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The Petrie symmetric functions

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Abstract. For any positive integer k and nonnegative integer m, we consider the symmetric function $G(k, m)$ defined as the sum of all monomials of degree m that contain no exponents larger than $k-1$. We call $G(k, m)$ a *Petrie symmetric function* in honor of Flinders Petrie, as the coefficients in its expansion in the Schur basis are determinants of Petrie matrices (and thus belong to $\{-1, 0, 1\}$ by a classical result of Gordon and Wilkinson). More generally, we prove a Pieri-like rule for expanding a product of the form $G(k, m) \cdot s_\mu$ in the Schur basis whenever μ is a partition; all coefficients in this expansion belong to $\{-1, 0, 1\}$. We show a further formula for $G(k, m)$ and prove that $G(k, 1), G(k, 2), G(k, 3), \ldots$ form an algebraically independent generating set for the symmetric functions when $1-k$ is invertible in the base ring. We prove a conjecture of Liu and Polo about the expansion of $G(k, 2k-1)$ in the Schur basis. We then take our Pieri-like rule as an impetus to pose a different question: What other symmetric functions f have the property that each product fs_μ expands in the Schur basis with all coefficients belonging to $\{-1, 0, 1\}$? We call this property *MNability* due to its most classical instance (besides the Pieri rules, which don't use -1 coefficients) being the Murnaghan–Nakayama rule. Surprisingly, we find a number of infinite families of MNable symmetric functions besides the classical ones.

Keywords: symmetric functions, Petrie matrices, Murnaghan–Nakayama rule, Pieri rules, Schur functions, determinants

1 Introduction

In the course of computing the cohomology of a line bundle in characteristic p, Liu and Polo [7] have encountered a symmetric function that can be defined as the sum of all monomials of degree $2p-1$ that contain no exponents larger than $p-1$. Using representation theory, they found a simple expansion of this function in the Schur basis [7, Corollary 1.4.4], which prompted them to ask whether this expansion also holds for non-prime p (in which case their argument no longer applies).

Indeed, it does (see Section 7 below). From a combinatorial point of view, it is natural to study an even more general family of symmetric functions: We fix a commutative ring $\mathbf{k}$. For any integers $k \geq 1$ and $m \geq 0$, we let $G(k, m)$ be the sum of all monomials (in $x_1, x_2, x_3, \ldots$) of degree m that contain no exponents larger than $k-1$. This $G(k, m)$

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is a symmetric function; moreover, if it is expanded in the Schur basis of the ring of symmetric functions, then the coefficients can be expressed as determinants of *Petrie matrices* (i.e., matrices with each column having the form $(0, 0, \ldots, 0, 1, 1, \ldots, 1, 0, 0, \ldots, 0)^T$). Thus, we call $G(k, m)$ a *Petrie function* in honor of Flinders Petrie, whose invention of contextual seriation gave birth to the notion of Petrie matrices. By a result of Gordon and Wilkinson [4], Petrie matrices are unimodular; thus, the coefficients in the Schur expansion of $G(k, m)$ belong to $\{-1, 0, 1\}$.

More generally, if μ is any partition, then we can expand the product $G(k, m) \cdot s_\mu$ in the Schur basis; all coefficients in this Pieri-like expansion are determinants of Petrie matrices as well (and thus belong to $\{-1, 0, 1\}$). At least for $\mu = \varnothing$, the coefficients have a combinatorial interpretation.

We show a further formula for $G(k, m)$ in terms of Frobenius homomorphisms $\mathbf{f}_n$ (also known as plethysm by the power-sum function p_n), and we use it to show that $G(k, 1), G(k, 2), G(k, 3), \ldots$ form an algebraically independent generating set for the symmetric functions when $1 - k$ is invertible in $\mathbf{k}$.

We then revisit our expansion of $G(k, m) \cdot s_\mu$ to ask a more general question (Section 8): What other symmetric functions f have the property that each product $f s_\mu$ expands in the Schur basis with all coefficients belonging to $\{-1, 0, 1\}$? We call such f *MNable*; examples of MNable symmetric functions are the classical functions h_m, e_m, p_m (by the Pieri and the Murnaghan–Nakayama rule, the latter of which gave MNability its name) and the Petrie functions $G(k, m)$ (by the above). Surprisingly, we have found several other MNable symmetric functions, such as the products $p_i p_j$ with $i \neq j$, or the differences $h_m - e_m$, $h_m - p_m$ and $h_m - p_m - e_m$ for even m.

