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# Independence Posets

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**Abstract.** Let $G$ be an acyclic directed graph. For each vertex $g \in G$, we define an involution on the independent sets of $G$. We call these involutions *flips*, and use them to define the *independence poset* for $G$—a new partial order on independent sets of $G$. Our independence posets are a generalization of distributive lattices, eliminating the lattice requirement: an independence poset that is a graded lattice is always a distributive lattice. Many well-known posets turn out to be special cases of our construction.

**Résumé.** Soit $G$ un graphe orienté acyclique. Pour chaque sommet $g \in G$, nous définissons une involution sur les ensembles indépendants de $G$. Nous appelons ces involutions des *flips*. Nous les utilisons pour définir une structure d'ensemble ordonné sur les ensembles indépendants de $G$, que nous nommons le *poset d'indépendance* de $G$. Nos posets d'indépendance généralisent les treillis distributifs en soustrayant la condition d'être un treillis: un poset d'indépendance qui est un treillis gradué est un treillis distributif. Il s'avère que plusieurs ensembles ordonnés bien connus peuvent être exprimés comme des posets d'indépendance.

**Keywords:** Poset, lattice, independent set, rowmotion.

## 1 Introduction

A *trim lattice* is an extremal left-modular lattice [11]. Trim lattices were introduced to serve as analogues of distributive lattices without the graded hypothesis: a graded trim lattice is a distributive lattice, and every distributive lattice is trim.

In Lemma 3.4, we define the *independence poset* $\text{top}(G)$ on the set of independent sets of a directed acyclic graph $G$, using an explicit description of cover relations as certain *flips*. Our independence posets further generalize distributive lattices by removing the lattice requirement: an independence poset that is a lattice is always a trim lattice. In other words, the common intersection of lattices and independence posets are exactly the trim lattices.[1]

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[1] An interactive poster version of this extended abstract can be found online at [14].

**Theorem 1.1.** *If* $\operatorname{top}(G)$ *is a lattice, then it is a trim lattice. Every trim lattice can be realized as an independence poset for a unique (up to isomorphism) acyclic directed graph* $G$*.*

The class of independence posets therefore includes all distributive lattices, Tamari lattices, Cambrian lattices, Fuss-Cambrian lattices, and torsion pairs of finite type tilted Artin algebras. We construct the underlying graph $G$ for distributive and Cambrian lattices in Section 3.4.

Many of these examples have naturally-defined cyclic actions on their elements; Equation (5.1) defines *rowmotion* on independence posets, a common generalization of these cyclic actions. We show in Theorem 5.1 that rowmotion can be computed in two other ways: as a walk on the Hasse diagram of the independence poset, and as a composition of reorientations of the underlying directed graph. This first method computes rowmotion within a *fixed* independence poset (*rowmotion in slow motion*), while the second relies on a sequence of bijections between independence posets for different orientations of the same underlying undirected graph (*rowmotion by deformotion*).

## 2 Independent sets and tight orthogonal pairs

We always take $G$ to be a finite acyclic directed graph; by acyclic, we mean that $G$ contains no oriented cycles. The transitive closure of $G$ defines a poset, which we refer to as *G-order*. Our convention is that $g_1 \geq g_2$ in $G$-order if and only if there is a directed path in $G$ from $g_1$ to $g_2$; when we compare vertices of $G$, we will always mean a comparison in $G$-order. We write $\ell$ for a linear extension of $G$-order, and $\ell'$ for a reverse linear extension of $G$-order. We write $\simeq$ for an isomorphism of posets.

Recall that an *independent set* $\mathcal{A} \subseteq G$ is a set of pairwise non-adjacent vertices of $G$. As we now explain, the orientation provided by $G$ allows us to complete an independent set to a pair of independent sets, either of which determines the other.

**Definition 2.1.** A pair $(\mathcal{D}, \mathcal{U})$ of independent sets of $G$ is called *orthogonal* if there is no edge in $G$ from an element of $\mathcal{D}$ to an element of $\mathcal{U}$. An orthogonal pair of independent sets $(\mathcal{D}, \mathcal{U})$ is called *tight* if whenever any element of $\mathcal{D}$ is increased (removed and replaced by a larger element with respect to $G$-order) or any element of $\mathcal{U}$ is decreased, or a new element is added to either $\mathcal{D}$ or $\mathcal{U}$, then the result is no longer an orthogonal pair of independent sets. We abbreviate **t**ight **o**rthogonal **p**air by *top*, and we write $\operatorname{top}(G)$ for the set of all tops of $G$.

