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A new order on integer partitions

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Abstract. Considering Schur positivity of differences of plethysms of homogeneous symmetric functions, we introduce a new relation on integer partitions. This relation is conjectured to be a partial order, with its restriction to one part partitions equivalent to the classical Foulkes conjecture. We establish some of the properties of this relation via the construction of explicit inclusion of modules whose characters correspond to the plethysms considered. We also prove some stability properties for the number of irreducible occurring in these modules as m grows.

Keywords: Representation theory, plethysm, symmetric group

It's been more than eighty years since Littlewood [9] introduced the plethysm operation on symmetric functions (although the name was only introduced in 1950). This binary operation, denoted by $f[g]$, plays a fundamental role in representation theory, and its calculation raises many interesting questions. Indeed, calculating the "structure" coefficients of plethysms of the Schur functions is considered by Stanley [14] as a key problem in algebraic combinatorics, and it also appears at the forefront of current research in Geometric Complexity Theory. A longstanding conjecture, stated by Foulkes [7] in 1953, simply concerns inequalities between some of these coefficients in simple cases. With the aim of setting up our own context, we cast the statement of his conjecture in terms of coefficients $a^{\lambda}_{\nu,\mu}$ of the Schur expansion of the plethysm

$$h_\nu[h_\mu] = \sum_\lambda a^{\lambda}_{\nu,\mu} s_\lambda,$$

of complete homogeneous symmetric functions, which are well known to be positive integers. Foulkes' conjecture states that for all $n \leq m$, and all partition λ of mn, we have $a^{\lambda}_{(n),(m)} \leq a^{\lambda}_{(m),(n)}$. This has been proven to hold for $n \leq 5$ [16] [5] [11] [4], and when m and n are far enough apart [3]. But it still remains open in full generality.

The question may naturally be extended to pairs of integer partitions μ and ν as follows. Consider the binary relation $\nu \trianglelefteq \mu$ on integer partitions, which holds if and only if $h_\mu[h_\nu] - h_\nu[h_\mu]$ is Schur positive, or equivalently $a^{\lambda}_{\nu,\mu} \leq a^{\lambda}_{\mu,\nu}$ for all λ. Clearly "$\trianglelefteq$" is reflexive and antisymmetric (if we exclude the partition (1)). The conjecture[1] here

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[1] This was first observed by F. Bergeron, personal communication.

is that it is also transitive; hence it gives us a partial order on integer partitions (see Figure 1). Foulkes conjecture corresponds to the statement that

$$(n) \trianglelefteq (m), \qquad \text{iff} \qquad n \leq m,$$

for one part partitions (m) and (n) associated to positive integers m and n.

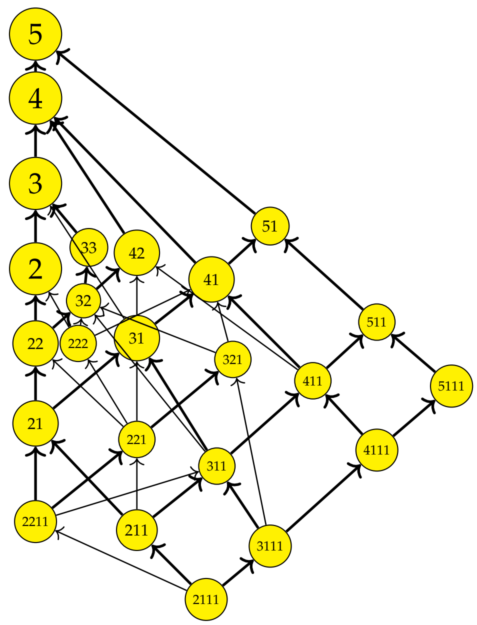

Figure 1: Poset structure on partitions

Our aim is to establish some basic results for the relation "$\trianglelefteq$" using an approach to plethysm via the representation theory of the symmetric group. Indeed, coefficients of Schur expansion of plethysm of the form $h_\mu[h_\nu]$ occur as multiplicities of irreducible representations in some $\mathfrak{S}_{nm}$-modules, denoted here by M_μ^ν. We construct them using similar techniques as [12], although their construction produce modules associated to plethysm of Schur functions.

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1 Main results

This extended abstract is a summary of our result in [15]. We skip here some of the proofs, but they can be found in this preprint, along with many examples of our definitions and constructions. For symmetric functions, notations used here are mostly those of [10]. Other classical combinatorial and representation theory notions are recalled in many texts, for example in [1].

We describe in section 2 the $\mathfrak{S}_{nm}$-modules M^{ν}_{μ}, whose Frobenius transform of its character is the plethysm $h_{\nu}[h_{\mu}]$, using techniques of [12]. The fact that $\nu \trianglelefteq \mu$ is then the existence of an injective morphism $M^{\nu}_{\mu} \hookrightarrow M^{\mu}_{\nu}$. In section 3, we show the following result:

Proposition 1.1. *If there is injective morphisms* $\mathcal{F}_1 : M^{\nu_1}_{\mu} \hookrightarrow M^{\mu}_{\nu_1}$ *and* $\mathcal{F}_2 : M^{\nu_2}_{\mu} \hookrightarrow M^{\mu}_{\nu_2}$*, we can construct an injective morphism* $\mathcal{F}_1 \odot \mathcal{F}_2 : M^{\nu_1 \uplus \nu_2}_{\mu} \hookrightarrow M^{\mu}_{\nu_1 \uplus \nu_2}$.