Most results in this abstract are proved in the draft [5].

Some results below (in particular, Theorems 4.4 and 4.6 in an equivalent form) have been independently found by H. Fu and Z. Mei [2].

2 Definitions

Our notations follow [6, Chapter 2]. We let $\mathbb{N} = \{0, 1, 2, \ldots\}$.

We fix a commutative ring $\mathbf{k}$. We let Λ denote the ring of symmetric functions (i.e., symmetric power series of bounded degree) in infinitely many variables $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$. This is a $\mathbf{k}$-subalgebra of the $\mathbf{k}$-algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of formal power series.

A *weak composition* means an infinite sequence $(\alpha_1, \alpha_2, \alpha_3, \ldots)$ of nonnegative integers such that only finitely many i satisfy $\alpha_i \neq 0$. If α is a weak composition, then α_i is the i-th entry of α (so that $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$), and $|\alpha|$ is the sum $\alpha_1 + \alpha_2 + \alpha_3 + \cdots \in \mathbb{N}$ (and is called the *size* of α). We let WC denote the set of all weak compositions.

For any weak composition α, we let $\mathbf{x}^\alpha$ denote the monomial $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots$. These monomials $\mathbf{x}^\alpha$ are all the monomials in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$.

A weak composition α will be identified with the ℓ-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ whenever $\ell \in \mathbb{N}$ satisfies $\alpha_{\ell+1} = \alpha_{\ell+2} = \alpha_{\ell+3} = \cdots = 0$.

A *partition* means a weak composition α such that $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots$. We let Par denote the set of all partitions. For each $n \in \mathbb{N}$, we let Par_n denote the set of all partitions α satisfying $|\alpha| = n$.

The $\mathbf{k}$-module Λ has several bases indexed by the set Par. The simplest one is the *monomial basis* $(m_\lambda)_{\lambda \in \operatorname{Par}}$, whose elements m_λ are the sums of the orbits of the monomials $\mathbf{x}^\alpha$ under the "permutation of variables" action of the infinite symmetric group. More precisely, for any partition λ, we can define the *monomial symmetric function* $m_\lambda \in \Lambda$ by

$$m_\lambda = \sum \mathbf{x}^\alpha,$$

where the sum ranges over all weak compositions $\alpha \in \mathrm{WC}$ that can be obtained from λ by permuting entries. For example,

$$m_{(2,2,1)} = \sum_{i<j<k} x_i^2 x_j^2 x_k + \sum_{i<j<k} x_i^2 x_j x_k^2 + \sum_{i<j<k} x_i x_j^2 x_k^2.$$

As λ ranges over all of Par, the symmetric functions m_λ form a basis of the $\mathbf{k}$-module Λ.

Other prominent symmetric functions in Λ are:

- the *complete homogeneous symmetric functions* h_n defined for all $n \in \mathbb{Z}$ by

  $$h_n = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n} = \sum_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = n}} \mathbf{x}^\alpha = \sum_{\lambda \in \operatorname{Par}_n} m_\lambda.$$

  (Thus, $h_0 = 1$ and $h_n = 0$ for all $n < 0$.)

- the *elementary symmetric functions* e_n defined for all $n \in \mathbb{Z}$ by

  $$e_n = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n} = \sum_{\substack{\alpha \in \mathrm{WC} \cap \{0,1\}^\infty; \\ |\alpha| = n}} \mathbf{x}^\alpha.$$

  (Thus, $e_0 = 1$ and $e_n = 0$ for all $n < 0$. If $n > 0$, then $e_n = m_{(1,1,\ldots,1)}$, where $(1, 1, \ldots, 1)$ is an n-tuple.)

- the *power-sum symmetric functions* p_n defined for all positive integers n by

  $$p_n = x_1^n + x_2^n + x_3^n + \cdots = m_{(n)}.$$

But most remarkable of all are the *Schur functions* s_λ for $\lambda \in \operatorname{Par}$. One way to define the Schur function s_λ corresponding to a partition λ is as follows:

$$s_\lambda = \sum \mathbf{x}_T,$$

where the sum ranges over all semistandard tableaux T of shape λ, and where $\mathbf{x}_T$ denotes the monomial obtained by multiplying the x_i for all entries i of T. The fact that $s_\lambda \in \Lambda$ is nontrivial (see, e.g., [6, Proposition 2.2.4]); the s_λ are rich in interesting and nontrivial properties ([6, Chapter 2], [8, Chapter 7], etc.). In particular, the family $(s_\lambda)_{\lambda \in \mathrm{Par}}$ is a basis of the $\mathbf{k}$-module Λ, known as the *Schur basis*.