Some examples are given in Figure 1. Theorem 2.3 shows that an independent set can be completed to a tight orthogonal pair in exactly two ways.

**Definition 2.2.** Let $g \in G$ and define $G_g$ to be the directed graph obtained by deleting the vertex $g$ from $G$ (along with all edges to $g$), and $G_g^\circ$ the directed graph obtained by deleting all vertices and edges adjacent to $g$ in $G$ (along with $g$ itself).

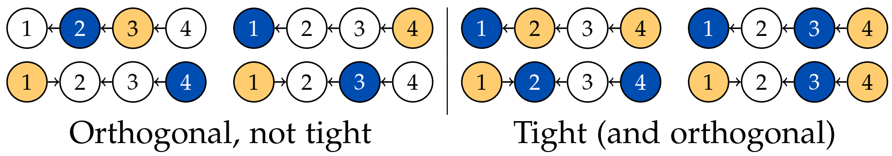


**Figure 1:** Eight pairs of independent sets $(\mathcal{D}, \mathcal{U})$ for two different orientations of a path graph. The blue vertices correspond to the elements of $\mathcal{D}$, while the orange vertices correspond to $\mathcal{U}$.

**Theorem 2.3.** *Let $\mathcal{I}$ be an independent set of a directed acyclic graph $G$. Then there exists a unique $(\mathcal{I}, \mathcal{U}) \in \mathsf{top}(G)$ and a unique $(\mathcal{D}, \mathcal{I}) \in \mathsf{top}(G)$.*

**Input:** An acyclic directed graph $G$ and an independent set $\mathcal{I}$.
**Output:** An element $(\mathcal{D}, \mathcal{I}) \in \mathsf{top}(G)$.
**set:** $\mathcal{D} = \{\}$
**for** $k$ *in* $\ell'$ **do**

  **if** $\left\{ \begin{array}{l} k \notin \mathcal{I} \\ i \to k \notin G \text{ for } i \in \mathcal{D} \\ k \to i \notin G \text{ for } i \in \mathcal{I} \end{array} \right\}$ **then** add $k$ to $\mathcal{D}$

**end**
**return** $(\mathcal{D}, \mathcal{I})$

**Algorithm 1:** The greedy construction of the unique $(\mathcal{D}, \mathcal{I}) \in \mathsf{top}(G)$ using any reverse linear extension $\ell'$ of $G$-order, given an independent set $\mathcal{I}$.

# 3 Independence posets

We specify cover relations to define a poset structure on the tight orthogonal pairs of $G$.

## 3.1 Flips

**Definition 3.1.** The *flip* of $(\mathcal{D}, \mathcal{U}) \in \mathsf{top}(G)$ at an element $g \in G$ is the tight orthogonal pair $\mathsf{flip}_g(\mathcal{D}, \mathcal{U})$ defined as follows (see Figure 3 for an example): if $g \notin \mathcal{D}$ and $g \notin \mathcal{U}$, the flip does nothing. Otherwise, preserve all elements of $\mathcal{D}$ that are not less than $g$ and all elements of $\mathcal{U}$ that are not greater than $g$ (and delete all other elements); after switching the set to which $g$ belongs, then greedily add elements to $\mathcal{D}$ and $\mathcal{U}$ (respecting the conditions to form an orthogonal pair) in the orders $\ell'$ and $\ell$, respectively.

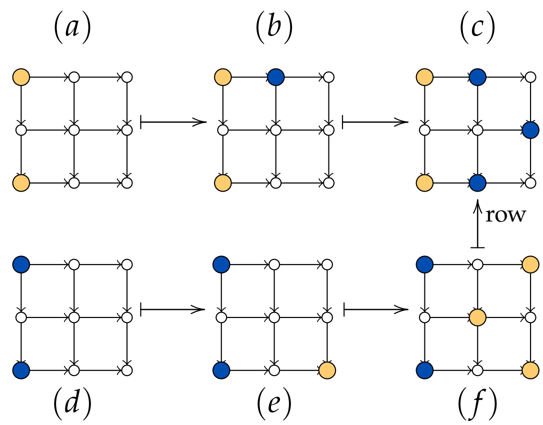


**Figure 2:** The progression $(a) \mapsto (b) \mapsto (c)$ is an illustration of Algorithm 1, which produces the tight orthogonal pair $(c)$ by greedily adding elements to the independent set $(a)$ in a reverse linear extension of $G$-order (from the top left to the bottom right). The progression $(d) \mapsto (e) \mapsto (f)$ illustrates the dual procedure, which greedily adds elements to the independent set $(d)$ in a linear extension of $G$-order (from the bottom right to the top left). Rowmotion, defined by Equation (5.1), sends $(f)$ to $(c)$.