We also construct in this section the generalized Foulkes-Howe map $\mathcal{F}_{\mu,\nu} : M^{\nu}_{\mu} \to M^{\mu}_{\nu}$. Using this map and proposition 1.1, we prove the following results in section 4:

Theorem 1.2. *(a) For any positive integer n and partition* μ*, there is an injection* $M^{1^n}_{\mu} \hookrightarrow M^{\mu}_{1^n}$.

(b) For any positive integer k and partition μ*, there is an injection* $M^{\mu \uplus 1^k}_{\mu} \hookrightarrow M^{\mu}_{\mu \uplus 1^k}$.

(c) For any positive integer k and partition μ*, there is an injection* $M^{\mu^k}_{\mu} \hookrightarrow M^{\mu}_{\mu^k}$.

(d) If the Foulkes' conjecture is true up to $n-1$*, then for all partition* μ *such that* $\mu_1 \leq n$*, there is an injection* $M^{\mu}_{(n)} \hookrightarrow M^{(n)}_{\mu}$.

In section 5, we describe semistandard homomorphisms as in [8], and we show how to use them in our setting. Using these morphisms, we prove the following stability properties in section 6:

Theorem 1.3. *For any partition* $\tilde{\mu}$ *of an integer* $\tilde{m}$*, if dim* $\mathrm{Hom}_{\mathfrak{S}_{nm}}(S^{\lambda}, M^{\nu}_{\mu}) = r$*, then we have dim* $\mathrm{Hom}_{\mathfrak{S}_{n(m+\tilde{m})}}(S^{\lambda+(n\tilde{m})}, M^{\nu}_{\mu+\tilde{\mu}}) \geq r$.

Theorem 1.4. *If the number of parts of* μ *is* $\tilde{m}$ *and dim* $\mathrm{Hom}_{\mathfrak{S}_{nm}}(S^{\lambda}, M^{\nu}_{\mu}) = r$*, then we have dim* $\mathrm{Hom}_{\mathfrak{S}_{(m+2\tilde{m})n}}(S^{\lambda+(2^{\tilde{m}n})}, M^{\nu}_{\mu+(2^{\tilde{m}})}) \geq r$.

2 Description of the modules

2.1 Representation theory of the symmetric group

As a means of deriving properties of plethysms, we consider representations of the symmetric group $\mathfrak{S}_n$. The necessary link between the two subjects of study is established via the classical Frobenius transform of characters. This is a linear transform which encodes a character as a symmetric function, with the property that irreducible characters are sent to the basis of Schur symmetric function. All the fundamental questions of representation theory of $\mathfrak{S}_n$ may be efficiently translated into calculation in the ring Λ of symmetric functions. We use a polynomial construction of these modules, as in [1].

A *diagram* is simply a finite subset $\boldsymbol{d}$ of $\mathbb{N} \times \mathbb{N}$, whose elements are called *cells*. A cell (i, j) is often geometrically represented in $\mathbb{N} \times \mathbb{N}$ by the *box* having vertices (i, j), $(i+1, j)$, $(i, j+1)$ and $(i+1, j+1)$. The k^{th} row (resp. column) of a diagram $\boldsymbol{d}$ is the subset of cells such that $j = k-1$ (resp. $i = k-1$). The *row-reading order* of the cells of $\boldsymbol{d}$ is the order such that $(i, j) < (i', j')$ if $j < j'$ or if $j = j'$ and $i < i'$.

A diagram is said to be a *Ferrers diagram* (using French convention) if for every $(i, j) \in \boldsymbol{d}$, the diagram also contains every $(i', j') \in \mathbb{N} \times \mathbb{N}$ such that $i' \leq i$ and $j' \leq j$. If λ_k is the number of cells in the k^{th} row of a Ferrers diagram, and $|\boldsymbol{d}| = n$, then $\lambda = (\lambda_1, \lambda_2, ..)$ is a decreasing sequence of positive integers who sums to n, i.e. a *partition* of n, denoted $\lambda \vdash n$. This sequence completely determines the diagram, and we write λ for both the partition and its associated Ferrers diagram.

Let Ω be a set. A *tableau* τ of *shape* λ with entries in Ω is a map $\tau : \lambda \to \Omega$. It is often displayed as a filing of the boxes of λ. The $\mathbb{C}$-vector space formally spanned by these tableaux is a $(\mathfrak{S}_\Omega \times \mathfrak{S}_\lambda)$-bimodule, where $\mathfrak{S}_\Omega$ acts on variables, *i.e.* $(\rho \cdot \tau)(i, j) = \rho(\tau(i, j))$ for $\rho \in \mathfrak{S}_\Omega$; and $\mathfrak{S}_\lambda$ acts on places (or cells), *i.e.* $(\tau \cdot \pi)(i, j) = \tau(\pi(i, j))$ for $\pi \in \mathfrak{S}_\lambda$. We then consider R_λ (resp. C_λ), the subgroup of permutations in $\mathfrak{S}_\lambda$ which only permutes cells lying in a same row (resp. column) in a tableau of shape λ. If, for $x \in \Omega$, we denote $\gamma_x = |\tau^{-1}(x)|$, the sequence $\gamma = (\gamma_x)_{x \in \Omega}$ is called the *content* of τ.

