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Elliptic and q-analogs of the Fibonomial numbers

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Abstract. In 2009, Sagan and Savage introduced a combinatorial model for the Fibonomial numbers, integer numbers that are obtained from the binomial coefficients by replacing each term by its corresponding Fibonacci number. In this paper, we present a combinatorial description for the q-analog and elliptic analog of the Fibonomial numbers. This is achieved by introducing some q-weights and elliptic weights to a slight modification of the combinatorial model of Sagan and Savage.

Keywords: Fibonomial, Fibonacci, q-analog, elliptic analog, weighted enumeration

1 Introduction

The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ is one of the most important and beautiful sequences in mathematics. It starts with the numbers $F_0 = 0$ and $F_1 = 1$, and is recursively defined by the formula $F_n = F_{n-1} + F_{n-2}$.

Fibonacci analogs of famous numbers, such as the binomial coefficients and Catalan numbers:

$$\binom{m+n}{n} = \frac{(m+n)!}{m! \cdot n!} \qquad \text{and} \qquad \frac{1}{n+1}\binom{2n}{n},$$

have intrigued some mathematicians over the last few years [1, 2, 3, 4, 9, 13]. The *Fibonomial* and *Fibo-Catalan* numbers are defined, respectively, as:

$$\binom{m+n}{n}_{\mathcal{F}} := \frac{F^!_{m+n}}{F^!_m \cdot F^!_n} \qquad \text{and} \qquad \frac{1}{F_{n+1}}\binom{2n}{n}_{\mathcal{F}},$$

where $F^!_n := \prod_{k=1}^{n} F_k$ is the Fibonacci analog of $n!$. These rational expressions turn out to be positive integers. In [9], Sagan and Savage introduced a combinatorial model to interpret the Fibonomial numbers in terms of certain tilings of an $m \times n$ rectangle. A *path-domino tiling* of an $m \times n$ rectangle is a tiling with monominos and dominos and a lattice path from $(0,0)$ to (m,n) is specified, and such that:

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- all tiles above the path are either monominos or horizontal dominos;
- all tiles below the path are either monominos or vertical dominos; and
- all tiles that touch the path from below are vertical dominos.

We call these last tiles touching the path from below *special vertical dominos*, and denote by $\mathcal{T}_{m,n}$ the collection of all path-domino tilings of an $m \times n$ rectangle. An example is illustrated on the left of Figure 1. The following result is a special case of [9, Theorem 3].[1]

Theorem 1.1 ([9]). *The Fibonomial number $\binom{m+n}{n}_{\mathcal{F}}$ counts the number of path-domino tilings of an $m \times n$ rectangle.*

The main objective of this paper is to present a q-analog and an elliptic analog generalization of this result. The resulting q-Fibonomial and elliptic Fibonomial numbers count the number of path-domino tilings of an $m \times n$ rectangle according to their q-weights and elliptic weights, respectively.

2 q-analog of the Fibonomial numbers

We denote by $\mathbb{N} := \{1, 2, 3, \dots\}$ the set of natural numbers. The *q-analog of $n \in \mathbb{N}$* is defined as

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1}.$$

The evaluation of this polynomial at $q = 1$ recovers the number n. To simplify notation, we sometimes omit the subindex q when it is clear from the context. Before studying the q-analog of the Fibonomial numbers, let us recall some useful and known straightforward lemmas.

Lemma 2.1. *For $m, n \in \mathbb{N}$, the following identities hold:*

$$[m+n]_q = [m]_q + q^m [n]_q \tag{2.1}$$

$$[m \cdot n]_q = [m]_q [n]_{q^m} \tag{2.2}$$

It is well known that the Fibonacci number F_n counts the number of tilings of an $(n-1)$-*strip* (a rectangle with diagonal endpoints $(0,0)$ and $(n-1,1)$) using dominos and monominos. Given such a tiling T, we define the *weight* $\omega(T)$ of T as the product of the weights of its tiles, where a monomino has weight 1 and a domino whose top-right coordinate is $(i,1)$ has weight q^{F_i}. The weight of a 0-strip is by definition equal to 1.

[1] The path-domino tilings here are a slight modification of the tilings used in [9]. The only difference is that in [9] the tiles below the path touching the bottom of the rectangle are required to be vertical dominos, while here this condition is required for the tiles below the path touching the path itself. This modification is essential to make our combinatorial model work.