Figure 3 illustrates a flip on a tight orthogonal pair in an orientation of $[7] \times [7]$.

**Proposition 3.2.** *For $g \in G$, $\mathrm{flip}_g(\mathcal{D}, \mathcal{U}) \in \mathrm{top}(G)$.*

*Proof.* The statement follows from the restriction of Theorem 2.3 to the elements of $G$ not less than $g$ and to the elements not greater than $g$. □

**Lemma 3.3.** *Let $g$ be an element of an acyclic directed graph $G$. Then $\mathrm{flip}_g^2(\mathcal{D}, \mathcal{U}) = (\mathcal{D}, \mathcal{U})$. If $h$ is incomparable with $g$ in $G$-order, then $\mathrm{flip}_g \circ \mathrm{flip}_h = \mathrm{flip}_h \circ \mathrm{flip}_g$.*

## 3.2 Independence Relations

For $G$ an acyclic directed graph, the *independence relations* on $\mathrm{top}(G)$ are the reflexive and transitive closure of the relations $(\mathcal{D}, \mathcal{U}) < (\mathcal{D}', \mathcal{U}')$ if there is some $g \in \mathcal{U}$ such that $\mathrm{flip}_g(\mathcal{D}, \mathcal{U}) = (\mathcal{D}', \mathcal{U}')$.

**Lemma 3.4.** *Independence relations are antisymmetric, and hence define an* independence poset, *denoted* $\mathrm{top}(G)$. *Flips and cover relations of* $\mathrm{top}(G)$ *coincide.*

By Lemma 3.4, the maximum element of $\mathrm{top}(G)$ is the unique tight orthogonal pair $\hat{1}$ of the form $(\mathcal{D}, \emptyset)$, and its minimum element $\hat{0}$ is of the form $(\emptyset, \mathcal{U})$.

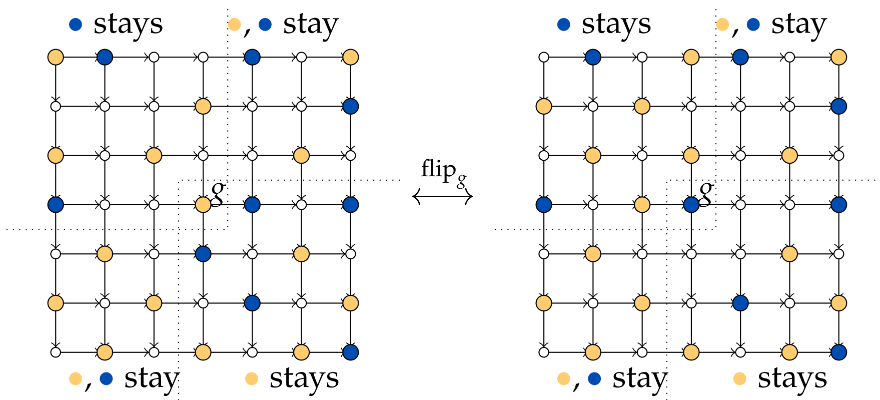


**Figure 3:** A flip on a top $(\mathcal{D}, \mathcal{U})$ in the $7 \times 7$ grid oriented from top left to bottom right. As in Figure 1, the blue vertices correspond to the elements of $\mathcal{D}$, while the orange vertices correspond to the elements of $\mathcal{U}$. Flipping at the vertex $g$ changes its color, and divides the grid into 5 connected regions (delineated by the dotted lines): the blue vertices not less than $g$ (i.e., not in the bottom right) and the orange vertices not greater than $g$ (i.e., not in the top left) are preserved by the flip. The orange vertices in the top left are filled in greedily from bottom right to top left; the blue vertices in the bottom right are filled in greedily from top left to bottom right.

## 3.3 Tight orthogonal pair recursion

For any $g \in G$, since $\{g\}$ is an independent set of $G$, by Theorem 2.3 there is a unique tight orthogonal pair $m_g$ of the form $(\mathcal{D}, \{g\})$, and a unique tight orthogonal pair $j_g$ of the form $(\{g\}, \mathcal{U})$. Write

$$\mathrm{top}_g(G) := [\hat{0}, m_g] \text{ and } \mathrm{top}^g(G) := [j_g, \hat{1}]. \tag{3.1}$$

We say that $g \in G$ is *extremal* if it is a minimal or maximal element of $G$-order.