If $|\Omega| = |\lambda| = n$, we may consider the set of bijective tableaux $t : \lambda \to \Omega$, usually denoted by lowercase Latin letters. Since, in that case, $\mathfrak{S}_\Omega \cong \mathfrak{S}_\lambda$, for our purpose it suffices to consider the action of $\mathfrak{S}_n$ on the $\mathbb{C}$-span of these bijective tableaux. Unless specified, we choose Ω to be the set $\boldsymbol{x} = \{x_1, ..., x_n\}$, but we can make isomorphic constructions for any set of size n.

For each bijective tableau t of shape λ, we associate the *tableau monomial*, which is $\boldsymbol{x}^t = \prod_{(i,j) \in [\lambda]} t(i, j)^j$. The $\mathbb{C}$-span of those monomials is called the *permutation module* associated to λ and denoted M^λ. In this module, we define the polynomials $\Delta_t = \sum_{\pi \in C_t} \varepsilon(\pi) \pi \cdot \boldsymbol{x}^t$. The $\mathbb{C}$-span of those polynomials, called the *Specht module* associated to λ, is denoted S^λ. We recall some well-known facts about these modules (see Sagan, [13]):

Proposition 2.1. • *The set $\{S^\lambda \mid \lambda \vdash n\}$ is a complete list of non-isomorphic irreducible representations of $\mathfrak{S}_n$;*

- *The Frobenius transform of the character of M^λ is h_λ;*
- *The Frobenius transform of the character of S^λ is s_λ.*

2.2 Wreath product

For $n, m \in \mathbb{N}$, let G (resp. H) be a subgroup of $\mathfrak{S}_m$ (resp. $\mathfrak{S}_n$). Also, let $f : H \to \mathrm{Aut}(G^n)$ be the morphism such that for every $h \in H$, $f(h)$ sends $(g_1, ..., g_n) \in G^n$ to $(g_{h^{-1}(1)}, ..., g_{h^{-1}(n)})$. We define the *wreath product* of G with H, denoted $G \wr H$, to be the semidirect product $G^n \rtimes_f H$.

Let V be a G-module and W be a H-module. The tensor product $V^{\otimes n}$ is naturally a G^n-module, and it is also a H-module, where H acts by permuting the components of a tensor. The interaction of the two actions correspond to the wreath product, so it is in fact a $G \wr H$-module. Also, there is a canonical surjection $G \wr H \twoheadrightarrow H$, so we can construct the inflated $G \wr H$-module $\mathrm{Inf}_H^{G \wr H} W$. We then define $V \oslash W$ to be the $G \wr H$-module given by $V^{\otimes n} \otimes \mathrm{Inf}_H^{G \wr H} W$. When $G = \mathfrak{S}_m$ and $H = \mathfrak{S}_n$, the operation $\oslash$ mimics the plethysm of symmetric functions, in the following sense:

Proposition 2.2. *Let V be a $\mathfrak{S}_m$-module such that the Frobenius transform of its character is f, and let W be a $\mathfrak{S}_n$-module such that the Frobenius transform of its character is g. Then, the Frobenius transform of the character of $(V \oslash W)\big\uparrow_{\mathfrak{S}_m \wr \mathfrak{S}_n}^{\mathfrak{S}_{nm}}$ is the plethysm $g[f]$.*

So, if we want to study the plethysms $h_\nu[h_\mu]$, we can do it by studying the modules $(M^\mu \oslash M^\nu)\big\uparrow_{\mathfrak{S}_m \wr \mathfrak{S}_n}^{\mathfrak{S}_{nm}}$.

2.3 The modules M_μ^ν

We now describe a combinatorial description of the modules $(M^\mu \oslash M^\nu)\big\uparrow_{\mathfrak{S}_m \wr \mathfrak{S}_n}^{\mathfrak{S}_{nm}}$ for $\mu \vdash m$ and $\nu \vdash n$. Let $\boldsymbol{x} = \{x_1, ..., x_{nm}\}$ and $\mathcal{M}_\mu$ be the set of injective tableaux $t : \mu \to \boldsymbol{x}$. Also, denote $\mathcal{N}_{\mu,\nu}$ the set of injective tableaux $\boldsymbol{T} : \nu \to \mathcal{M}_\mu$ such that the union of the entries of each entry t of $\boldsymbol{T}$ is $\boldsymbol{x}$. Thus, an element of $\mathcal{N}_{\mu,\nu}$ is a tableau of shape ν whose entries are tableaux of shape μ and such that each element of $\boldsymbol{x}$ appears exactly one time. By permuting all the entries at once, the $\mathbb{C}$-span of the elements of $\mathcal{N}_{\mu,\nu}$ is a $\mathfrak{S}_{nm}$-module, isomorphic to $\mathbb{C}[\mathfrak{S}_{nm}]$. For example, if $\mu = (2, 1)$ and $\nu = (2, 2)$, then an element of $\mathcal{N}_{\mu,\nu}$

is

$$\boldsymbol{T} = \begin{array}{|c|c|} \hline \begin{array}{|c|c} \hline x_4 & \\ \hline x_1 & \multicolumn{1}{|c|}{x_5} \\ \hline \end{array} & \begin{array}{|c|c} \hline x_9 & \\ \hline x_6 & \multicolumn{1}{|c|}{x_2} \\ \hline \end{array} \\ \hline \begin{array}{|c|c} \hline x_{12} & \\ \hline x_{10} & \multicolumn{1}{|c|}{x_3} \\ \hline \end{array} & \begin{array}{|c|c} \hline x_7 & \\ \hline x_9 & \multicolumn{1}{|c|}{x_{11}} \\ \hline \end{array} \\ \hline \end{array}.$$

