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Universal oriented matroids for subword complexes of Coxeter groups

Jean-Philippe Labbé*[1]

[1]*Institut für Mathematik, Freie Universität Berlin, Berlin, Germany*

Abstract. We study geometric realizations of subword complexes using multilinear algebra and combinatorics on reduced words. Namely, we provide an explicit description of partial oriented matroids using *parameter tensors* that encapsulate the necessary information to obtain geometric realizations of subword complexes, and show that every geometric realization is parametrized by a certain parameter tensor.

Résumé. Nous étudions les réalisations géométriques des complexes de sous-mots à l'aide d'algèbre multi-linéaire et de la combinatoire des mots réduits. Nous donnons une description explicite de matroïdes partiellement orientés appelés *tenseurs de paramètres* qui englobent l'information nécessaire afin d'obtenir des réalisations géométiques des complexes de sous-mots. De plus, nous démontrons que chaque réalisation géométrique est paramétrée par un tenseur de paramètres approprié.

Keywords: Subword complexes, Coxeter groups, sign function, Schur functions, oriented matroids, reduced expressions

1 Introduction

Multi-triangulations offer a broad generalization of triangulations of a convex polygon [14]. The simplicial complex whose facets are maximal k-crossing-free sets of diagonals of a convex polygon (i.e. multi-triangulations) is conjectured to be the boundary of a convex polytope that would generalize the associahedron. This conjecture first appeared in the Oberwolfach Book of Abstract, handwritten by Jonsson in 2003, which then did not appear in the official MFO Report [10]. The determination of the polytopality of spheres is famously known to be fraught with pitfalls, e.g. the recent discussions in [8, 9]. Currently, the only known non-trivial polytopal construction is for the 2-triangulations of the 8-gon [2, 4, 5]. However, the complex of multi-triangulations happens to be an instance of subword complexes [11]. Via their intrinsic relation with reduced words, subword complexes incarnate objects related to cluster algebras [6], toric geometry [7], and Hopf algebras [1], for example. Taking advantage of the combinatorics of reduced

*labbe@math.fu-berlin.de. Supported by the DFG Collaborative Research Center TRR 109 "Discretization in Geometry and Dynamics".

words, the notions of *sign function* and *signature matrices* are used to lay down necessary conditions for the polytopality of subword complexes in [2]. Therein, a combinatorial construction of signature matrices and complete simplicial fans for subword complexes of type A_3 and for certain cases in type A_4 is further obtained. In spite of these positive results, the reason *why* the construction works is still rather mysterious.

In this extended abstract, we introduce *parameter tensors* and show their universality for geometric realizations of subword complexes. Namely, we show that every realization of any spherical subword complex of type W as a simplicial fan delivers a realization of a (partially) oriented matroid, called *parameter tensor*, that depends solely on data derived from the group W. Section 2 gathers the required basic notions. In Section 3, we present a *theory of sign functions on words* that unifies the usual sign of permutations and the sign function presented in [2] . In Section 4, we define *parameter tensors* and *model matrices* as a tool to factorize the determinant of "partial alternant matrices". Finally, in Section 5, we show the universality of *parameter tensors*.

2 Preliminaries

2.1 Linear algebra, Vandermonde matrix and partial Schur functions

Let $d \geq 1$ and V_d be a d-dimensional real vector space and denote its dual space by V_d^*. Given a basis $\{\mathbf{e}^1, \ldots, \mathbf{e}^d\}$ of V_d and a basis $\{\mathbf{f}_1, \ldots, \mathbf{f}_d\}$ of V_d^*, a $(d \times d)$-matrix $M^i{}_j = (m_{i,j}) = (m^i_j)$ represents the tensor

$$\mathcal{M}^i{}_j := \sum_{i=1}^{d} \sum_{j=1}^{d} m^i_j \mathbf{e}^i \otimes \mathbf{f}_j \in V_d \otimes V_d^*.$$

We view the product of $(d_1 \times d_2)$-matrices with $(d_2 \times d_3)$-matrices using tensors via the following linear map:

$$\begin{aligned} \left(V_{d_1} \otimes V_{d_2}^*\right) \times \left(V_{d_2} \otimes V_{d_3}^*\right) &\to V_{d_1} \otimes V_{d_3}^* \\ ((\mathbf{x} \otimes \mathbf{f}), (\mathbf{y} \otimes \mathbf{g})) &\mapsto \mathbf{f}(\mathbf{y}) \cdot (\mathbf{x} \otimes \mathbf{g}). \end{aligned} \tag{2.1}$$