Lemma 2.2 (*cf.* [9])**.** *For $n \in \mathbb{N}$, the q-analog of the Fibonacci numbers*[2] *can be computed as*

$$[F_n]_q = \sum_T \omega(T),$$

where $[F_n]_q = 1 + q + q^2 + \cdots + q^{F_n - 1}$ and the sum ranges over all tilings of an $(n-1)$-strip using dominos and monominos.

Proof. The result is clearly true for $n = 1, 2$. Let $n > 2$, applying (2.1) from Lemma 2.1 we get:

$$[F_n]_q = [F_{n-1} + F_{n-2}]_q = [F_{n-1}]_q + q^{F_{n-1}}[F_{n-2}]_q$$

By induction, the first term of this sum corresponds to the tilings of an $(n-1)$-strip that finish with a monomino, while the second term to the tilings of an $(n-1)$-strip that finish with a domino. $\square$

Lemma 2.3. *For $m, n \in \mathbb{N}$, the following identities hold:*

$$F_{m+n} = F_n F_{m+1} + F_m F_{n-1} \tag{2.3}$$

$$[F_{m+n}]_q = [F_n]_q [F_{m+1}]_{q^{F_n}} + q^{F_n F_{m+1}} [F_m]_q [F_{n-1}]_{q^{F_m}} \tag{2.4}$$

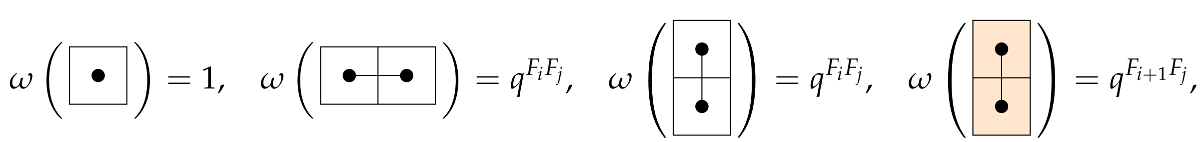
Proof. (2.3) is a well known identity for Fibonacci numbers; see for instance [9, Lemma 1]. Applying Lemma 2.1 to (2.3) leads to (2.4). $\square$

For $m, n \in \mathbb{N}$, the *q-analog of the Fibonomial number* is defined as

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}} := \frac{[F_{m+n}]_q^!}{[F_m]_q^! \cdot [F_n]_q^!}, \tag{2.5}$$

where $[F_n]_q^! := \prod_{k=1}^n [F_k]_q$ is the q-Fibonacci analog of $n!$. Surprisingly, this rational expression turns out to be a polynomial. Our objective is to present a combinatorial model to describe it. In order to achieve this, we will introduce some q-weights associated to path-domino tilings of an $m \times n$ rectangle (with m columns and n rows).

Let $T \in \mathcal{T}_{m,n}$ be a path-domino tiling of an $m \times n$ rectangle. The *q-weights* of the possible tiles in T are defined as follows:



where (i, j) denotes the coordinate of the top-right corner of the tile, and the shaded vertical domino represents a special vertical domino touching the path from below. The *q-weight* of T is defined as the product of the weights of its tiles; see an example in Figure 1. The following theorem is one of our main results.

[2]The elliptic and q-analogs of the Fibonacci numbers we use are different from the one considered *e.g.* in [12].

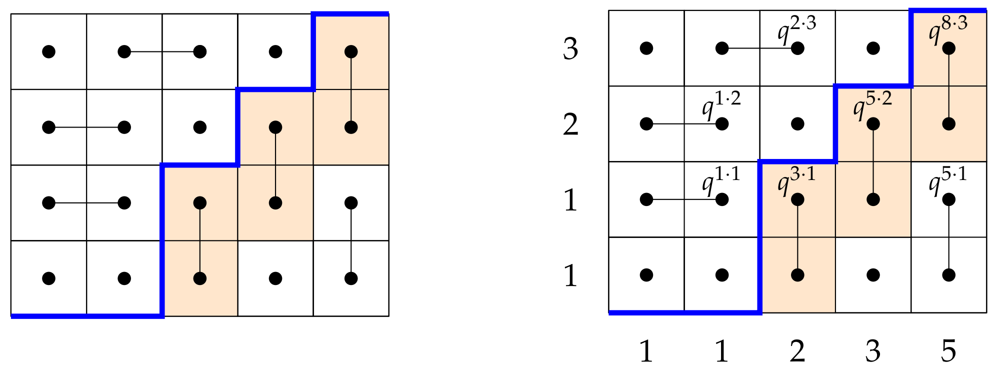


Figure 1: A path-domino tiling T of a 5×4 rectangle (left), and the q-Fibonacci weights of its tiles (right). The weight of the tiling is the product of the weights of its tiles, $\omega(T) = q^{1+2+6+3+10+24+5} = q^{51}$.