We would like to have a monomial that is invariant under the row permutations of both $\boldsymbol{T}$ and each of its entries. First, if we consider the tableau monomial of an entry t of $\boldsymbol{T}$ as defined earlier, it is invariant under row permutations of t, but it does not keep track of the entries of t in the first row. This is a problem, because we have to know which variables are in the first row of each entry of $\boldsymbol{T}$. We can multiply the tableau monomial by each variable x_i that appears in t, so that the $\mathfrak{S}_m$-module structure is preserved. We obtain what we call the *higher tableau monomial*, which is

$$\xi^t = \left(\prod_{(i,j)\in\mu} t(i,j) \right) \boldsymbol{x}^t = \prod_{(i,j)\in\mu} t(i,j)^{j+1}.$$

In order to use each ξ^t to construct a monomial that is invariant under the row permutations of $\boldsymbol{T}$, consider the ring $\mathbb{C}\left[X \mid X \text{ is a monomial of } \mathbb{C}[\boldsymbol{x}]\ \right]$ (i.e. we consider each monomial of $\mathbb{C}[\boldsymbol{x}]$ as a variable of this polynomial ring), and denote $*$ its multiplication. In this ring, we can define the tableau monomial of $\boldsymbol{T}$, named *plethystic tableau monomial*, to be

$$\xi^{*\boldsymbol{T}} = \mathop{*}_{(i,j)\in\nu} (\xi^{\boldsymbol{T}(i,j)})^{*(j+1)}.$$

For example, the plethystic tableau monomial of $\boldsymbol{T}$ as above is

$$\xi^{*\boldsymbol{T}} = (x_{10}x_3x_{12}^2) * (x_9x_{11}x_7^2) * (x_1x_5x_4^2)^{*2} * (x_6x_2x_9^2)^{*2}.$$

Let M_μ^ν be the $\mathfrak{S}_{nm}$-module generated by the $\xi^{*\boldsymbol{T}}$ for $\boldsymbol{T} \in \mathcal{N}_{\mu,\nu}$.

We have the following result, which is implicit in [12]:

Proposition 2.3. *The representation M_μ^ν is isomorphic to the representation $(M^\mu \oslash M^\nu)\uparrow_{\mathfrak{S}_m \wr \mathfrak{S}_n}^{\mathfrak{S}_{nm}}$.*

So, we can think of an element of M_μ^ν as a monomial with exponents corresponding to ν whose variables are monomials with exponents corresponding to μ and such that this "monomial of monomials" contains each variable in $\boldsymbol{x} = \{x_1, ..., x_{nm}\}$ exactly once. Note that $M_{(1)}^\mu \cong M^\mu$, the usual permutation module.

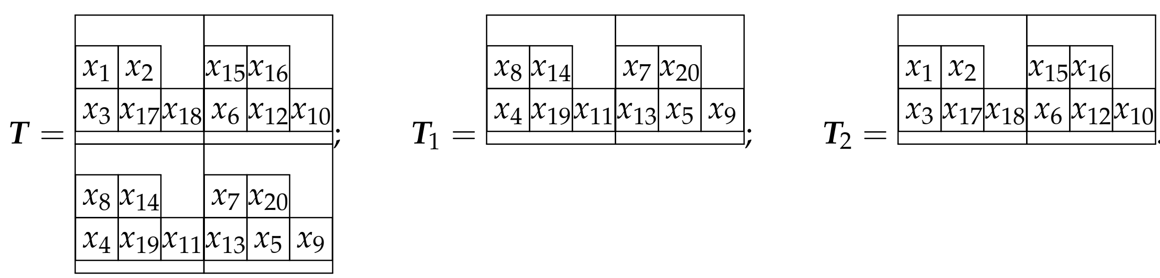
3 The morphisms

3.1 Decomposition via the tensor product

For two partitions ν_1, ν_2, we define their union $\nu_1 \uplus \nu_2$ to be the partition that has the parts of μ_1 followed by the parts of μ_2, and then reordered to obtain a partition. For example, $(2,2) \uplus (3,1) = (3,2,2,1)$. Also, if t_1 is a tableau of shape ν_1 and t_2 is a tableau of shape ν_2, we define their union $t_1 \uplus t_2$ to be the tableau obtained by stacking t_1 and t_2, and then reordering the rows so that it is a tableau of shape $\nu_1 \uplus \nu_2$. We show in [15] the following results:

Proposition 3.1. *Let μ be a partition of m and ν be a partition of n. Suppose that there is ν_1, ν_2 such that $\nu = \nu_1 \uplus \nu_2$, with $\nu_1 \vdash n_1$ and $\nu_2 \vdash n_2$. Then, we have an isomorphism $M_\mu^\nu \cong \left(M_\mu^{\nu_1} \otimes M_\mu^{\nu_2}\right) \uparrow_{\mathfrak{S}_{mn_1} \times \mathfrak{S}_{mn_2}}^{\mathfrak{S}_{mn}}$.*

For example, if $\mu = (3,2)$ and $\nu = (2,2)$, then $\nu = (2) \uplus (2)$, and if we consider