More generally, given a tensor $\mathcal{T}^i{}_j \in V_{d_1} \otimes V_{d_2}^*$ and a tensor $\mathcal{U}^j{}_k \in V_{d_2} \otimes V_{d_3}^*$, we write the tensor contraction as $\mathcal{V}^i{}_k := \mathcal{T}^i{}_j \cdot \mathcal{U}^j{}_k$, using the rule given in Equation (2.1). Contraction of higher rank tensors is defined similarly, by matching the appropriate pairs of indices. The Vandermonde matrix of size d is

$$\mathrm{Vander}(d) := \sum_{i=1}^{d} \sum_{j=1}^{d} x_j^{i-1} \mathbf{e}^i \otimes \mathbf{f}_j = (x_j^{i-1})_{(i,j) \in [d] \times [d]},$$

and its determinant is

$$\det \mathrm{Vander}(d) = \prod_{1\le i<j\le d}(x_j - x_i) = \sum_{\pi\in\mathfrak{S}_d} \mathrm{sign}(\pi) x_1^{\pi(1)-1}\cdots x_d^{\pi(d)-1}.$$

The Vandermonde matrix is also obtained as the product of two rectangular matrices as follows. Let $W_d := \mathcal{W}^i{}_{j,k} := \bigoplus_{i=1}^d \mathrm{Id}_d$ be the augmentation of d identity matrices by concatenating them columnwise. This matrix can be rewritten as a tensor in $V_d \otimes V_d^* \otimes V_d^*$ as $\sum_{i=1}^d \left(\mathbf{e}^i \otimes \left(\sum_{j=1}^d \mathbf{f}_j\right) \otimes \mathbf{f}_i\right)$. Further, we define the tensor $\mathcal{X}^{k,j}{}_l \in V_d \otimes V_d \otimes V_d^*$ as

$$X_d := \mathcal{X}^{k,j}{}_l := \sum_{j=1}^d \left(\sum_{k=1}^d x_j^{k-1}\right) \mathbf{e}^k \otimes \mathbf{e}^j \otimes \mathbf{f}_j.$$

From the definitions of W_d and X_d, and the properties of product of tensors, we get $\mathrm{Vander}(d) = W_d X_d = \mathcal{W}^i{}_{j,k} \cdot \mathcal{X}^{k,j}{}_l$. Altering W_d and X_d defined above in particular ways lead to generalizations of the Vandermonde matrix. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$ be a partition with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_d \ge 0$, and J be an ordered set of indices of cardinality d. The *Schur function* $\int_{\lambda,J}$ in the variables $\{x_j : j \in J\}$ is the quotient

$$\int_{\lambda,J} := \frac{\det(x_j^{i-1+\lambda_{d-i+1}})_{(i,j)\in[d]\times J}}{\det \mathrm{Vander}_J(d)},$$

where $\mathrm{Vander}_J(d)$ is $\mathrm{Vander}(d)$ with variables indexed by J [12, Section 1.3]. When $J = [n]$ for some $n \ge 1$ we omit the subscript J and simply write $\int_\lambda$.

Definition 2.1 (Partial Schur functions)**.** Let $m \ge 1$, $P = (p_i)_{i=1}^n$ be an ordered set partition of $[m]$ where parts may be empty, and $\Lambda = (\lambda_i)_{i=1}^n$ be a sequence of partitions such that the number of parts of λ_i is the cardinality of p_i. The *partial Schur function* with respect to P and Λ is

$$\mathcal{S}_{\Lambda,P} := \prod_{i=1}^n \int_{\lambda_i,p_i}.$$

Example 2.2. Let $m = 4$, $P = (\{1,3\},\{2,4\})$, and $\Lambda = ((3,1),(2,0))$. The partial Schur function $\mathcal{S}_{\Lambda,P}$ with respect to P and Λ is

$$\mathcal{S}_{\Lambda,P} = \int_{(3,1),\{1,3\}} \cdot \int_{(2,0),\{2,4\}} = x_1x_3(x_1^2 + x_1x_3 + x_3^2)\cdot(x_2^2 + x_2x_4 + x_4^2).$$