Theorem 2.4. *For $m, n \in \mathbb{N}$, the q-analog of the Fibonomial numbers is a polynomial in q with non-negative integer coefficients. It can be computed as*

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}} = \sum_{T \in \mathcal{T}_{m,n}} \omega(T).$$

Proof. Let us start proving the result for the initial cases $m = 1$ or $n = 1$.

For $n = 1$, we have $\left[\begin{smallmatrix} m+1 \\ 1 \end{smallmatrix}\right]_{\mathcal{F}} = [F_{m+1}]_q$. The collection $\mathcal{T}_{m,1}$ coincides with the tilings of an m-strip with dominos and monominos, since only the last step of the specified lattice path can be a north step because of the special vertical domino condition. The weight of a domino in a tiling, whose top-right corner has coordinate $(i, 1)$, is $q^{F_i F_1} = q^{F_i}$. Therefore, the result follows from Lemma 2.2.

For $m = 1$, we have $\left[\begin{smallmatrix} 1+n \\ n \end{smallmatrix}\right]_{\mathcal{F}} = [F_{n+1}]_q$. The collection $\mathcal{T}_{1,n}$ can be identified with the collection of tilings of a vertical n-strip with dominos and monominos, where the topmost domino has a special weight. The weight of a usual vertical domino, whose top-right corner has coordinate $(1, j)$, is $q^{F_1 F_j} = q^{F_j}$, while the weight of a special vertical domino located at the same place is $q^{F_{1+1} F_j} = q^{F_j}$. Therefore, the result also follows from Lemma 2.2.

Now assume the result holds when $m = 1$ or $n = 1$. Letting $m, n > 1$ and using Equation (2.4) in the following equation we obtain:

$$\begin{aligned} \begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}} &= \frac{[F_{m+n}]_q [F_{m+n-1}]_q^!}{[F_m]_q^! \cdot [F_n]_q^!} \\ &= [F_{m+1}]_{q^{F_n}} \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_{\mathcal{F}} + q^{F_n F_{m+1}} [F_{n-1}]_{q^{F_m}} \begin{bmatrix} m-1+n \\ n \end{bmatrix}_{\mathcal{F}} \end{aligned}$$

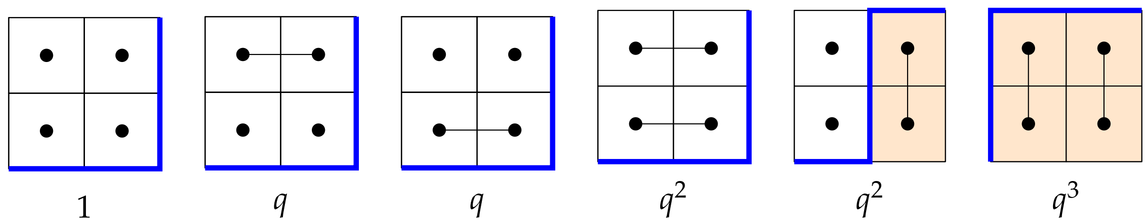


Figure 2: The six path-domino tilings of a 2×2 rectangle, and their contribution to the q-Fibonomial $\left[{2+2 \atop 2}\right]_{\mathcal{F}} = \frac{[F_4][F_3]}{[F_2][F_1]} = \frac{[3][2]}{[1][1]} = 1 + 2q + 2q^2 + q^3$ when computed as a generating function $\sum_T \omega(T)$.