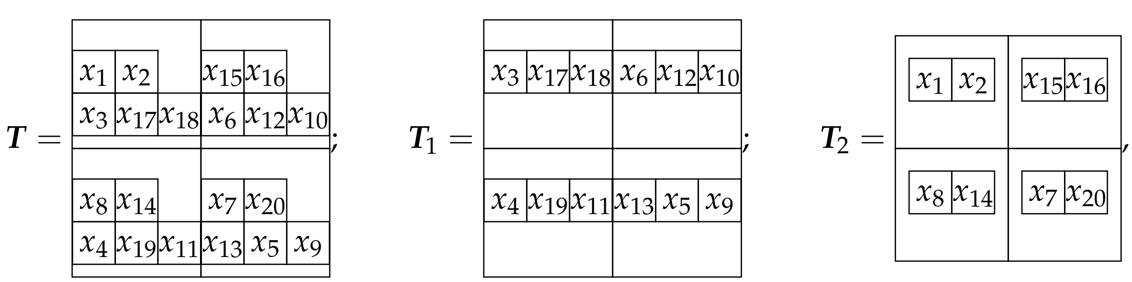


then the elements $\xi^{*\boldsymbol{T}}$ and $\xi^{*\boldsymbol{T}_1} \otimes \xi^{*\boldsymbol{T}_2}$ carry the same information.

When we decompose μ in the same way, we have a weaker result:

Proposition 3.2. *Let μ be a partition of m and ν be a partition of n. Suppose that there is μ_1, μ_2 such that $\mu = \mu_1 \uplus \mu_2$, with $\mu_1 \vdash m_1$ and $\mu_2 \vdash m_2$. Then, we have an injection $\left(M_{\mu_1}^\nu \otimes M_{\mu_2}^\nu\right) \uparrow_{\mathfrak{S}_{nm_1} \times \mathfrak{S}_{nm_2}}^{\mathfrak{S}_{nm}} \hookrightarrow M_\mu^\nu$.*

For example, if $\mu = (3,2)$ and $\nu = (2,2)$ as before, then $\mu = (3) \uplus (2)$, and if we have



then $\xi^{*\boldsymbol{T}_1} \otimes \xi^{*\boldsymbol{T}_2}$ is sent to $\dfrac{1}{4} \displaystyle\sum_{\sigma \in R_{\boldsymbol{T}_1} \times R_{\boldsymbol{T}_2}} \sigma \cdot \xi^{*\boldsymbol{T}}$.

By combining these two results, we obtain a proof of 1.1 in [15].

3.2 Decomposition using permutation modules

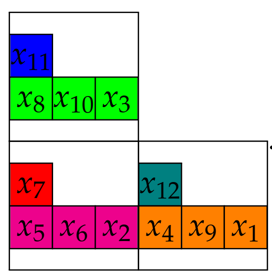
For a partition μ, let μ^n be $\underbrace{\mu \uplus ... \uplus \mu}_{n \text{ times}}$, and consider the permutation module M^{μ^n}. By proposition 3.1 (for $\nu = (1)$), this module is isomorphic to the module $(M^{\mu})^{\otimes n}\uparrow_{(\mathfrak{S}_n)^m}^{\mathfrak{S}_{nm}}$. So, we can write an element of M^{μ^n} as $\xi^{t_1} \otimes ... \otimes \xi^{t_n}$, where each t_i is a tableau of shape μ and each variable of $\boldsymbol{x} = \{x_1, .., x_{nm}\}$ is used exactly once. Let $\boldsymbol{T} \in \mathcal{N}_{\mu,\nu}$ be such that if (i,j) is its ℓ^{th} cell in row-reading order, then, $\boldsymbol{T}(i,j) = t_\ell$. We then define the projection

$$\begin{array}{cccc} \phi : & M^{\mu^n} & \twoheadrightarrow & M^{\nu}_{\mu} \\ & \xi^{t_1} \otimes ... \otimes \xi^{t_n} & \longmapsto & \xi^{*\boldsymbol{T}} \end{array}.$$

Reciprocally, for $\xi^{*\boldsymbol{T}} \in M^{\nu}_{\mu}$, if we denote $\boldsymbol{T}(i,j) = t_\ell$ when (i,j) is the ℓ^{th} cell of $\boldsymbol{T}$ in row-reading order, there is an injective homomorphism

$$\begin{array}{cccc} \widetilde{\phi} : & M^{\nu}_{\mu} & \hookrightarrow & M^{\mu^n} \\ & \xi^{*\boldsymbol{T}} & \longmapsto & \dfrac{1}{\nu!} \displaystyle\sum_{\sigma \in R_\nu} \xi^{t_{\sigma(1)}} \otimes ... \otimes \xi^{t_{\sigma(n)}}. \end{array},$$

We can easily see that $\phi \circ \widetilde{\phi} = \mathrm{Id}$. For example, let $\mu = (3,1)$ and $\nu = (2,1)$. An element of $\mathcal{N}_{\mu,\nu}$ is

$\boldsymbol{T} =$ .