2.2 Combinatorics on words, Coxeter groups and Subword Complexes

Let $S = \{s_1, \dots, s_n\}$ be a finite alphabet of *letters*. Let $S^* := \bigoplus_{i\in\mathbb{N}} S^i$ be the free monoid generated by elements in S by concatenation, and call its elements *words*. Let $w = w_1 \cdots w_m \in S^m$ be a word of *length* m. When there exists two words $u, v \in S^*$ such that $w = ufv$, the word $f \in S^*$ is called a *factor* of w. An *occurrence* of a factor f of length k in w is a set of positions $\{i, \dots, i+(k-1)\}$ with $i \in [m-k+1]$ such that $w = w_1 \cdots w_{i-1} f w_{i+k} \cdots w_m$, i.e. $f = w_i \cdots w_{i+(k-1)}$. We denote by $\{w\}_i$ the *set of occurrences of the letter* $s_i \in S$ in w. We associate the ordered set partition $\Omega_w = (\{w\}_1, \dots, \{w\}_n)$ of $[m]$ to w. We denote the cardinality of the set $\{w\}_i$ by $|w|_i := \#\{w\}_i$. The vector $\alpha_w = (|w|_1, \dots, |w|_n)$ is a weak composition called the *abelian vector* of w. Let $k \leq m$ and $w : [m] \to S$ be a word of length m, a *subword* v of length k of w is a word obtained by the composition $v := w \circ u$, for some strictly increasing function $u : [k] \to [m]$.

Let $(W, S = \{s_1, \dots, s_n\})$ be a *finite irreducible Coxeter system* with Coxeter matrix $M = (m_{i,j})_{i,j \in [n]}$. The function $\ell : W \to \mathbb{N}$ denotes the *length function* sending an element $w \in W$ to the length of its reduced words, the symbol $w_\circ$ denotes the *longest element* of W, and $N := \ell(w_\circ)$. Given an element $w \in W$, we denote the *set of reduced words* of w by $\mathcal{R}(w)$.

Problem 2.3. Let $w \in W$.
1. Characterize the set of abelian vectors $\{\alpha_v : v \in \mathcal{R}(w)\}$.
2. Give precise bounds on the number

$$\nu(w) := \max \big\{ \max\{\alpha_v(s) : s \in S\} : v \in \mathcal{R}(w) \big\},$$

   which is *the maximum number of occurrences of a letter in any reduced word of w.*
3. Describe the vector

$$\mu(w) := \big(\min\{\alpha_v(s) : v \in \mathcal{R}(w)\} \big)_{s \in S},$$

   which gives *the minimum number of occurrences of each letter in any reduced word of w.*

In the symmetric group case $\mathfrak{S}_{n+1} = A_n$ and taking $w = w_\circ$, Problem 2.3(2) is related to the *k-set problem* or *halving line problem* via duality between points and pseudolines on the plane, see [14, Section 3.1] for a contextual explanation. We denote $\nu := \nu(w_\circ)$ and refer to this value as the *höchstfrequenz* of the group W. The graph whose vertices are reduced words of w and edges represent braid moves between words is denoted $\mathcal{G}(w)$. This graph is well-known to be connected [3, Theorem 3.3]. Furthermore, $\mathcal{G}(w_\circ)$ and the graph obtained by contracting odd-length braid moves $\mathcal{G}^{\text{even}}(w_\circ)$ are bipartite [2, Theorem 3.1].

Let $p \in S^m$, the *subword complex* $\Delta_W(p)$ is the simplicial complex on the set $[m]$ whose facets are complements of occurrences of reduced words for $w_\circ$ in the word p [11, Definition 2.1]. Knutson and Miller originally asked whether spherical subword complexes can be realized as the boundary of convex polytopes [11, Question 6.4]. So far, the realized subword complexes include famous polytopes: simplices, even-dimensional cyclic polytopes, polar dual of generalized associahedra, see [6, Section 6] for a survey on the related conjectures and the references therein. Subword complexes of type A are intimately related to multi-triangulations, a generalization of usual triangulations of a convex polygon [10, 14], and certain cases are known to be realizable as fans [2, 13]. The only "non-classical" instance which is realized is a 6-dimensional polytope with 12 vertices realizing the simplicial complex of 2-triangulations of the 8-gon, see [4, 5, 2]. Further, fan realizations have been provided for type A_3 and two cases in A_4 [2] and for 2-triangulations (type A) with rank $5, 6, 7$ and 8 [13].

2.3 Fans and Gale duality

Given a finite totally ordered label set J of cardinality m, a *vector configuration* in $\mathbb{R}^d$ is a finite set $\mathbf{A} := \{\mathbf{p}_j \mid j \in J\}$ of labeled vectors $\mathbf{p}_j \in \mathbb{R}^d$. We write vector configurations as matrices in $\mathbb{R}^{d\times m}$. A *fan* supported by $\mathbf{A}$ is a family $\mathcal{F} = \{K_1, K_2, \ldots, K_k\}$ of nonempty polyhedral cones generated by vectors in $\mathbf{A}$ such that:

- Every nonempty face of a cone in $\mathcal{F}$ is also a cone in $\mathcal{F}$.
- The intersection of any two cones in $\mathcal{F}$ is a face of both.