**Lemma 3.5.** *Let $G$ be an acyclic directed graph. If $g$ is an extremal element of $G$, then* $\mathrm{top}(G) = \mathrm{top}_g(G) \sqcup \mathrm{top}^g(G)$ *. Furthermore,*

- *If $g$ is minimal, $(\mathcal{D}, \mathcal{U}) \in \mathrm{top}_g(G)$ if and only if $g \in \mathcal{U}$, and*
- *If $g$ is maximal, $(\mathcal{D}, \mathcal{U}) \in \mathrm{top}^g(G)$ if and only if $g \in \mathcal{D}$.*

*In particular, if $x \in \mathrm{top}^g(G)$ and $y \in \mathrm{top}_g(G)$, then $x \not\leq y$.*

**Lemma 3.6.** *For any element $g \in G$, if $g \in \mathcal{D}$ then $(\mathcal{D}, \mathcal{U}) \in \mathrm{top}^g(G)$; and if $g \in \mathcal{U}$ then $(\mathcal{D}, \mathcal{U}) \in \mathrm{top}_g(G)$.*

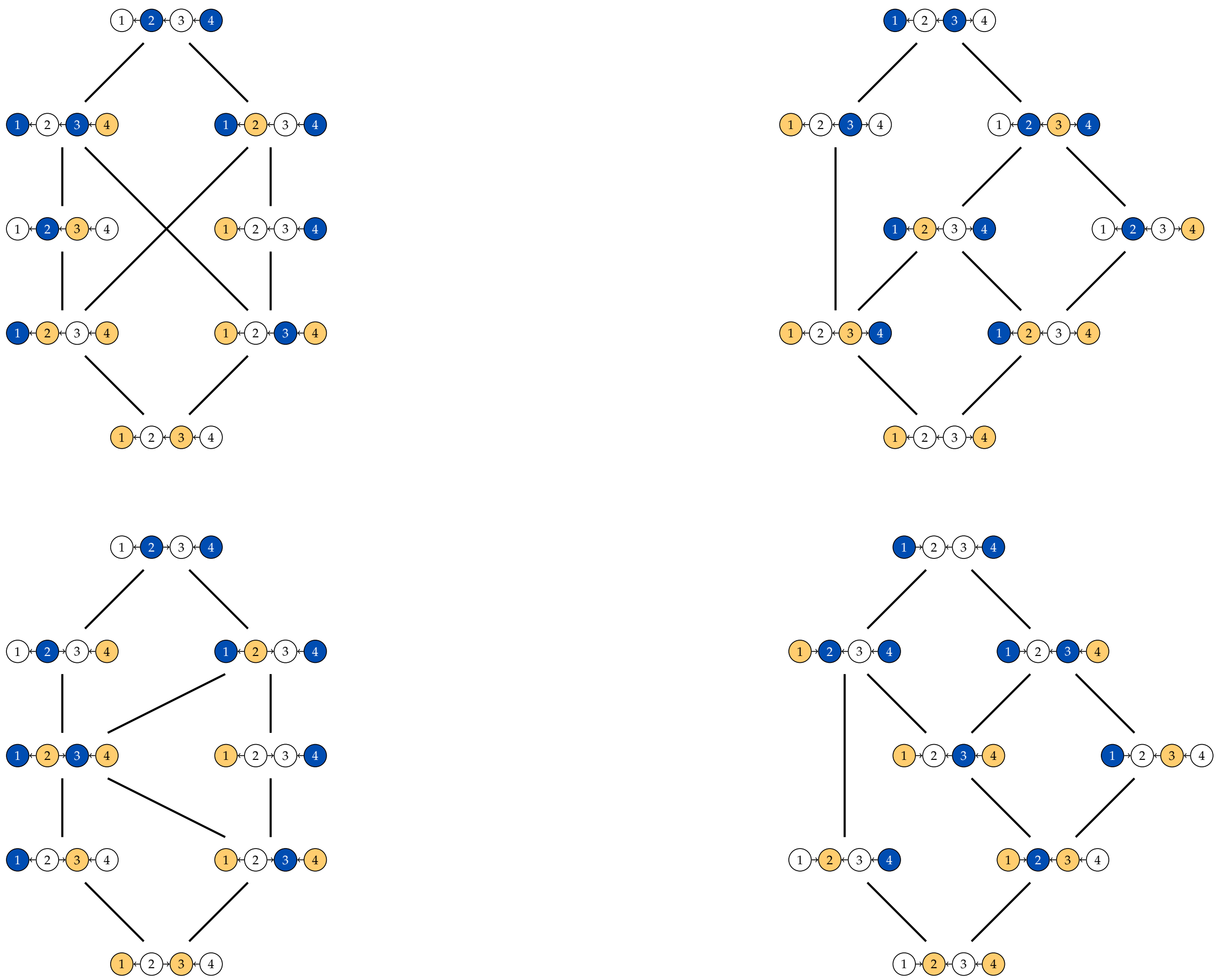

**Figure 4:** Independence posets for four orientations of a path of length 4. Each poset has eight elements (the tops drawn in blue and orange as in Figures 1 and 3), corresponding to the eight independent sets in the underlying undirected graph. The top left poset is not a lattice, the bottom left poset is a distributive lattice, and both posets on the right are trim lattices.

**Lemma 3.7.** *Let G be a directed acyclic graph.*

- *If $g$ is minimal and $(\mathscr{D},\mathscr{U}) \in \mathrm{top}_g(G)$, then $\mathrm{flip}_g(\mathscr{D},\mathscr{U}) = (\mathscr{D} \cup \{g\}, \mathscr{U}')$ for some $\mathscr{U}'$.*
- *If $g$ is maximal and $(\mathscr{D},\mathscr{U}) \in \mathrm{top}^g(G)$, then $\mathrm{flip}_g(\mathscr{D},\mathscr{U}) = (\mathscr{D}', \mathscr{U} \cup \{g\})$ for some $\mathscr{D}'$.*

*Proof.* This follows from the definition of flip; when $g$ is minimal, all of $\mathscr{D}$ is preserved since every element of $\mathscr{D}$ is not less than $g$. Similarly, when $g$ is maximal, all of $\mathscr{U}$ is preserved, since every element of $\mathscr{U}$ is not greater than $g$. □