The corresponding element of M^{ν}_{μ} is $\xi^{*\boldsymbol{T}} = (x_5x_6x_2x_7^2) * (x_4x_9x_1x_{12}^2) * (x_8x_{10}x_3x_{11}^2)^{*2}$. We then have:

$$\widetilde{\phi}(\xi^{*\boldsymbol{T}}) = \frac{1}{2}\left(x_5x_6x_2x_7^2 \otimes x_4x_9x_1x_{12}^2 \otimes x_8x_{10}x_3x_{11}^2 + x_4x_9x_1x_{12}^2 \otimes x_5x_6x_2x_7^2 \otimes x_8x_{10}x_3x_{11}^2\right)$$
$$\phi(\widetilde{\phi}(\xi^{\boldsymbol{T}})) = \frac{1}{2}\Big((x_5x_6x_2x_7^2) * (x_4x_9x_1x_{12}^2) * (x_8x_{10}x_3x_{11}^2)^{*2}$$
$$+ (x_4x_9x_1x_{12}^2) * (x_5x_6x_2x_7^2) * (x_8x_{10}x_3x_{11}^2)^{*2}\Big),$$

which is exacty $\xi^{\boldsymbol{T}}$. These morphisms are essential in our setting, as we use them in sections 3.3 and 5.

3.3 Generalized Foulkes-Howe map

We have already seen that the fact that $\nu \trianglelefteq \mu$ is equivalent to the existence of an injective homomorphism $M^{\nu}_{\mu} \hookrightarrow M^{\mu}_{\nu}$. We describe in the previous section a morphism from M^{ν}_{μ} to M^{μ^n}, and another from M^{ν^m} to M^{μ}_{ν}. The only missing part is a morphism from M^{μ^n} to M^{ν^m}.

Let $\xi^{t_1} \otimes ... \otimes \xi^{t_n}$ be an element of $(M^{\mu})^{\otimes n}\uparrow^{\mathfrak{S}_{nm}}_{(\mathfrak{S}_n)^m}$, which is isomorphic to M^{μ^n}. Denote $x_{i,j}$ the j^{th} entry (in row-reading order) of t_i. By construction, the set of all $x_{i,j}$ is equal to $\boldsymbol{x}$, so it is only a relabelling of the variables. Then, for $1 \leq j \leq m$, consider the tableau $s_j : \nu \to \{x_{1,j}, ..., x_{m,j}\}$ in row-reading order. We define the following morphism:

$$\begin{array}{cccc} \Psi : & M^{\mu^n} & \longrightarrow & M^{\nu^m} \\ & \xi^{t_1} \otimes ... \otimes \xi^{t_n} & \longmapsto & \dfrac{1}{\mu!} \displaystyle\sum_{\sigma \in R_{t_1} \times ... \times R_{t_n}} \sigma \cdot \xi^{s_1} \otimes ... \otimes \xi^{s_m} \end{array} .$$

For example, if $\mu = (2,2)$ and $\nu = (2,1)$, take the monomial $x_1 x_2 x_5^2 x_6^2 x_9^3 x_{10}^3 x_3^4 x_4^4 x_7^5 x_8^5 x_{11}^6 x_{12}^6$ of M^{μ^n}, which corresponds to the tensor product $x_1 x_2 x_3^2 x_4^2 \otimes x_5 x_6 x_7^2 x_8^2 \otimes x_9 x_{10} x_{11}^2 x_{12}^2$. It can be written as $\xi^{t_1} \otimes \xi^{t_2} \otimes \xi^{t_3}$, with

$$t_1 = \begin{array}{|c|c|} \hline x_3 & x_4 \\ \hline x_1 & x_2 \\ \hline \end{array}, \quad t_2 = \begin{array}{|c|c|} \hline x_7 & x_8 \\ \hline x_5 & x_6 \\ \hline \end{array}, \quad t_3 = \begin{array}{|c|c|} \hline x_{11} & x_{12} \\ \hline x_9 & x_{10} \\ \hline \end{array}.$$

We then construct the following tableaux:

$$s_1 = \begin{array}{|c|c} \hline x_9 & \\ \hline x_1 & \multicolumn{1}{c|}{x_5} \\ \hline \end{array}, \quad s_2 = \begin{array}{|c|c} \hline x_{10} & \\ \hline x_2 & \multicolumn{1}{c|}{x_6} \\ \hline \end{array}, \quad s_3 = \begin{array}{|c|c} \hline x_{11} & \\ \hline x_3 & \multicolumn{1}{c|}{x_7} \\ \hline \end{array}, \quad s_4 = \begin{array}{|c|c} \hline x_{12} & \\ \hline x_4 & \multicolumn{1}{c|}{x_8} \\ \hline \end{array},$$

We can conclude that the image of the monomial by Ψ is

$$\frac{1}{4} \sum_{\sigma \in R_{t_1} \times R_{t_2} \times R_{t_3}} \sigma \cdot x_1 x_5 x_2^2 x_6^2 x_3^3 x_7^3 x_4^4 x_8^4 x_9^5 x_{10}^6 x_{11}^7 x_{12}^8.$$

Now consider the composition of homomorphisms $\phi \circ \Psi \circ \widetilde{\phi} : M^{\nu}_{\mu} \to M^{\mu}_{\nu}$. When $\mu = (m)$, $\nu = (n)$, this map is the *Foulkes-Howe map* states in the language of symmetric groups. This map was developed as a tool to prove the Foulkes' conjecture. For this reason, we will call this homomorphism the *generalized Foulkes-Howe map*, and we denote it $\mathcal{F}_{\mu,\nu}$. We use this map in the next section.