A fan $\mathcal{F}$ is *simplicial* if every $K \in \mathcal{F}$ is a simplicial cone, and it is *complete* if the union $K_1 \cup \cdots \cup K_k$ is $\mathbb{R}^d$. A *Gale transform* $\mathbf{B} \in \mathbb{R}^{(m-d)\times m}$ of $\mathbf{A}$ is vector configuration whose rowspan equals the right-kernel of $\mathbf{A}$. We denote by $\mathrm{Gale}(\mathbf{A})$ the set of all Gale transforms of $\mathbf{A}$.

3 Sign functions of words

The sign of a permutation is defined using the parity of its number of pairwise inversions; even permutations being "$+$" and odd permutations being "$-$". There are many possible extensions of this notion on subsets of words of S^*: Let $M \subseteq S^*$ be a non-empty set of words in S. A function from M to the multiplicative group $\mathbb{Z}_2$ is a *sign function* on M. The set of sign functions on M equipped with the binary operation

$$\begin{aligned} {\mathbb{Z}_2}^M \times {\mathbb{Z}_2}^M &\to {\mathbb{Z}_2}^M \\ (\phi, \psi) &\mapsto \phi\psi(m) := \begin{cases} 1, & \text{if } \phi(m) = \psi(m), \\ -1, & \text{else,} \end{cases} \end{aligned}$$

forms a group: the *group of sign functions on M*.

The T-sign function. Let $w_\circ$ be the longest element of W. The *T-sign function* is defined as

$$\begin{aligned} \tau : \mathcal{R}(w_\circ) &\to \{+1, -1\} \\ w &\mapsto \tau(w), \end{aligned}$$

whereby if w and w' are two reduced words of $w_\circ$ related by a braid move of length $m_{i,j}$, then $\tau(w) = (-1)^{m_{i,j}-1}\tau(w')$ [2, Definition 3.5]. Therein, it is proved that this function is well-defined and unique up to a global multiplication by "-1".

Convention in Figures. Classes of reduced words with more than one element are written as $12\{13\}21 := \{121321, 123121\}$. The length of braid moves with continuous edges are even, those with dashed edges are odd, and doubled edges are used for multiples of 4.

Example 3.1 (T-sign function in type A_3)**.** Since a braid move of length 3 does not change the sign, we contract the edges in $\mathcal{G}(w_\circ)$ corresponding to them to obtain $\mathcal{G}^{\text{even}}(w_\circ)$. We can give "$+$" and "$-$" signs to the vertices of $\mathcal{G}^{\text{even}}(w_\circ)$ and we get a function on $\mathcal{G}^{\text{even}}(w_\circ)$ that change along its edges, see Figure 1.

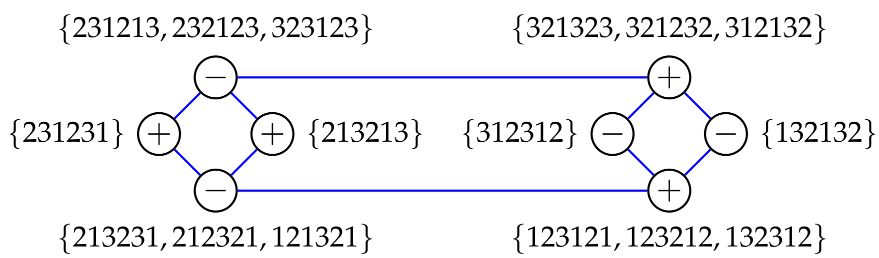

Figure 1: T-sign function on reduced words of $w_\circ$ in type A_3

The T-sign function is central to a construction of complete simplicial fans for subword complexes in [2]. This construction is based on signature matrices. Let $p \in S^m$, a matrix $\mathbf{B} \in \mathbb{R}^{N\times m}$ is a *signature matrix* of type W for p, if for every occurrence Z of every reduced word v of $w_\circ$ in p, the equality $\operatorname{sign}(\det[\mathbf{B}]_Z) = \tau(v)$ holds [2, Definition 9]. Let $\mathbf{A} \in \mathbb{R}^{(m-N)\times m}$, we denote by $\mathcal{F}_{p,\mathbf{A}}$ the collection of cones spanned by sets of columns of $\mathbf{A}$ that correspond to faces of the subword complex $\Delta_W(p)$.