3 Definition of the Petrie functions

We are now ready to define the functions we will study:

Definition 3.1. (a) For any positive integer k, we let

$$G(k) = \sum_{\substack{\alpha \in \mathrm{WC}; \\ \alpha_i < k \text{ for all } i}} \mathbf{x}^\alpha .$$

This is a symmetric formal power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ (but does not lie in Λ in general, since it contains monomials of arbitrarily high degrees).
(b) For any positive integer k and any $m \in \mathbb{N}$, we let

$$G(k, m) = \sum_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = m; \\ \alpha_i < k \text{ for all } i}} \mathbf{x}^\alpha \in \Lambda .$$

For example,

$$\begin{aligned} G(3,4) &= \sum_{i<j<k<\ell} x_i x_j x_k x_\ell + \sum_{i<j<k} x_i^2 x_j x_k + \sum_{i<j<k} x_i x_j^2 x_k + \sum_{i<j<k} x_i x_j x_k^2 + \sum_{i<j} x_i^2 x_j^2 \\ &= m_{(1,1,1,1)} + m_{(2,1,1)} + m_{(2,2)} . \end{aligned}$$

We suggest to name $G(k)$ and $G(k, m)$ the *Petrie functions*, for reasons that Theorem 4.4 and Corollary 4.5 will elucidate. We begin with some easy facts:

Proposition 3.2. *Let k be a positive integer. Then,*

$$G(k) = \sum_{\substack{\alpha \in \mathrm{WC}; \\ \alpha_i < k \text{ for all } i}} \mathbf{x}^\alpha = \sum_{\substack{\lambda \in \mathrm{Par}; \\ \lambda_i < k \text{ for all } i}} m_\lambda = \prod_{i=1}^{\infty} \left(x_i^0 + x_i^1 + \cdots + x_i^{k-1} \right) .$$

Proposition 3.3. *Let k be a positive integer. Let $m \in \mathbb{N}$.*

(a) *The symmetric function $G(k, m)$ is the m-th degree homogeneous component of $G(k)$.*

(b) *We have*

$$G(k,m) = \sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=m;\\ \alpha_i<k \text{ for all } i}} \mathbf{x}^{\alpha} = \sum_{\substack{\lambda\in\mathrm{Par};\\ |\lambda|=m;\\ \lambda_i<k \text{ for all } i}} m_{\lambda}.$$

(c) *If $k > m$, then $G(k,m) = h_m$.*

(d) *If $k = 2$, then $G(k,m) = e_m$.*

Parts **(c)** and **(d)** of Proposition 3.3 show that the Petrie functions $G(k,m)$ can be seen as interpolating between the h_m and the e_m. Another easily established identity is $G(m,m) = h_m - p_m$ for each positive integer m.

It is also not hard to see that the comultiplication Δ of the Hopf algebra Λ (see [6, Section 2.1] for its definition) satisfies

$$\Delta(G(k,m)) = \sum_{i=0}^{m} G(k,i) \otimes G(k,m-i)$$

for each $k > 0$ and $m \in \mathbb{N}$.

The Petrie function $G(3)$ has appeared in [8, Exercise 7.3], where it was expanded as a polynomial in $e_1, e_2, e_3, \ldots$ (a result of Gessel). We shall now expand $G(k)$ and $G(k,m)$ in terms of Schur functions. For this, we need to define some notations.

4 The Schur expansions of $G(k)$ and $G(k,m)$

If $\mathcal{A}$ is any logical statement, then $[\mathcal{A}]$ shall denote the *truth value* of $\mathcal{A}$ (that is, 1 if $\mathcal{A}$ is true, and 0 if $\mathcal{A}$ is false). We use the notation $(a_{i,j})_{1\leq i\leq \ell,\ 1\leq j\leq \ell}$ for the $\ell\times\ell$-matrix whose (i,j)-th entry is $a_{i,j}$ for each $i,j \in \{1,2,\ldots,\ell\}$.