By induction (and using again Lemma 2.2), the first term of the sum is the weighted counting of the path-domino tilings of the $m \times n$ rectangle whose specified path ends with a north step, while the second term is the weighted counting of those finishing with an east step. Indeed, the path-domino tilings whose path ends with a north step have an extra contribution $[F_{m+1}]_{q^{F_n}}$ corresponding to the tilings of the last row with horizontal dominos and monominos. The path-domino tilings whose path ends with an east step have an extra contribution $q^{F_n F_{m+1}}[F_{n-1}]_{q^{F_m}}$; this corresponds to the weight of the forced special vertical domino ($q^{F_n F_{m+1}}$) and the tilings of the remaining $(n-2)$-strip in the last column ($[F_{n-1}]_{q^{F_m}}$). □

We have checked the following conjecture by computer for $m, n \leq 10$:

Conjecture 2.5. *The polynomials* $\left[{m+n \atop n}\right]_{\mathcal{F}}$ *are unimodal.*

Example 2.6. Figure 2 illustrates an example of the q-Fibonomial for $m = n = 2$.

Example 2.7 ($n = 2$)**.** Let $m \in \mathbb{N}$ and $n = 2$. Theorem 2.4 leads to the identity:

$$[F_{m+2}][F_{m+1}] = \sum_{k=1}^{m+1} q^{c_k^m} [F_k]^2, \tag{2.6}$$

where $c_k^m = \sum_{i=k}^{m} F_{i+1}$.

The left hand side comes from the equality $\left[{m+2 \atop 2}\right]_{\mathcal{F}} = [F_{m+2}][F_{m+1}]$. The right hand side is the sum of the weights of all path-domino tilings of an $m \times 2$ rectangle. In fact, the term $q^{c_k^m}[F_k]^2$ indicates the sum of the weights of the path-domino tilings whose specified path is $E^{k-1}N^2E^{m-(k-1)}$: $q^{c_k^m}$ is the product of the weights of the special vertical dominos, and $[F_k]^2$ is the weight of the two horizontal rows above the path. Since there are no more possibilities for the specified path due to the special vertical domino condition, the identity (2.6) follows. The evaluation at $q = 1$ recovers

$$F_{m+2}F_{m+1} = \sum_{k=1}^{m+1} F_k^2. \tag{2.7}$$

Remark 2.8. (2.7) is a well known identity due to its relation with the golden ratio and golden spirals in nature, see for instance [5, 8]. The left hand side of the equation is the area of a $F_{m+2} \times F_{m+1}$ rectangle, which can be subdivided into a sequence of squares, with side lengths $F_1, F_2, \ldots, F_{m+1}$, forming a spiral as illustrated in Figure 3 (left). This Fibonacci spiral is an approximation of the golden spiral, a special case of logarithmic spirals which describe the shape of various natural phenomena such as galaxies, nautilus shells and hurricanes. On the other hand, (2.6) also has a natural geometric interpretation. The left hand side represents the weighted area of a $F_{m+2} \times F_{m+1}$ rectangle, where a unit square whose bottom-left corner is located at (i,j) has weight q^{i+j}. This rectangle can be subdivided into a sequence of squares, with side lengths $F_1, F_2, \ldots, F_{m+1}$, in the north-east direction as illustrated in Figure 3 (right). The sum of their weighted areas is exactly the right hand side of (2.6). This sum can also be interpreted as the "mass" of the rectangle, where the F_k-square has density $d(F_k) = q^{c_k^m}$. This density increases according to the ratio $\frac{d(F_k)}{d(F_{k+1})} = q^{F_{k+1}}$, satisfying the initial condition $d(F_{m+1}) = 1$. It would be interesting to assign these densities to the squares giving rise to the Fibonacci spiral on Figure 3 (left), and see if the resulting equation has some physical meaning. Or even more interesting, to have a continuous version of the equation representing the mass of the Fibonacci spiral (or golden spiral), in order to describe some physical phenomenon in nature. For instance, it is quite natural to think that the density of galaxies grows exponentially as it approaches the center of the spiral. In Example 3.6, we additionally provide a generalization of (2.6) using elliptic weight functions.