Theorem 3.2 ([5, Section 3.1, Theorem 3.7], see also [2, Theorem 3])**.** *Let $p \in S^m$ and $\mathbf{A} \in \mathbb{R}^{(m-N)\times m}$. The collection of cones $\mathcal{F}_{p,\mathbf{A}}$ is a complete simplicial fan realizing the subword complex $\Delta_W(p)$ if and only if*

(S) the Gale dual $\mathbf{B} \in \operatorname{Gale}(\mathbf{A})$ is a signature matrix for p, (Signature) and

(I) there is a facet of $\Delta_W(p)$ such that the interior of its associated cone in $\mathcal{F}_{p,\mathbf{A}}$ does not intersect any other cone of $\mathcal{F}_{p,\mathbf{A}}$. (Injectivity)

The S-sign function. The S-sign function is an integral part of the sign of $\det[\mathbf{B}]_Z$. Let $w \in S^m$ be a word with abelian vector $\alpha_w = (c_i)_{s_i\in S}$. The word $\overline{w} := s_1^{c_1}\dots s_n^{c_n}$ is the *lexicographic normal form* of α_w. Permutations in $\mathfrak{S}_m$ acts on the letters of w as $\pi\cdot w := w_{\pi(1)}\cdots w_{\pi(m)}$, where $\pi \in \mathfrak{S}_m$. The permutation of $\mathfrak{S}_m$ with exactly the same inversions as w is called its *standard permutation* and is denoted $\operatorname{std}(w)$. The standard permutation is the minimal length permutation whose inverse sorts w: $\operatorname{std}(w)^{-1}\cdot w = \overline{w}$. The *inversion number* $\operatorname{inv}(w)$ of w is the number of inversions of $\operatorname{std}(w)$.

Definition 3.3 (S-sign of a word)**.** Let $w \in S^*$. The *S-sign* $\sigma(w)$ of w is

$$\sigma(w) := (-1)^{\operatorname{inv}(w)},$$

where $\operatorname{inv}(w)$ is the inversion number of w.

Example 3.4 (Sign of permutations)**.** Let $w \in S^*$ with abelian vector $\alpha_w = (1, \ldots, 1)$. Reading the indices of the letters of w from left to right gives a permutation of $[n]$ whose inversion number $\operatorname{inv}(w)$ is the number of pairs of distinct letters in S which are unordered in w.

Theorem 3.5. *Let $1 \le i < j \le n$, and $u, v \in S^*$ be two words. Further, define*

$$b_{i,j} := s_i s_j s_i \ldots \text{ of length } m_{i,j}, \quad \kappa := \sum_{i<k\le j} |u|_k, \quad \text{and } \mu := \sum_{i\le k<j} |v|_k.$$

In other words, the number κ is the number of occurrences of letters s_k in u such that $i < k \le j$ and μ is the number of occurrences of letters s_k in v such that $i \le k < j$. The S-sign function σ satisfies

$$\sigma(ub_{i,j}v) = \begin{cases} (-1)^{\frac{m_{i,j}}{2}} \sigma(ub_{j,i}v), & \text{if } m_{i,j} \text{ is even,} \\ (-1)^{\kappa+\mu} \sigma(ub_{j,i}v), & \text{if } m_{i,j} \text{ is odd.} \end{cases}$$

Example 3.6 (Dihedral Group $I_2(m)$)**.** Let $W = I_2(m)$, with $m \ge 2$. The S-sign function for the reduced words is determined by the residue of m mod 4, see Figure 2.

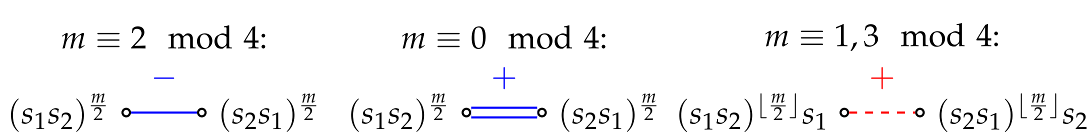

Figure 2: The S-sign for reduced words of $w_\circ$ for the dihedral group $I_2(m)$. Since the S-sign values vary within the same residue class, we label the edge by the product of the S-signs of its vertices.

Example 3.7 (Symmetric group $\mathfrak{S}_4 = A_3$)**.** Let $W = A_3$. The S-sign function for the reduced words of $w_\circ$ *is not* equal to the T-sign, see Figure 3.

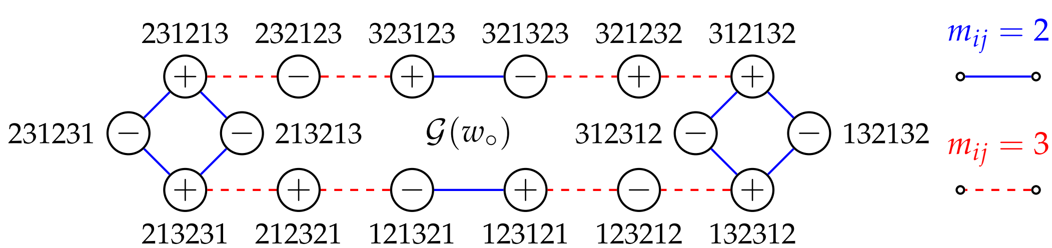

Figure 3: The S-sign for reduced words of $w_\circ$ in type A_3

The punctual sign function.