**Theorem 3.8.** *Let $g$ be an extremal element of an acyclic directed graph G. Then*

$$(\mathscr{D},\mathscr{U}) \mapsto (\mathscr{D},\mathscr{U} \setminus \{g\}) \text{ is a bijection } \begin{cases} \mathrm{top}_g(G) \simeq \mathrm{top}(G_g^\circ) & \text{if } g \text{ minimal} \\ \mathrm{top}_g(G) \simeq \mathrm{top}(G_g) & \text{if } g \text{ maximal} \end{cases}$$

$$(\mathscr{D},\mathscr{U}) \mapsto (\mathscr{D} \setminus \{g\},\mathscr{U}\} \text{ is a bijection } \begin{cases} \mathrm{top}^g(G) \simeq \mathrm{top}(G_g) & \text{if } g \text{ minimal} \\ \mathrm{top}^g(G) \simeq \mathrm{top}(G_g^\circ) & \text{if } g \text{ maximal} \end{cases}$$

*Proof.* We only prove the results for $g$ minimal, the case for $g$ maximal being analogous. We first show $\mathrm{top}_g(G) \simeq \mathrm{top}(G_g^\circ)$; by Lemma 3.5, $(\mathscr{D},\mathscr{U}) \in \mathrm{top}_g(G)$ if and only if $g \in \mathscr{U}$. But since $(\mathscr{D},\mathscr{U})$ is a tight orthogonal pair, no element of $\mathscr{D}$ or of $\mathscr{U}$ can be adjacent to $g$, from which we conclude the result by definition of $G_g^\circ$. We now show $\mathrm{top}^g(G) \simeq \mathrm{top}(G_g)$; since $\mathrm{top}(G) = \mathrm{top}_g(G) \sqcup \mathrm{top}^g(G)$, by Lemma 3.5 elements of $\mathrm{top}^g(G)$ consist of those tight orthogonal pairs of $G$ with either $g \in \mathscr{D}$ or $g \notin \mathscr{U} \cup \mathscr{D}$. Each tight orthogonal pair of $G_g$ can be uniquely extended to such a tight orthogonal pair. □

## 3.4 Examples

Let $P$ be a poset with corresponding distributive lattice $J(P)$. Then the dual of the comparability graph $G$ of $P$—defined by $p_2 \to p_1$ in $G$ if and only if $p_1 \leq p_2$ in $P$—satisfies $\mathrm{top}(G) \simeq J(P)$.