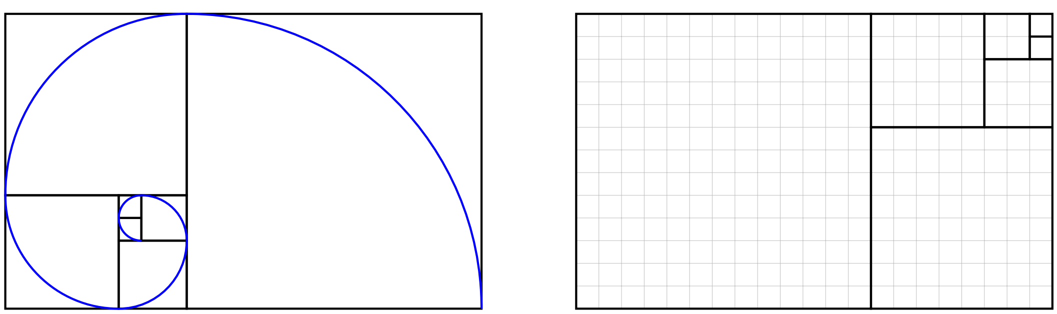


Figure 3: Geometric interpretation of Equations (2.6) and (2.7).

3 Elliptic analog of the Fibonomial numbers

As in the previous section, the elliptic analog of the Fibonomial number is obtained by replacing each term in the binomial coefficient by its corresponding Fibonacci "elliptic number". The elliptic number used here is a slight modification of the elliptic number introduced by Schlosser and Yoo in [11], motivated by work of Schlosser on elliptic binomial coefficients [10]. The elliptic number is an elliptic function that generalizes the q-analog of a number, and plays an important role in the theory of hypergeometric series and special functions.

An *elliptic function* is a function defined over the complex numbers that is meromorphic and doubly periodic. It is well known (*cf. e.g.* [14]) that elliptic functions can be obtained as quotients of *modified Jacobi theta functions*. These are defined as

$$\theta(x;p) := \prod_{j\geq 0}\left((1-p^j x)(1-\frac{p^{j+1}}{x})\right), \qquad \theta(x_1,\dots,x_\ell;p) = \prod_{k=1}^{\ell}\theta(x_k;p),$$

where $x, x_1, \dots, x_\ell \neq 0$ and $|p| < 1$. The *elliptic analog* of a natural number $n \in \mathbb{N}$ (or simply *elliptic number*) is defined as:

$$[n]_{a,b;q,p} := \frac{\theta(q^n, aq^n, bq, \frac{a}{b}q; p)}{\theta(q, aq, bq^n, \frac{a}{b}q^n; p)}. \tag{3.1}$$

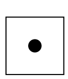
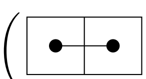
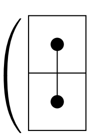
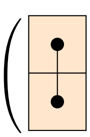
This definition corresponds to the definition in [11] subject to the substitution $b \mapsto bq^{-1}$. Taking the limit $p \to 0$, then $a \to 0$ and then $b \to 0$, one recovers the q-analog $[n]_q$. We simply define the *elliptic analog of the Fibonacci number* F_n by $[F_n]_{a,b;q,p}$.

For $m, n \in \mathbb{N}$, the *elliptic analog of the Fibonomial number* is defined as

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}_{a,b;q,p}} := \frac{[F_{m+n}]^!_{a,b;q,p}}{[F_m]^!_{a,b;q,p} \cdot [F_n]^!_{a,b;q,p}}, \tag{3.2}$$

where $[F_n]^!_{a,b;q,p} := \prod_{k=1}^{n}[F_k]_{a,b;q,p}$ is the elliptic Fibonacci analog of $n!$.

Similarly as before, the elliptic Fibonomial number counts path-domino tilings of an $m \times n$ rectangle according to certain elliptic weights. For $T \in \mathcal{T}_{m,n}$, the *elliptic weights* of the possible tiles in T are defined as follows:

$\widetilde{\omega}\Bigg($  $\Bigg) = 1,\ \widetilde{\omega}\Bigg($  $\Bigg) = \omega_1(i,j),\ \widetilde{\omega}\Bigg($  $\Bigg) = \omega_1(j,i),\ \widetilde{\omega}\Bigg($  $\Bigg) = \omega_2(i,j),$

where (i,j) denotes the coordinate of the top-right corner of the tile, the shaded vertical domino represents a special vertical domino touching the path from below, and

$$\omega_1(i,j) := v_{a,b;q^{F_j},p}(F_i, F_{i-1}), \tag{3.3}$$

$$\omega_2(i,j) := v_{a,b;q,p}(F_{i+1}F_j, F_iF_{j-1}), \tag{3.4}$$

are defined in terms of the following expression:

$$v_{a,b;q,p}(m,n) := \frac{\theta(aq^{2m+n}, b, bq^n, \frac{a}{b}q^n, \frac{a}{b}; p)}{\theta(aq^n, bq^m, bq^{m+n}, \frac{a}{b}q^m, \frac{a}{b}q^{m+n}; p)}q^m. \tag{3.5}$$