Definition 4.1. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \mathrm{Par}$, and let k be a positive integer. Then, the *k-Petrie number* $\mathrm{pet}_k(\lambda)$ of λ is the integer defined by

$$\mathrm{pet}_k(\lambda) = \det\left(([0 \leq \lambda_i - i + j < k])_{1\leq i\leq \ell,\ 1\leq j\leq \ell}\right).$$

Note that this integer does not depend on the choice of ℓ (in the sense that it does not change if we enlarge ℓ by adding trailing zeroes to the representation of λ).

Example 4.2. Let λ be the partition $(3,1,1) \in \mathrm{Par}$, let $\ell = 3$, and let k be a positive integer. Then, the definition of $\mathrm{pet}_k(\lambda)$ yields

$$\begin{aligned}
\mathrm{pet}_k(\lambda) &= \det\begin{pmatrix} [0\leq \lambda_1 < k] & [0\leq \lambda_1+1<k] & [0\leq\lambda_1+2<k] \\ [0\leq\lambda_2-1<k] & [0\leq\lambda_2<k] & [0\leq\lambda_2+1<k] \\ [0\leq\lambda_3-2<k] & [0\leq\lambda_3-1<k] & [0\leq\lambda_3<k]\end{pmatrix} \\
&= \det\begin{pmatrix} [0\leq 3<k] & [0\leq 4<k] & [0\leq 5<k] \\ [0\leq 0<k] & [0\leq 1<k] & [0\leq 2<k] \\ [0\leq -1<k] & [0\leq 0<k] & [0\leq 1<k]\end{pmatrix}
\end{aligned}$$

(since $\lambda_1 = 3$ and $\lambda_2 = 1$ and $\lambda_3 = 1$). Thus, taking $k = 4$, we obtain

$$\operatorname{pet}_4(\lambda) = \det \begin{pmatrix} [0 \leq 3 < 4] & [0 \leq 4 < 4] & [0 \leq 5 < 4] \\ [0 \leq 0 < 4] & [0 \leq 1 < 4] & [0 \leq 2 < 4] \\ [0 \leq -1 < 4] & [0 \leq 0 < 4] & [0 \leq 1 < 4] \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 0.$$

On the other hand, taking $k = 5$, we obtain

$$\operatorname{pet}_5(\lambda) = \det \begin{pmatrix} [0 \leq 3 < 5] & [0 \leq 4 < 5] & [0 \leq 5 < 5] \\ [0 \leq 0 < 5] & [0 \leq 1 < 5] & [0 \leq 2 < 5] \\ [0 \leq -1 < 5] & [0 \leq 0 < 5] & [0 \leq 1 < 5] \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = -1.$$

Proposition 4.3. *Let $\lambda \in \operatorname{Par}$, and let k be a positive integer. Then, $\operatorname{pet}_k(\lambda) \in \{-1, 0, 1\}$.*

Proof sketch. We will use the concept of Petrie matrices (see [4, Theorem 1]). Each row of the matrix $([0 \leq \lambda_i - i + j < k])_{1 \leq i \leq \ell,\ 1 \leq j \leq \ell}$ has the form

$$(\underbrace{0, 0, \ldots, 0}_{a \text{ zeroes}}, \underbrace{1, 1, \ldots, 1}_{b \text{ ones}}, \underbrace{0, 0, \ldots, 0}_{c \text{ zeroes}}) \qquad \text{for some } a, b, c \in \mathbb{N} \text{ (where any of } a, b, c \text{ can be 0).}$$

Thus, the matrix $([0 \leq \lambda_i - i + j < k])_{1 \leq i \leq \ell,\ 1 \leq j \leq \ell}$ is the transpose of a Petrie matrix. Hence, its determinant belongs to $\{-1, 0, 1\}$ (since [4, Theorem 1] shows that the determinant of any square Petrie matrix belongs to $\{-1, 0, 1\}$). □

We can now expand the Petrie symmetric functions $G(k)$ in the basis $(s_\lambda)_{\lambda \in \operatorname{Par}}$ of Λ:

Theorem 4.4. *Let k be a positive integer. Then,*

$$G(k) = \sum_{\lambda \in \operatorname{Par}} \operatorname{pet}_k(\lambda)\, s_\lambda.$$

Corollary 4.5. *Let k be a positive integer. Let $m \in \mathbb{N}$. Then,*

$$G(k, m) = \sum_{\lambda \in \operatorname{Par}_m} \operatorname{pet}_k(\lambda)\, s_\lambda.$$

These two results are particular cases of more general facts stated below (Theorem 5.3 and Corollary 5.4).