Definition 3.8 (Punctual sign function ⧖)**.** The *punctual sign function* ⧖ is defined as

$$\begin{aligned} \text{⧖} : \mathcal{R}(w_\circ) &\to \{+1, -1\} \\ w &\mapsto \sigma(w) \cdot \tau(w), \end{aligned}$$

where σ is the S-sign function on S^* and τ is the T-sign function on $\mathcal{R}(w_\circ)$.

We henceforth fix the T-sign of the lexicographically first reduced subword of $w_\circ$ occuring in $(s_1 \cdots s_n)^\infty$ to have positive sign. Set $w = ub_{i,j}v$ and $w' = ub_{j,i}v$ with $\ell(b_{i,j}) = m_{i,j}$ as in Theorem 3.5, then

$$\text{⧖}(w) = \begin{cases} \text{⧖}(w') & \text{if } m_{i,j} \equiv 2 \mod 4, \\ -\text{⧖}(w') & \text{if } m_{i,j} \equiv 0 \mod 4, \\ (-1)^{\kappa+\mu}\text{⧖}(w') & \text{if } m_{i,j} \equiv 1 \text{ or } 3 \mod 4. \end{cases}$$

Example 3.9 (Dihedral Group $I_2(m)$)**.** Let $W = I_2(m)$, with $m \geq 2$. The punctual sign function for the reduced words of $w_\circ$ is determined by the residue of m mod 4, see Figure 4.

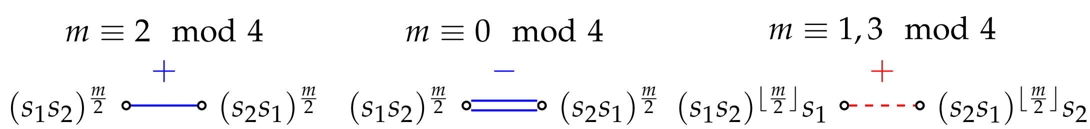

Figure 4: The punctual signs for reduced words of the dihedral group $I_2(m)$. Since the punctual sign values varies within the same residue class, we label the edge by the product of the punctual signs of its vertices.

Example 3.10 (Symmetric group $\mathfrak{S}_4 = A_3$)**.** Let $W = A_3$. The punctual sign function is illustrated in Figure 5.

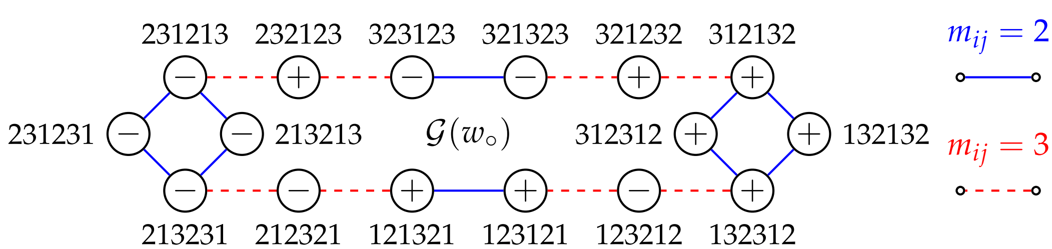

Figure 5: The punctual sign function for reduced words of the group A_3

4 Parameter tensors and model matrices for reduced words

In this section, we give a factorization formula for the determinant of matrices of the form

$$\Big(f_{i,j}(x_j)\Big)_{i,j\in[k]},$$

where $f_{i,j}(x_j)$ is a polynomial in $\mathbb{R}[x_j]$ and certain subsets of columns are equal up to a change of variable depending on some reduced word for $w_\circ$. To do so, we fix a certain Coxeter group W and define certain tensors. A *parameter tensor* $\mathcal{P}^i{}_{s,k}(W)$ is a tensor in $V_N \otimes {V_n}^* \otimes {V_\nu}^*$ over $\mathbb{R}$. Given a parameter tensor $\mathcal{P} = (p^i{}_{s,j})_{(i,j,s)\in[N]\times S\times\{0,\ldots,\nu-1\}}$ and a reduced word $v = v_1v_2\ldots v_N \in \mathcal{R}(w_\circ)$, the *model matrix* of v with respect to $\mathcal{P}$ is the $(N\times N)$-matrix

$$M^i{}_l(v,\mathcal{P}) := \mathcal{C}^i{}_{j,k}(v,\mathcal{P}) \cdot \mathcal{T}^{k,j}{}_l(\nu,N),$$

where

$$\mathcal{C}^i{}_{j,k}(v,\mathcal{P}) := \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{d} p^i{}_{v_j,k-1}\,\mathbf{e}^i\otimes\mathbf{f}_j\otimes\mathbf{f}_k,$$

and

$$\mathcal{T}^{k,j}{}_l(\nu,N) := \sum_{j=1}^{N}\sum_{l=j}^{j}\left(\sum_{k=1}^{\nu} x_l^{k-1}\right)\mathbf{e}^k\otimes\mathbf{e}^j\otimes\mathbf{f}_l.$$