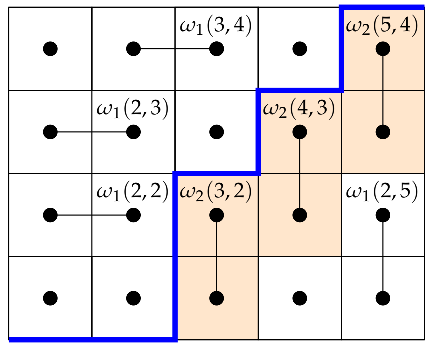


Figure 4: The path-domino tiling T from Figure 1 with the elliptic Fibonacci weights of its tiles. The weight of the tiling is the product of the weights of its tiles.

Note that the weight of a "regular" vertical domino is evaluated at (j, i) instead of (i, j). This transposition does not make any difference for the q-analog of the Fibonomial numbers, but it does for the elliptic case. The *elliptic weight* $\widetilde{\omega}(T)$ of T is defined as the product of the weights of its tiles; see an example in Figure 4. The elliptic weight is a generalization of the q-weight, since we obtain the q-weight by taking the limit $p \to 0$, $a \to 0$ and $b \to 0$ in this order.

Theorem 3.1. *For $m, n \in \mathbb{N}$, the elliptic analog of the Fibonomial numbers can be computed as*

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}_{a,b;q,p}} = \sum_{T \in \mathcal{T}_{m,n}} \widetilde{\omega}(T).$$

The proof of this theorem follows the same steps as the proof of Theorem 2.4. The proofs of the technical lemmas and examples use some basic properties of theta functions summarized in the following proposition, which are essential in the theory of elliptic hypergeometric series.

Proposition 3.2 (*cf.* [14, p. 451, Example 5])**.** *The theta function satisfies the following basic properties:*

$$\theta(x; 0) = 1 - x, \tag{3.6}$$

$$\theta(xy, \tfrac{x}{y}, uz, \tfrac{u}{z}; p) = \theta(uy, \tfrac{u}{y}, xz, \tfrac{x}{z}; p) + \tfrac{x}{z}\theta(zy, \tfrac{z}{y}, ux, \tfrac{u}{x}; p). \tag{3.7}$$

Before proving Theorem 3.1, let us again prove some straightforward lemmas:

Lemma 3.3. *For $m, n \in \mathbb{N}$, the following identities hold:*

$$[m+n]_{a,b;q,p} = [m]_{a,b;q,p} + v_{a,b;q,p}(m,n)[n]_{a,b;q,p} \tag{3.8}$$

$$[m \cdot n]_{a,b;q,p} = [m]_{a,b;q,p}[n]_{a,b;q^m,p} \tag{3.9}$$

Proof. Equation (3.8) can be verified using Equation (3.7), and Equation (3.9) follows from simple cancellations. □

Using the same arguments as in the q-case and replacing the weight of a domino whose top-right coordinate is $(i,1)$ by $\omega_1(i,1)$, we obtain the following lemmas.

Lemma 3.4. *For $n \in \mathbb{N}$, the elliptic analog of the Fibonacci numbers can be computed as*

$$[F_n]_{a,b;q,p} = \sum_T \widetilde{\omega}(T),$$

where the sum ranges over all tilings of an $(n-1)$-strip using dominos and monominos.

Proof. The result is clearly true for $n = 1, 2$. Let $n > 2$, applying (3.8) from Lemma 3.3 we obtain:

$$[F_n]_{a,b;q,p} = [F_{n-1}]_{a,b;q,p} + \omega_1(n-1,1)[F_{n-2}]_{a,b;q,p}.$$

By induction, the first term of this sum corresponds to the tilings of an $(n-1)$-strip that finish with a monomino, while the second term to the tilings of an $(n-1)$-strip that finish with a domino. □

Lemma 3.5. *For $m, n \in \mathbb{N}$, the following identity holds:*

$$[F_{m+n}]_{a,b;q,p} = [F_n]_{a,b;q,p}[F_{m+1}]_{a,b;q^{F_n},p} + \omega_2(m,n)[F_m]_{a,b;q,p}[F_{n-1}]_{a,b;q^{F_m},p}. \tag{3.10}$$

Proof. Applying Lemma 3.3 to (2.3) leads to (3.10). □

Now, we have all tools to prove the main theorem in this section.