4 First results

We can use the tools we just developed to prove theorem 1.2. Full proofs are given in [15]. Part (a) is the following proposition:

Proposition 4.1. *For any positive integer n and partition μ, there is an injection $M_{\mu}^{1^n} \hookrightarrow M_{1^n}^{\mu}$.*

For example, if $\mu = (2,1)$ and $n = 2$, we have that (using the generalized Foulkes-Howe map)

$$\mathcal{F}_{\mu,1^n}\left((x_2x_3x_4^2) * (x_5x_6x_1^2)^{*2}\right) = \frac{1}{4}\sum_{\sigma\in\mathfrak{S}_{\{2,3\}}\times\mathfrak{S}_{\{5,6\}}} \sigma\cdot(x_2x_5^2)*(x_3x_6^2)*(x_4x_1^2)^{*2},$$

and we can see that the image completely determine the element we chose from $M_{\mu}^{1^n}$. As a corollary, we obtain part (b) of theorem 1.2:

Corollary 4.2. *For any positive integer k and partition μ, there is an injection $M_{\mu}^{\mu\uplus 1^k} \hookrightarrow M_{\mu\uplus 1^k}^{\mu}$.*

Using the proposition 1.1, we also prove in [15] the two last parts of theorem 1.2:

Proposition 4.3. *For any positive integer k and partition μ, there is an injection $M_{\mu}^{\mu^k} \hookrightarrow M_{\mu^k}^{\mu}$.*

Proposition 4.4. *If the Foulkes' conjecture is true up to $n-1$, then for all partition μ such that $\mu_1 \le n$, there is an injection $M_{(n)}^{\mu} \hookrightarrow M_{\mu}^{(n)}$.*

5 Decomposition via semistandard homomorphisms

5.1 Semistandard homomorphisms

We use here the description of [8] for a basis of the homomorphisms from S^λ to a permutation module M^γ, and mostly use its notations.

Let $\lambda \vdash n$ and $\tau : \lambda \to \mathbb{N}^*$ be a tableau of content γ. We denote $T(\lambda,\gamma)$ the set of such tableaux. Remark that up to a relabelling of the entries, we can always choose γ to be in decreasing order, so that it corresponds to a partition. Also, using the identification $\mathfrak{S}_\lambda \cong \mathfrak{S}_n$, the $\mathfrak{S}_n$-module $\mathbb{C}[T(\lambda,\gamma)]$ is isomorphic to M^γ, where we now consider this module to be generated by the monomials ξ^t (instead of the $\boldsymbol{x}^t$, as in the usual definition). In effect, for any bijective tableau $t : \gamma \to \boldsymbol{x}$, the map $f_t : \mathbb{C}[T(\lambda,\gamma)] \to M^\gamma$ such that $f_t(\tau) = \prod_{(i,j)\in\lambda} t(i,j)^{\tau(i,j)}$ is an isomorphism.

Denote by r_τ the number of $\sigma \in R_\tau$ such that $\tau\cdot\sigma = \tau$. For every $\tau \in T(\lambda,\gamma)$, we define a homomorphism $\widehat{\Theta}_\tau$ by setting

$$\begin{aligned}\widehat{\Theta}_\tau :\ M^\lambda &\longrightarrow M^\gamma\\ \xi^t &\mapsto \frac{1}{r_\tau}\sum_{\sigma\in R_\lambda} f_t(\tau\cdot\sigma)\end{aligned}$$

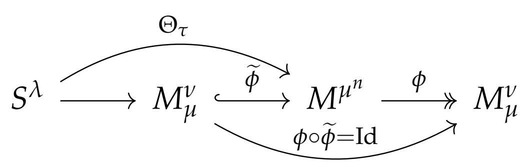
If i is the inclusion $i : S^\lambda \hookrightarrow M^\lambda$, we define $\Theta_\tau : S^\lambda \to M^\gamma$ to be the morphism $\widehat{\Theta}_\tau \circ i$. Note that $\Theta_\tau = 0$ if and only if τ has a column containing two equal entries.

We say that a tableau τ with entries in $\mathbb{N}$ is *semistandard* if it is weakly increasing in the rows (from left to right) and strictly increasing in the columns (from bottom to top). When $\tau \in T(\lambda, \gamma)$ is semistandard, we say that Θ_τ is a *semistandard homomorphism*. These homomorphisms have the nice following property:

Proposition 5.1. *The set* $\{\Theta_\tau \mid \tau \in T(\lambda, \gamma)$ *semistandard*$\}$ *is a basis for* $Hom_{\mathfrak{S}_r}(S^\lambda, M^\gamma)$.

5.2 Application for the modules M^ν_μ

We want to use the semistandard homomorphisms to study the modules M^ν_μ. First, combining the semistandard homomorphisms and the morphisms of section 3.2, we obtain the following setting:



So for λ a partition of nm and τ a semistandard tableau of shape λ and content μ^n, if we define $\overline{\Theta}_\tau = \phi \circ \Theta_\tau$, we have that $\overline{\Theta}_\tau$ is a generating set of $\mathrm{Hom}_{\mathfrak{S}_{nm}}(S^\lambda, M^\nu_\mu)$. Hence, we can have a combinatorial description of the decomposition of the modules M^ν_μ. However, as ϕ is a projection, this set is far from being linearly independent, and this tool is generally not efficient to calculate this decomposition, although it can be a good tool to make proofs about these modules.