We proceed to express $\det M(v,\mathcal{P})$ for some reduced word $v\in\mathcal{R}(w_\circ)$ in terms of maximal minors of $\mathcal{P}$. Columns of $\mathcal{P}$ are labeled by tuples (s_i,r_j) where $s_i\in S$ and $r_j\in\{0,\ldots,\nu-1\}$. Let $\alpha_v=(c_i)_{s_i\in S}$ be the abelian vector of v and define the following subsets of columns of $\mathcal{P}$:

$$\mathfrak{Z}_{\alpha_v} := \{\mathfrak{z}\subseteq S\times\{0,\ldots,\nu-1\} : \mathfrak{z} \text{ contains exactly } c_i \text{ elements } (s_i,\cdot), \forall s_i\in S\}.$$

Given $\mathfrak{z}\in\mathfrak{Z}_{\alpha_v}$, for every $i\in[n]$, order decreasingly the elements of $R_i := [r_j : \text{if } v_j = s_i]_{j=1}^N$ and substract $c_i - j$ to the element at position j (starting at $j=1$) to obtain the *standard partition* $\lambda_{\mathfrak{z},i}$. Let $\Lambda_{\mathfrak{z}}$ denote the sequence of partitions $(\lambda_{\mathfrak{z},i})_{i=1}^n$. Finally, let

$$\mathcal{V}(v) := \prod_{\substack{s_i\in S\\ c_i\geq 2}}\ \prod_{\substack{v_j=v_k=s_i\\ j<k}} (x_k - x_j).$$

We denote by $[\mathcal{P}]_{\mathfrak{z}}$ the submatrix of $\mathcal{P}$ formed by the columns indexed by the set $\mathfrak{z}$.

Theorem 4.1. *Let $\mathcal{P}$ be a parameter tensor for a Coxeter system (W,S), $v\in\mathcal{R}(w_\circ)$, and $\Omega_v := (\{v\}_1,\ldots,\{v\}_n)$ be the ordered set partition of $[N]$ determined by v. The determinant of the model matrix $M(v,\mathcal{P})$ of v with respect to $\mathcal{P}$ is the multivariate polynomial*

$$\det M(v,\mathcal{P}) = \sigma(v)\mathcal{V}(v)\sum_{\mathfrak{z}\in\mathfrak{Z}_{\alpha_v}} \det[\mathcal{P}]_{\mathfrak{z}}\mathcal{S}_{\Lambda_{\mathfrak{z}},\Omega_v},$$

where $\Lambda_\mathfrak{z} = (\lambda_{\mathfrak{z},1}, \dots, \lambda_{\mathfrak{z},n})$, *and* $\mathcal{S}_{\Lambda_\mathfrak{z},\Omega_v}$ *is the partial Schur function with respect to* $\Lambda_\mathfrak{z}$ *and* Ω_v, *as defined in Section 2.1.*

Corollary 4.2. *If* $x_i > 0$ *for all* $i \in [N]$ *and* $x_j > x_i$ *whenever* $i < j$ *and* $v_i = v_j$, *then*

$$\operatorname{sign}(\det M(v, \mathcal{P})) = \sigma(v) \operatorname{sign}\left(\sum_{\mathfrak{z} \in \mathfrak{Z}_{\alpha_v}} \det[\mathcal{P}]_\mathfrak{z} \mathcal{S}_{\Lambda_\mathfrak{z},\Omega_v}\right).$$

In the article [2], the construction of fans is based on *counting matrices*, whose entries enumerate occurrences of certain subwords contained in a fixed word. These counting matrices turn out to be signature matrices. The factorizations of the determinants of counting matrices were critical in order to prove the correctness of the construction. Theorem 4.1 shed some new light on the factorization formulas of counting matrices: it gives a complete description of the factorizations of counting matrices. Furthermore, for any finite irreducible Coxeter group, it precisely dictates how one may obtain signature matrices through parameter tensors.