Proof of Theorem 3.1. For $n = 1$, we have $\left[{m+1 \atop 1}\right]_{\mathcal{F}_{a,b;q,p}} = [F_{m+1}]_{a,b;q,p}$. The collection $\mathcal{T}_{m,1}$ coincides with the tilings of an m-strip with dominos and monominos, since only the last step of the specified lattice path can be a north step because of the special vertical domino condition. Therefore, the result follows from Lemma 3.4.

For $m = 1$, we have $\left[{1+n \atop n}\right]_{\mathcal{F}_{a,b;q,p}} = [F_{n+1}]_{a,b;q,p}$. The collection $\mathcal{T}_{1,n}$ can be identified with the collection of tilings of a vertical n-strip with dominos and monominos, where the topmost domino has a special weight. The weight of a usual vertical domino, whose top-right corner has coordinate $(1, j)$, is $\omega_1(j, 1)$ while the weight of a special vertical domino located at the same place is $\omega_2(1, j)$. Since $\omega_2(1, j) = \omega_1(j, 1)$, the result also follows from Lemma 3.4.

Now assume the result holds when $m = 1$ or $n = 1$. Letting $m, n > 1$ and using (3.10) in the following equation we obtain:

$$\begin{aligned}\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}_{a,b;q,p}} &= [F_{m+1}]_{a,b;q^{F_n},p} \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_{\mathcal{F}_{a,b;q,p}} \\ &\quad + \omega_2(m,n)[F_{n-1}]_{a,b;q^{F_m},p} \begin{bmatrix} m-1+n \\ n \end{bmatrix}_{\mathcal{F}_{a,b;q,p}}.\end{aligned}$$

By induction (and using again Lemma 3.4), the first term of the sum is the weighted counting of the path-domino tilings of the $m \times n$ rectangle whose specified path ends with a north step, while the second term is the weighted counting of those finishing with an east step.

Indeed, the path-domino tilings whose path ends with a north step have an extra contribution $[F_{m+1}]_{a,b;q^{F_n},p}$. This corresponds to the weighted enumeration of the tilings of the last row with horizontal dominos and monominos. This follows from the fact that both quantities satisfy the same initial conditions and recurrence relation, which is obtained by applying (3.8) to $F_m + F_{m-1}$. The path-domino tilings whose path ends with an east step have an extra contribution $\omega_2(m,n)[F_{n-1}]_{a,b;q^{F_m},p}$. This corresponds to the weight of the forced special vertical domino ($\omega_2(m,n)$) and the tilings of the remaining $(n-2)$-strip in the last column ($[F_{n-1}]_{a,b;q^{F_m},p}$). $\square$

Example 3.6 ($n = 2$)**.** The identitiy (2.6) in Example 2.7 for $m \in \mathbb{N}$ and $n = 2$ generalizes in the elliptic case to

$$[F_{m+2}]_{a,b;q,p}[F_{m+1}]_{a,b;q,p} = \sum_{k=1}^{m+1} \Omega_k^m [F_k]^2_{a,b;q,p}, \tag{3.11}$$

where $\Omega_k^m = \prod_{i=k}^m \omega_2(i,2)$ is the product of the weights of the special vertical dominos, and $[F_k]^2_{a,b;q,p} = [F_k]_{a,b;q,p} \cdot [F_k]_{a,b;q^{F_2},p}$ is the weight of the two horizontal rows above the path.

Example 3.7 ($a, b; p \to 0$)**.** By computing the limits $p \to 0$, $a \to 0$ and $b \to 0$ (in this order) of $\widetilde{\omega}(T)$ and $\left[{m+n \atop n}\right]_{\mathcal{F}_{a,b;q,p}}$ we obtain that Theorem 2.4 is a special case of Theorem 3.1 for $|q| < 1$.

