(n, m, p)$ defined by

$$\rho_{\xi \to \tau} := \tau_1(\partial/\partial \xi_1) + \cdots + \tau_n(\partial/\partial \xi_n). \tag{5.6}$$

This operator lowers ξ-degree by 1 and raises τ-degree by 1.

Let $V(n, k, m, p)$ be the smallest linear subspace of $S(n, m, p)$ which contains the superspace Vandermonde $\delta_{n,k}$ and is closed under all commuting partial derivatives $\partial/\partial x_i$ as well as all possible polarization operators. Considering commuting multidegree alone, $V(n, k, 2, 1)$ is a bigraded $\mathfrak{S}_n$-module. Rhoades and Wilson conjectured [9] that

$$\mathrm{grFrob}(V(n, k, 2, 1); q, t) = \Delta'_{e_{k-1}} e_n. \tag{5.7}$$

This is similar in form to Zabrocki's conjecture (5.2). Unlike (5.2), (5.7) is proven at $t = 0$. Theorem 5.2 extends to the setting of (5.7).

Theorem 5.3. *Let $m, p, k \geq 0$ and let $\alpha \in (\mathbb{Z}_{\geq 0})^m$ and $\beta \in (\mathbb{Z}_{\geq 0})^p$ be multidegrees. The sequence $(V(n, k, m, p)_{\alpha,\beta})_{n \geq 0}$ is representation stable with respect to some sequence of maps f_n.*

Proof. The identity

$$(\partial/\partial x_1)(\partial/\partial x_2) \cdots (\partial/\partial x_n) \delta_{n+1,k} = \delta_{n,k} \tag{5.8}$$

implies that the map $S(n, m, p) \hookrightarrow S(n+1, m, p)$ sends $V(n, k, m, p)$ into $V(n+1, k, m, p)$. Now apply Proposition 5.1 and Theorem 2.1. □

We close with a very difficult problem which serves as a final illustration of our introductory leitmotif.

Problem 5.4. Find the Schur expansion of any of the following symmetric functions:

$$\Delta'_{e_{k-1}} e_n, \quad \mathrm{grFrob}(R(n,m,p);q,t,z), \quad \mathrm{grFrob}(V(n,k,m,p);q,t). \tag{5.9}$$

In the case $k = n$, the first of these is ∇e_n, where ∇ is the Bergeron-Garsia nabla operator.

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