The k-Petrie numbers can be described more explicitly:

Theorem 4.6. *Let $\lambda \in \operatorname{Par}$, and let k be a positive integer. Let $\mu = \lambda^t$ be the* conjugate *of λ (that is, the partition μ defined by setting $\mu_i = |\{j \geq 1 \mid \lambda_j \geq i\}|$ for all i).*

(a) *If $\mu_k \neq 0$, then $\operatorname{pet}_k(\lambda) = 0$. From now on, let us assume $\mu_k = 0$. For each $i \in \{1, 2, \ldots, k-1\}$, let γ_i be the unique element of $\{1, 2, \ldots, k\}$ that is congruent to $\mu_i - i$ modulo k.*

(b) *If the $k-1$ numbers $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ are not distinct, then $\operatorname{pet}_k(\lambda) = 0$.*

(c) *If the $k-1$ numbers $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ are distinct, then $\operatorname{pet}_k(\lambda)$ is a certain power of -1 (see [5] for details).*

5 A "Pieri" rule

It turns out that Theorem 4.4 can be generalized. For that, we need to define a "relative" version of Petrie numbers:

Definition 5.1. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \operatorname{Par}$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \in \operatorname{Par}$, and let k be a positive integer. Then, the *k-Petrie number* $\operatorname{pet}_k(\lambda, \mu)$ of λ and μ is the integer defined by

$$\operatorname{pet}_k(\lambda, \mu) = \det\left(\left([0 \leq \lambda_i - \mu_j - i + j < k]\right)_{1 \leq i \leq \ell,\ 1 \leq j \leq \ell}\right).$$

Note that this integer does not depend on the choice of ℓ (in the sense that it does not change if we enlarge ℓ by adding trailing zeroes to the representations of λ and μ).

The following proposition generalizes (and is proved similarly to) Proposition 4.3:

Proposition 5.2. *Let $\lambda \in \operatorname{Par}$ and $\mu \in \operatorname{Par}$, and let k be a positive integer. Then, $\operatorname{pet}_k(\lambda, \mu) \in \{-1, 0, 1\}$.*

Now, we have the following generalizations of Theorem 4.4 and Corollary 4.5:

Theorem 5.3. *Let k be a positive integer. Let $\mu \in \operatorname{Par}$. Then,*

$$G(k) \cdot s_\mu = \sum_{\lambda \in \operatorname{Par}} \operatorname{pet}_k(\lambda, \mu) s_\lambda.$$

Corollary 5.4. *Let k be a positive integer. Let $m \in \mathbb{N}$. Let $\mu \in \operatorname{Par}$. Then,*

$$G(k, m) \cdot s_\mu = \sum_{\lambda \in \operatorname{Par}_{m+|\mu|}} \operatorname{pet}_k(\lambda, \mu) s_\lambda.$$

We have two proofs of Theorem 5.3: one using the skew Cauchy and the Jacobi–Trudi identities, and another using the approach to Schur polynomials via alternants. (See [5] for the second proof.) Corollary 5.4 easily follows from Theorem 5.3.

We are not aware of any combinatorial rules for $\operatorname{pet}_k(\lambda, \mu)$ other than the (general, but recursive and rather indirect) algorithmic description given in [4] for determinants of arbitrary square Petrie matrices.

6 The Frobenius formula and Petrie generating sets

We shall next state another formula for the Petrie symmetric functions $G(k, m)$. For this formula, we need the following definition ([6, Exercise 2.9.9]):

Definition 6.1. Let n be a positive integer. We define a map $\mathbf{f}_n : \Lambda \to \Lambda$ by setting

$$\mathbf{f}_n(a) = a\left(x_1^n, x_2^n, x_3^n, \ldots\right) \qquad \text{for each } a \in \Lambda.$$

This map $\mathbf{f}_n$ is called the *n-th Frobenius endomorphism* of Λ.

It is known (e.g., [6, Exercise 2.9.9(d)]) that this map $\mathbf{f}_n : \Lambda \to \Lambda$ is a $\mathbf{k}$-algebra endomorphism of Λ (and even a Hopf algebra endomorphism, using the appropriate Hopf structure). In terms of plethysm ([8, Ch. 7, Definition A.2.6]), it is simply described by $\mathbf{f}_n(a) = a[p_n]$ for each $a \in \Lambda$ (and also by $\mathbf{f}_n(a) = p_n[a]$ if $\mathbf{k} = \mathbb{Z}$).