5 Universality of parameter tensors

We show that parameter tensors are universal in the following sense: Given a complete simplicial fan $\mathcal{F}$ supported by a vector configuration $\mathbf{A}$ realizing a subword complex $\Delta_W(p)$ and a Gale dual $\mathbf{B} \in \operatorname{Gale}(\mathbf{A})$, there exists a parameter tensor $\mathcal{P}_\mathbf{A}$ that parametrizes $\mathbf{B}$. Concretely, consider some matrix $\mathbf{A} \in \mathbb{R}^{(m-N)\times m}$, whose column i correspond to the i-th letter p_i of p and a Gale dual $\mathbf{B} \in \operatorname{Gale}(\mathbf{A})$. For each letter $s_j \in S$, consider the columns $\{p\}_j$ of $\mathbf{B}$. For each coordinate $k \in [N]$ of the columns, it is possible to find a polynomial of degree at most $|p|_j - 1$ that interpolates the values at these occurrences. There are many possibilities to do so; to get a specific choice, we consider the two-dimensional points $(i, \mathbf{B}(k,i))$ for the k-th coordinate, where $\mathbf{B}(k,i)$ denotes the the k-th entry of the i-th column of $\mathbf{B}$. This way, as i increases, so does the first entry $x_i := i$. Doing this for each letter $s_j \in S$, we get a parameter tensor $\mathcal{P}_\mathbf{B}$ with ν replaced by $\max\{|p|_j - 1 \ : \ j \in [n]\}$. In order to know if $\mathbf{B}$ is a signature matrix for p, we use Corollary 4.2.

Theorem 5.1. *Let* $p \in S^m$ *and* $\mathbf{A} \in \mathbb{R}^{(m-N)\times m}$. *Further let* $\mathcal{P}_\mathbf{B}$ *denote the parameter tensor associated to a Gale dual* $\mathbf{B} \in \operatorname{Gale}(\mathbf{A})$ *as above. In particular, assume that* $x_i > 0$ *for all* $i \in [N]$ *and* $x_j > x_i$ *whenever* $i < j$ *and* $v_i = v_j$. *The matrix* $\mathbf{B}$ *is a signature matrix for* p *if and only if*

$$\operatorname{sign}\left(\sum_{\mathfrak{z} \in \mathfrak{Z}_{\alpha_v}} \det[\mathcal{P}_\mathbf{B}]_\mathfrak{z} \mathcal{S}_{\Lambda_\mathfrak{z},\Omega_v}\right) = \Xi(v) = \sigma(v)\tau(v)$$

for every occurrence of each reduced word v *of* $w_\circ$ *which is a subword of* p.

After doing a braid move of length 2 on v, the variables x_i's on the left-hand side of the equation get relabeled, the minors $[\mathcal{P}_{\mathbf{B}}]_{\mathfrak{z}}$'s remain unchanged, and the right-hand side remains invariant. This yields exactly one equation up to labeling of the x_i's to be fulfilled for each commutation class. Furthermore, two reduced words with the same abelian vector have equal left-hand sides up to relabeling of the x_i's. Hence for each abelian vector α_v, there is either 1 or 2 equalities up to relabeling of the x_i's that have to be fulfilled depending on whether the punctual sign is constant for the abelian vector α_v.

The definition of signature matrix involves checking an equation for *every* occurrence of *every* reduced word directly on the Gale dual $\mathbf{B}$. Under the condition that the numbers $\{x_i\}_{i\in[r]}$ increase on occurrences of letters in S, these conditions can be expressed with at most two explicit conditions per abelian vectors and the values of the x_i's play a less significant role. Thus, the previous theorem allows to reduce the study of signature matrices to that of minors of parameter tensors.

Theorem 5.2 (Universality of parameter tensors)**.** *Let $p \in S^m$ and $\mathcal{F}_{p,\mathbf{A}}$ be a complete simplicial fan realizing the subword complex $\Delta_W(p)$ for some matrix $\mathbf{A} \in \mathbb{R}^{(m-N)\times m}$. There exist a parameter tensor $\mathcal{P}_{\mathbf{B}}$, and m real numbers $x_i > 0$, with $i \in [m]$, such that*

- *$i < j$ and $p_i = p_j$ implies $x_i < x_j$, and*
- *for every occurrence of each reduced word v of $w_\circ$ which is a subword of p, the following equality holds*
$$\operatorname{sign}\left(\sum_{\mathfrak{z}\in\mathfrak{Z}_{\alpha_v}} \det[\mathcal{P}_{\mathbf{B}}]_{\mathfrak{z}}\mathcal{S}_{\Lambda_{\mathfrak{z}},\Omega_v}\right) = \mathbb{X}(v) = \sigma(v)\tau(v).$$

Thus, in order to obtain geometric realizations (either as fans or polytopes) of subword complexes, the oriented matroids described by parameter tensors constitute natural and universal objects to study. Finally, this theorem reveals how both the S- and T-sign functions, and partial Schur functions *lay at the heart of geometrical realizations of cyclic polytopes, (generalized) associahedra, and subword complexes*.

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