6 Stability properties

We use semistandard homomorphisms to prove that some sequences of plethysm coefficients are increasing. To do so, we generalize the arguments of de Boeck in [2]. We only put here the lemmas needed to show theorems 1.3 and 1.4. The proofs of these lemmas and theorems are in [15].

For two tableaux τ_1, τ_2, we define the *join* of the two tableaux, denote $\tau_1 \vee \tau_2$, to be the tableau such that the row i, when read from left to right, consists of the row i of τ_1 followed by the row i of τ_2. If τ_1 is of shape μ_1 and τ_2 is of shape μ_2, then $\tau_1 \vee \tau_2$ is of shape $\mu_1 + \mu_2$, where we use the componentwise addition of integer vectors.

We use in this first lemma the tableau $\tau_{(nm'),\mu'^n}$ for $\mu' \vdash m'$, which is the only semistandard tableau of shape (nm') and content μ'^n. For any $\mu \vdash m$ and $\nu \vdash n$, we have the following lemma:

Lemma 6.1. *Let $\tau \in T(\lambda, \mu^n)$ such that $\overline{\Theta}_\tau : S^\lambda \to M^\nu_\mu$ is non-zero. Then, for any partition $\tilde{\mu}$ of an integer $\tilde{m}$, the tableau $\tilde{\tau} = \tau \vee \tau_{(n\tilde{m}),\tilde{\mu}^n}$ is such that $\overline{\Theta}_{\tilde{\tau}} : S^{\lambda+(n\tilde{m})} \to M^\nu_{\mu+\tilde{\mu}}$ is also non-zero.*

This lemma is the main ingredient to prove theorem 1.3.

We also generalize another stability property, originally due to Dent [6]. For a tableau τ' of shape (2^k) and another tableau τ, we have that $\tau' \vee \tau$ corresponds to adding two columns of length k at the left of τ. Moreover, we write $\tau_{(2^k),(2^k)}$ for the only semistandard tableau of shape and content (2^k). Using this, we prove the following lemma:

Lemma 6.2. *Let $\tau \in T(\lambda, \mu^n)$ such that $\overline{\Theta}_\tau : S^\lambda \to M^\nu_\mu$ is non-zero. Then, if the number of parts of μ is $\tilde{m}$, the tableau $\tilde{\tau} = \tau_{(2^{\tilde{m}n}),(2^{\tilde{m}n})} \vee \tau$ is such that $\overline{\Theta}_{\tilde{\tau}} : S^{\lambda+(2^{\tilde{m}n})} \to M^\nu_{\mu+(2^{\tilde{m}})}$ is also non-zero.*

This lemma is also the main ingredient to prove theorem 1.4, by a similar proof than the one of 1.3.

References

[1] F. Bergeron. *Algebraic Combinatorics and Invariant Spaces*. CRC Press, 2009.

[2] M. de Boeck. "On the structure of Foulkes modules for the symmetric group". PhD thesis. University of Kent, 2015. Link.

[3] M. Brion. "Stable Properties of Plethysm: on Two Conjectures of Foulkes". *Manuscripta mathematica* **80**.4 (1993), pp. 347–372.

[4] M.-W. Cheung, C. Ikenmeyer, and S. Mkrtchyan. "Symmetrizing Tableaux and the 5th Case of the Foulkes Conjecture". *Journal of Symbolic Computation* **80** (Sept. 2015).

[5] S. Dent and J. Siemons. "On a Conjecture of Foulkes". *Journal of Algebra* **226**.1 (2000), pp. 236 –249.

[6] S. C. Dent. "Incidence structures of partitions". PhD thesis. University of East Anglia, 1997. Link.

[7] H. O. Foulkes. "Concomitants of the Quintic and Sextic Up To Degree Four in the Coefficients of the Ground Form". *Journal of the London Mathematical Society* **s1-25**.3 (1950), pp. 205–209.

[8] G. D. James. *The Representation Theory of the Symmetric Groups*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 1978.

[9] D. E. Ltttlewood. "Polynomial Concomitants and Invariant Matrices". *Journal of the London Mathematical Society* **s1-11**.1 (1936), pp. 49–55.

[10] I. G. MacDonald and A. V. Zelevinsky. *Symmetric functions and Hall polynomials; 2nd ed.* Oxford classic texts in the physical sciences. Oxford University Press, 1998.

[11] T. McKay. "On Plethysm Conjectures of Stanley and Foulkes". *Journal of Algebra* **319** (Mar. 2008), pp. 2050–2071.

[12] R. Paget and M. Wildon. "Generalized Foulkes modules and maximal and minimal constituents of plethysms of Schur functions". *Proceedings of the London Mathematical Society* **118**.5 (2019), pp. 1153–1187.

[13] B. E. Sagan. *The symmetric group - representations, combinatorial algorithms, and symmetric functions*. Wadsworth, 1991.

[14] R. Stanley. "Positivity Problems and Conjectures in Algebraic Combinatorics". *in Mathematics: Frontiers and Perspectives* (1999), pp. 295–319.

[15] É. Tétreault. "A new order on integer partitions". 2020. arXiv:2010.15333.

[16] R. M. Thrall. "On Symmetrized Kronecker Powers and the Structure of the Free Lie Ring". *American Journal of Mathematics* **64**.1 (1942), pp. 371–388.