We now have a new formula for $G(k, m)$:

Theorem 6.2. *Let k be a positive integer. Let $m \in \mathbb{N}$. Then,*

$$G(k,m) = \sum_{i \in \mathbb{N}} (-1)^i h_{m-ki} \cdot \mathbf{f}_k(e_i).$$

(The sum on the right hand side of this equality is well-defined, since all sufficiently large $i \in \mathbb{N}$ satisfy $m - ki < 0$ and thus $h_{m-ki} = 0$.)

This theorem can be proved by an inclusion-exclusion-like computation or using generating functions (the latter proof is given in [5]).

Theorem 6.2 can be used to derive the following:

Theorem 6.3. *Fix a positive integer k. Assume that $1 - k$ is invertible in $\mathbf{k}$. Then, the family $(G(k,m))_{m \geq 1} = (G(k,1), G(k,2), G(k,3), \ldots)$ is an algebraically independent generating set of the commutative $\mathbf{k}$-algebra Λ.*

Thus, products of several elements of this family form a basis of Λ (if $1 - k$ is invertible in $\mathbf{k}$). These bases remain to be studied.

7 The Liu–Polo conjecture

We now sketch the answer to the question posed in [7, Remark 1.4.5] by Liu and Polo. By studying cohomology in positive characteristic, they have proved the following identity for all prime numbers n:

Theorem 7.1. *Let n be an integer such that $n > 1$. Then,*

$$\sum_{\substack{\lambda \in \mathrm{Par}_{2n-1}; \\ (n-1,n-1,1) \triangleright \lambda}} m_\lambda = \sum_{i=0}^{n-2} (-1)^i s_{(n-1,n-1-i,1^{i+1})}. \tag{7.1}$$

Here, the symbol $\triangleright$ stands for *dominance* of partitions (also known as majorization); i.e., for two partitions λ and μ satisfying $|\lambda| = |\mu|$, we have

$$\lambda \triangleright \mu \qquad \text{if and only if} \qquad (\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i \text{ for all } i).$$

Also, the power-like notation "1^{i+1}" in the partition on the right hand side of (7.1) stands for a sequence of $i+1$ entries all equal to 1. Thus, $\left(n-1, n-1-i, 1^{i+1}\right) = (n-1, n-1-i, \underbrace{1, 1, \ldots, 1}_{i+1 \text{ times}})$.

We can prove Theorem 7.1 for all n as follows. The first step is to recognize that the left hand side of (7.1) is $G(n, 2n-1)$, because the partitions $\lambda \in \operatorname{Par}_{2n-1}$ satisfying $(n-1, n-1, 1) \rhd \lambda$ are precisely the partitions $\lambda \in \operatorname{Par}_{2n-1}$ satisfying $\lambda_i < n$ for all i. Theorem 4.4 gives an expansion of $G(n, 2n-1)$ in the Schur basis, if we content ourselves with knowing that the coefficients are n-Petrie numbers. However, we want to know their exact values in order to prove (7.1). Thus, we proceed differently. An application of Theorem 6.2 (or a simple combinatorial argument) yields

$$G(n, n+k) = h_{n+k} - h_k p_n \qquad \text{for each } k \in \{0, 1, \ldots, n-1\}.$$

Thus, in particular, $G(n, 2n-1) = h_{2n-1} - h_{n-1} p_n$.

Now, we recall the *skewing operations* $f^\perp : \Lambda \to \Lambda$ for all $f \in \Lambda$ as defined in [6, Section 2.8] (and in various other places). All we need to know about them is that for each $i \in \mathbb{N}$, the skewing operation $e_i^\perp : \Lambda \to \Lambda$ is the $\mathbf{k}$-linear map that sends each Schur function s_λ to the skew Schur function $s_{\lambda / \left(1^i\right)}$.