Following [13], a Cambrian lattice may be realized as $\mathrm{top}(G)$ for $G$ constructed as follows. An example appears in Figure 5; non-experts may ignore the description below—Figure 5 can still be appreciated as an example of an independence poset without knowing where the underlying graph comes from. For $c = a_1 a_2 \cdots a_n$ a permutation of the simple reflections of a $W$, Reading's *c-sorting word* for the long element of $W$ is the leftmost reduced word for $w_\circ$ in the word $c^\infty$: $w_\circ(c) = a_1 a_2 \cdots a_N$ with each $a_i \in S$. $G$ then has vertex set $[N]$ (indexing the reflections of $W$) and a directed edge $j \to i$ iff $j > i$ and $(a_i \cdots a_{j-1}) a_j (a_{j-1} \cdots a_i)$ does not lie in the parabolic subgroup of $W$ generated by all simple reflections except $a_i$.

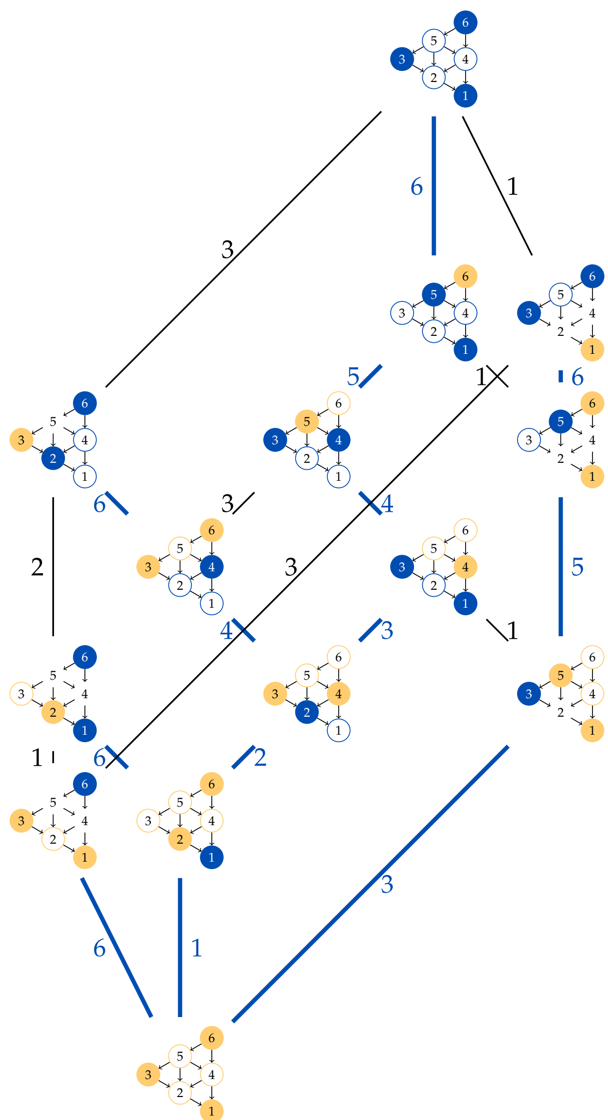

**Figure 5:** The Tamari lattice with 14 elements, realized as an independence poset. The thick blue edges indicate the tree structure provided by the natural labelling, giving an efficient method to generate all independent sets of the underlying graph. The filling of the vertices of the graph specify the tight orthogonal pairs, while the color of the boundaries specify the maximal orthogonal pair.

# 4 Trim lattices and Maximal Orthogonal Pairs

## 4.1 Extremal lattices

A *chain* in a poset is a sequence of elements $x_0 < x_1 < \cdots < x_r$, of *length* $r$. The poset $\mathrm{top}(G)$ has a maximal chain of length $|G|$ obtained by starting at $\hat{0}$ and flipping the elements of $G$ in the order of a linear extension of $G$-order.

**Lemma 4.1.** *For $G$ an acyclic directed graph with $g_1, \ldots, g_{|G|}$ a linear extension of $G$-order, the sequence*

$$\hat{0} \lessdot \mathrm{flip}_{g_1}(\hat{0}) \lessdot (\mathrm{flip}_{g_2} \circ \mathrm{flip}_{g_1})(\hat{0}) \lessdot \ldots \lessdot (\mathrm{flip}_{g_{|G|}} \circ \cdots \circ \mathrm{flip}_{g_1})(\hat{0}) = \hat{1}$$

*is a maximal chain in* $\mathrm{top}(G)$*.*

*Proof.* Write $(\mathcal{D}_i, \mathcal{U}_i)$ for the $i$th element of the sequence. By Lemma 3.4, this sequence is unrefinable. By induction, after the $i$th step all elements of $\mathcal{U}_i$ lie above $\{g_1, \ldots, g_i\}$ and all elements of $\mathcal{D}_i$ are contained in $\{g_1, \ldots, g_i\}$. Furthermore, since each $\mathcal{U}_i$ is computed greedily in linear extension order, $g_{i+1} \in \mathcal{U}_i$. The sequence must end with $\hat{1}$, because the only way for all elements of $\mathcal{U}_{|G|}$ to lie above $\{g_1, \ldots, g_{|G|}\}$ is for $\mathcal{U}_{|G|}$ to be empty. □

An *extremal lattice* is a lattice whose longest chain is of length equal to the number of its join irreducible elements and to the number of its meet irreducible elements. As motivation for our main result of this section, we have the following easy statement (we will refine it in Theorem 1.1).