4 The q-Fibonacci analog of the rational Catalan numbers

Given a pair of relatively prime numbers $m, n \in \mathbb{N}$ (that is, such that their greatest common divisor is $(m,n) := \gcd\{m,n\} = 1$), the *m,n-Catalan number* is defined as

$$\mathrm{Cat}_{m,n} := \frac{1}{m+n}\binom{m+n}{n}.$$

This number is equal to the number of lattice paths from $(0,0)$ to (m,n) that stay weakly above the main diagonal of the $m \times n$ rectangle. The study of these numbers (also in the non-coprime case), and their q-analog and q,t-analog generalizations, has produced a substantial amount of research related to topics including rectangular diagonal harmonics, Shi hyperplane arrangements, affine Weyl groups, affine Hecke algebras, knot theory, and representation theory of Cherednik algebras.

The *q-Fibonacci analog of the m,n-Catalan number* is defined as

$$[\mathrm{FCat}_{m,n}] := \frac{1}{[F_{m+n}]}\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}} = \frac{[F_{m+n-1}]_q^!}{[F_m]_q^![F_n]_q^!}.$$

Surprisingly, this rational expression also turns out to be a polynomial when the greatest common divisor $(m,n) \in \{1,2\}$. Before proving this, we need the following lemma.

Lemma 4.1 ([6]). *For $m,n \in \mathbb{N}$, we have $(F_m, F_n) = F_{(m,n)}$.*

Proposition 4.2 (Shu Xiao Li [7]). *If the greatest common divisor $(m,n) \in \{1,2\}$, then $[\mathrm{FCat}_{m,n}]$ is a polynomial in q with integer coefficients.*

Proof. First note that

$$[F_n]\begin{bmatrix} m+n \\ n \end{bmatrix}_{\mathcal{F}} = [F_{m+n}]\begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_{\mathcal{F}}.$$

Since all the terms involved in this identity are polynomials, we know $[F_{m+n}]$ divides $[F_n]\left[\begin{smallmatrix} m+n \\ n \end{smallmatrix}\right]_{\mathcal{F}}$. But $(F_{m+n}, F_n) = F_{(m+n,n)} = F_{(m,n)} = 1$ whenever (m,n) is equal to 1 or 2. Since the roots of $[n]$ are the n-th roots of unity that are different to 1, the polynomials $[F_{m+n}]$ and $[F_n]$ have no roots in common. Thus, $[F_{m+n}]$ divides $\left[\begin{smallmatrix} m+n \\ n \end{smallmatrix}\right]_{\mathcal{F}}$. □

Computational experimentation for $m,n \leq 15$ suggests that the coefficients of these polynomials are non-negative integers. However, we do not have a proof nor a combinatorial model to describe them.

Remark 4.3. The model of Sagan and Savage in [9] gives a combinatorial interpretation of the Lucas analog of the binomial coefficients. The *Lucas polynomials* $\{n\}$ generalize the Fibonacci numbers, and are defined by the initial conditions $\{0\} = 0$, $\{1\} = 1$ and the recurrence $\{n\} = s\{n-1\} + t\{n-2\}$ for some variables s,t. It was proven in [3, Section 6.2], that the Lucas analog of the rational m,n-Catalan numbers is also polynomial in s,t with integer coefficients.[3] However, to the best of our knowledge, no q-analogs of the Lucas-binomial coefficients have been studied in the literature. The proof of Proposition 4.2 was originally found by our colleague Shu Xiao Li, during discussions about this topic in the Algebraic Combinatorics Seminar at the Fields Institute in 2015. At this Seminar, our colleague Farid Aliniaeifard showed that Conjecture 2.5 implies the positivity of the coefficients.

Remark 4.4. Given a crystallographic Coxeter group W with Coxeter exponents $e_1 < e_2 < \ldots < e_n$, the rational W-Catalan number is defined as $C_W(a) = \prod_{i=1}^n \frac{a+e_i}{e_i+1}$, and this is an integer when a is relatively prime to $e_n + 1$. We can define a q-Fibonacci analog as follows:

$$C_{W,\mathcal{F}}(a) = \prod_{i=1}^{n} \frac{[F_{a+e_i}]}{[F_{e_i+1}]}.$$

[3]Their proof is reproduced from the proof of Proposition 4.2 with our permission.

We have checked that this is a polynomial with positive integer coefficients when a and $e_n + 1$ are relatively prime, for each type and various values of a.

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