For any $m \in \mathbb{N}$, we define a map $\mathbf{B}_m : \Lambda \to \Lambda$ by setting

$$\mathbf{B}_m(f) = \sum_{i \in \mathbb{N}} (-1)^i h_{m+i} e_i^\perp f \qquad \text{for all } f \in \Lambda.$$

It is known ([6, Exercise 2.9.1(a)]) that this map $\mathbf{B}_m$ is well-defined and $\mathbf{k}$-linear. Moreover, [6, Exercise 2.9.1(b)] shows that if $\lambda \in \operatorname{Par}$ and $m \in \mathbb{Z}$ satisfy $m \geq \lambda_1$, then

$$\mathbf{B}_m(s_\lambda) = s_{(m, \lambda_1, \lambda_2, \lambda_3, \ldots)}. \tag{7.2}$$

(This map $\mathbf{B}_m$ is known as the *m-th Bernstein operator* [10, Section 4.20(a)] or — in honor of (7.2) — a *Schur row-adder* [3].) On the other hand, it is not hard to see that

$$\mathbf{B}_m(h_n) = h_m h_n - h_{m+1} h_{n-1} \qquad \text{and} \qquad \mathbf{B}_m(p_n) = h_m p_n - h_{m+n}$$

for each positive integer n and each $m \in \{0, 1, \ldots, n\}$. Using these two equalities, we readily see that

$$\mathbf{B}_{n-1}(h_n - p_n) = h_{2n-1} - h_{n-1} p_n = G(n, 2n-1). \tag{7.3}$$

On the other hand, [6, Exercise 5.4.12(g)] (or the Murnaghan–Nakayama rule) yields

$$p_n = \sum_{i=0}^{n-1} (-1)^i s_{\left(n-i, 1^i\right)}.$$

Subtracting this from $h_n = s_{(n)} = s_{(n-0,1^0)}$, we find

$$h_n - p_n = \sum_{i=0}^{n-2} (-1)^i s_{\left(n-1-i,1^{i+1}\right)}.$$

Applying the map $\mathbf{B}_{n-1}$ to this equality, we obtain

$$\mathbf{B}_{n-1}(h_n - p_n) = \sum_{i=0}^{n-2} (-1)^i \mathbf{B}_{n-1}\left(s_{\left(n-1-i,1^{i+1}\right)}\right) = \sum_{i=0}^{n-2} (-1)^i s_{\left(n-1,n-1-i,1^{i+1}\right)}$$

(by (7.2)). Comparing this with (7.3), we find

$$G(n, 2n-1) = \sum_{i=0}^{n-2} (-1)^i s_{\left(n-1,n-1-i,1^{i+1}\right)}.$$

Since the left hand side of (7.1) is $G(n, 2n-1)$, we have thus proved Theorem 7.1.

8 MNable symmetric functions

Let us now take Corollary 5.4 as inspiration to identify a property of some symmetric functions that appears to have been hitherto unstudied.

Let $\mathbf{k} = \mathbb{Z}$ throughout this section. We recall the Hall inner product $(\cdot,\cdot) : \Lambda \times \Lambda \to \mathbf{k}$; it is the unique $\mathbf{k}$-bilinear form on Λ that satisfies $\left(s_\lambda, s_\mu\right) = \delta_{\lambda,\mu}$ for all $\lambda, \mu \in \operatorname{Par}$. (In other words, it is the unique $\mathbf{k}$-bilinear form on Λ that makes $(s_\lambda)_{\lambda\in\operatorname{Par}}$ into an orthonormal basis.) See [6, Definition 2.5.12] or [8, Section 7.9] (where it is denoted by $\langle\cdot,\cdot\rangle$) for its further properties.

Definition 8.1. **(a)** A symmetric function $f \in \Lambda$ will be called *signed multiplicity-free* if f can be expanded as a linear combination of distinct Schur functions with all coefficients in $\{-1, 0, 1\}$. (That is, if the Hall inner product $\left(f, s_\mu\right)$ is -1, 0 or 1 for each partition μ.)
(b) A symmetric function $f \in \Lambda$ will be called *MNable* if for each partition μ, the product $f s_\mu$ is signed multiplicity-free.

For example, the symmetric function $h_3 p_2$ is signed multiplicity-free, since $h_3 p_2 = s_{(5)} + s_{(3,2)} - s_{(3,1,1)}$; but it is not MNable, since the product

$$h_3 p_2 s_{(2)} = -s_{(3,2,1,1)} + s_{(3,2,2)} - s_{(4,1,1,1)} + s_{(4,3)} - s_{(5,1,1)} + 2 s_{(5,2)} + s_{(6,1)} + s_{(7)}$$

is not signed multiplicity-free (due to the coefficient of $s_{(5,2)}$ being 2).

Roughly speaking, an $f \in \Lambda$ is MNable if and only if there is a **M**urnaghan-**N**akayama-like rule for multiplying Schur functions by f. Thus, the name "**MN**able".