**Lemma 4.2.** *If* $\mathrm{top}(G)$ *is a lattice, then it is an extremal lattice.*

*Proof.* A lattice with a chain of length $n$ must have at least $n$ join-irreducible elements and at least $n$ meet-irreducible elements (since each element of the chain is the join of the join-irreducibles beneath it and the meet of the meet-irreducibles above it). Suppose $\mathrm{top}(G)$ is a lattice; since it only has $|G|$ join-irreducible and $|G|$ meet-irreducible elements, and since it has a chain of length $|G|$ by Lemma 4.1, it is extremal. □

Any acyclic directed graph $G$ gives rise to an extremal lattice $\mathcal{L}(G)$, as follows [13, 7, 8]: for $X, Y \subseteq G$ with $X \cap Y = \emptyset$, we say $(X, Y)$ is an *orthogonal pair* if there is no edge from any $i \in X$ to any $k \in Y$, and we say it is a *maximal orthogonal pair* if $X$ and $Y$ are maximal with that property. We abbreviate **m**aximal **o**rthogonal **p**air by *mop*.

The extremal lattice $\mathcal{L}(G)$ is equivalently given by *either* of $(X, Y) \leq (X', Y')$ if and only if $X \subseteq X'$, or $(X, Y) \leq (X', Y')$ if and only if $Y' \subseteq Y$. Furthermore, the join is computed by intersecting the second terms, while meet is given by the intersection of the first terms. Conversely, we can associate an acyclic directed grath $G(\mathcal{L})$ to any extremal lattice called its *Galois graph* with the property that $\mathcal{L}(G(\mathcal{L})) \simeq \mathcal{L}$.

If $x$ is an element of an extremal lattice $\mathcal{L}(G)$ with corresponding maximal orthogonal pair $(X, Y)$, we write $x_{\mathcal{J}} = X$ and $x_{\mathcal{M}} = Y$—that is, $x_{\mathcal{J}}$ corresponds to the join-irreducible elements below $x$, while $x_{\mathcal{M}}$ corresponds to the meet-irreducible elements above $x$. We refer to [13] for further details on extremal lattices, including Markowsky's generalization of Birkhoff's fundamental theorem of distributive lattices to extremal lattices.

## 4.2 Trim lattices

An element $x$ of a lattice $\mathcal{L}$ is called *left modular* if for any $y \leq z$ we have the equality $(y \vee x) \wedge z = y \vee (x \wedge z)$. A lattice is called *left modular* if it has a maximal chain of left modular elements. A *trim lattice* is an extremal left-modular lattice. We have already shown that if an independence poset is a lattice, then it is extremal. Our goal is to prove that it is actually trim. We say that a relation $y < z$ in an extremal lattice $\mathcal{L}(G)$ is *overlapping* if $y_{\mathcal{M}} \cap z_{\mathcal{J}} \neq \emptyset$.

**Theorem 4.3** ([13, Theorem 3.4]). *An extremal lattice $\mathcal{L}(G)$ is trim if and only if every relation is overlapping if and only if every cover relation is overlapping.*

If a cover relation is overlapping, then it overlaps in a unique element. We may define the *downward* and *upward labels* of $y \in \mathcal{L}(G)$ as

$$\begin{aligned} \mathrm{D}(y) &:= \{\text{the unique element of } x_{\mathcal{M}} \cap y_{\mathcal{J}} : \text{ all } x \text{ such that } x \lessdot y\} \text{ and} \\ \mathrm{U}(y) &:= \{\text{the unique element of } y_{\mathcal{M}} \cap z_{\mathcal{J}} : \text{ all } z \text{ such that } y \lessdot z\}. \end{aligned}$$

The downward and upward labels actually associate two independent sets to each element of $\mathcal{L}$ (we show in Section 4.3 that they together form a tight orthogonal pair).

**Theorem 4.4** ([13, Corollary 5.6]). *For $\mathcal{L}$ a trim lattice, $\mathrm{D}$ and $\mathrm{U}$ are both bijections from $\mathcal{L}$ to the set of independent sets of $G(\mathcal{L})$.*

In a trim lattice $\mathcal{L}(G)$, there is a unique meet-irreducible element $m_g$ with $\mathrm{U}(m_g) = \{g\}$, and a unique join-irreducible element $j_g$ with $\mathrm{D}(j_g) = \{g\}$. The following analogues for trim lattices of Lemma 3.5 and Theorem 3.8 were shown in [13].

**Theorem 4.5** ([13, Lemma 3.10, Proposition 3.11, Proposition 3.12]). *Let $g$ be minimal in an acyclic directed graph $G$, and write $\mathcal{L}_g(G) := [\hat{0}, m_g]$ and $\mathcal{L}^g(G) := [j_g, \hat{1}]$. Then: $\mathcal{L}(G) = \mathcal{L}_g(G) \sqcup \mathcal{L}^g(G)$, $\mathcal{L}^g(G) \simeq \mathcal{L}(G_g)$, $\mathcal{L}_g(G) \simeq \mathcal{L}(G_g^\circ)$, and $x \in \mathcal{L}_g(G)$ if and only if $g \in \mathrm{U}(x)$.*

## 4.3 Trim Lattices and Independence Posets

We restate our main theorem relating trim lattices and independence posets, referring to [12] for proofs.

**Theorem 1.1.** *If* $\operatorname{top}(G)$ *is a lattice, then it is trim. Every trim lattice can be realized as an independence poset for a unique (up to isomorphism) acyclic directed graph $G$.*

The cover relations in the lattice of order ideals of a finite poset $\mathcal{P}$ are naturally labelled by elements of $\mathcal{P}$; the map that associates an order ideal to the set of labels on its downward covers is a bijection to the set of antichains of $\mathcal{P}$. Similarly, the cover relations in a Cambrian lattice on the $c$-sortable elements of a finite Coxeter group $W$ are naturally labelled by reflections; the map that associates a $c$-sortable element to the labels on its downward covers gives a bijection to the set of $c$-noncrossing partitions of $W$. Theorem 1.1 allows us to simultaneously generalize both of these bijections.

**Theorem 4.6.** *If $\mathcal{L}(G)$ is a trim lattice and $x \in \mathcal{L}(G)$, then $\phi(x) = (\mathrm{D}(x), \mathrm{U}(x))$ is a tight orthogonal pair. Furthermore, if $\mathcal{L}(G)$ is trim—or, equivalently, if* $\operatorname{top}(G)$ *is a lattice—then $\phi$ is an isomorphism $\mathcal{L}(G) \simeq \operatorname{top}(G)$.*

# 5 Rowmotion on Independence Posets

Since both components of a top are independent sets, and each independent set can be completed to a top in two ways, it is natural to define *rowmotion* by sending one completion to the other:

$$\operatorname{row}(\mathcal{D}, \mathcal{U}) := \text{the unique } (\mathcal{D}', \mathcal{U}') \in \operatorname{top}(G) \text{ with } \mathcal{D} = \mathcal{U}'. \tag{5.1}$$

This definition gives a common generalization of the usual notion of rowmotion for distributive lattices and the Kreweras complement for Cambrian lattices. It turns out that there are two equally natural (but slower) ways to compute rowmotion [4, 5, 10] as a composition of flips (*rowmotion in slow motion*) and as a composition of toggles (*rowmotion by deformotion*). Here, for $g$ a minimal or maximal element of $G$, the *toggle* $\operatorname{tog}_g$ reverses every edge incident to $g$; this operation induces a bijection between $\operatorname{top}(G)$ and $\operatorname{top}(\operatorname{tog}_g(G))$ whose effect essentially interchanges the relative order of a decomposition of $\operatorname{top}(G)$ into two intervals using the element $g$. (Note that by composing a sequence of toggles that involves every vertex of $G$, the orientation of every edge of $G$ is reversed twice, giving back $G$ itself.)

**Theorem 5.1.** *Let $G$ be a directed acyclic graph. Then rowmotion can be computed in slow motion and by deformotion—that is,* $\operatorname{row} = \prod_{g \in \ell} \operatorname{flip}_g = \prod_{g \in \ell'} \operatorname{tog}_g$ *for any linear extension $\ell$ and reverse linear extension $\ell'$ of $G$-order.*

The full version of this extended abstract appeared in [12]; it contains complete proofs, expanded discussion, as well as connections between independence posets and the representation theory of certain finite-dimensional directed algebras.

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