Question 8.2. Which symmetric functions are MNable?

It is not clear whether a full characterization of MNable symmetric functions is even possible. However, there are many. Here is a non-exhaustive list:

Theorem 8.3. (a) *The functions h_i and e_i are MNable for each $i \in \mathbb{N}$.*
(b) *The function p_i is MNable for each positive integer i.*
(c) *The Petrie function $G(k,m)$ and the difference $G(k,m) - h_m$ are MNable for any integers $k \geq 1$ and $m \geq 0$.*
(d) *The differences $h_i - p_i$ and $h_i - e_i$ are MNable for each positive integer i. (This includes $h_1 - e_1 = 0$.)*
(e) *The difference $h_i - p_i - e_i$ is MNable for each **even** positive integer i.*
(f) *The product $p_i p_j$ is MNable whenever $i > j > 0$.*
(g) *The function $m_{(k^n)}$ as well as the differences $h_{nk} - m_{(k^n)}$ and $e_{nk} - (-1)^{(k-1)n} m_{(k^n)}$ are MNable for any positive integers n and k (where (k^n) denotes the n-tuple $(k,k,\ldots,k)$).*
(h) *If some $f \in \Lambda$ is MNable, then so are $-f$ and $\omega(f)$, where $\omega : \Lambda \to \Lambda$ is the* fundamental involution *of Λ (see [6, Section 2.4] or [8, Section 7.6]).*
(i) *A symmetric function $f \in \Lambda$ is MNable if and only if all its homogeneous components are MNable.*
(j) *If $f \in \Lambda$ is MNable and k is a positive integer, then $\mathbf{f}_k(f)$ is MNable. (See Definition 6.1 for the meaning of $\mathbf{f}_k$.)*
(k) *A symmetric function $f \in \Lambda$ is MNable if and only if $(f, s_{\lambda/\mu}) \in \{-1, 0, 1\}$ for each skew partition λ/μ.*

A few telegraphic remarks on the proofs are in order. Part **(a)** of Theorem 8.3 follows from the Pieri and dual Pieri rules, as part **(b)** does from the Murnaghan–Nakayama rule. The $G(k,m)$ claim in part **(c)** follows from Corollary 5.4; the $G(k,m) - h_m$ claim relies on the fact that $\operatorname{pet}_k(\lambda,\mu) \in \{0,1\}$ if λ/μ is a horizontal strip. Parts **(d)** and **(e)** can be shown by analyzing the rare cases in which a skew partition can be two of "horizontal strip", "vertical strip" and "rim hook" at once. Part **(f)** follows from a study of rim hook tableaux. Part **(h)** follows from the facts that ω is an algebra automorphism and sends s_λ to s_{λ^t}. Part **(k)** is easy to see using skewing operators (or simply using the fact that the same Littlewood–Richardson coefficients appear in the formulas $s_\mu s_\nu = \sum_{\lambda \in \operatorname{Par}} c^\lambda_{\mu,\nu} s_\lambda$ and $s_{\lambda/\mu} = \sum_{\nu \in \operatorname{Par}} c^\lambda_{\mu,\nu} s_\nu$). Part **(i)** is easy. Part **(j)** follows from part **(k)** and the SXP algorithm in [1]. The $m_{(k^n)}$ claim in part **(g)** follows from part **(j)** (since $m_{(k^n)} = \mathbf{f}_k(e_n)$); the rest of **(g)** follows by studying skew partitions again.

Note that Theorem 8.3 **(k)** shows that the MNability of a symmetric function can be tested in finite time: For each $d \in \mathbb{N}$, there are only finitely many skew Schur functions $s_{\lambda/\mu}$ of degree d.

The families in parts **(a)–(h)** and **(j)** of Theorem 8.3 cover all MNable homogeneous symmetric functions of degree < 4. In degree 4, we have two further MNable symmetric functions that we were unable to "explain" (i.e., embed in any infinite family):

$$s_{(1,1,1,1)} - s_{(3,1)} + s_{(4)} \qquad \text{and} \qquad s_{(4)} - s_{(2,2)}.$$

While Question 8.2 seems wide open, several particular cases appear manageable: for example, which products of h_i's (or p_i's) are MNable? Note that the only MNable Schur functions are $h_i = s_{(i)}$ and $e_i = s_{(1,1,\ldots,1)}$.

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