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Tableau Stabilization and Lattice paths (extended abstract)

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Abstract. If one attaches shifted copies of a skew tableau to the right of itself and rectifies, at a certain point the copies no longer experience vertical slides, a phenomenon called tableau stabilization. While tableau stabilization was originally developed to construct the sufficiently large rectangular tableaux fixed by given powers of promotion, the purpose of this extended abstract is to improve the original bound on tableau stabilization to the number of rows of the skew tableau. In order to prove this bound, we encode increasing subsequences as lattice paths and show that various operations on these lattice paths weakly increase the maximum combined length of the increasing subsequences.

Keywords: Young Tableau, RSK, Green's theorem, Lattice paths, Increasing sequences, Jeu-De-Taquin

1 Introduction

Tableau stabilization was introduced in [1] in order to construct sufficiently large rectangular tableaux fixed by given powers of promotion. Rhoades first counted the number of rectangular tableaux fixed by the powers of promotion by exhibiting a remarkable cyclic sieving phenomenon [6] for the action of promotion on rectangular tableaux [7]. Alexandersson, OÄ§uz, and Linusson have recently found similar results for certain families of semistandard Young tableaux, like stretched hooks, disjoint rectangles, and special cases of small ribbons [4]. Moreover, Alexandersson, Pfannerer, Rubey, and Uhlin showed that whenever the fake degree polynomial $f^{\lambda/\mu}(q)$ associated to a skew shape λ/μ evaluates to a nonnegative integer at roots of unity, then there is a CSP triple $(\mathrm{SSYT}(\lambda/\mu), C_n, f^{\lambda/\mu}(q))$ for some cyclic action C_n on $\mathrm{SSYT}(\lambda/\mu)$, not necessarily promotion [2].

But while many of these CSPs count fixed points of promotion, they say nothing about what the actual fixed points are. Purbhoo first found all rectangular tableaux of shape (a^b) fixed by a promotions for $b \geq a$ [5]. More recently, Ahlbach exhibited all sufficiently large rectangular tableaux fixed by a given power of promotion by applying the rectification operator to skew tableaux formed by attaching shifted copies of a skew tableaux to itself [1]. This naturally gives rise to the notion of tableaux stabilization.

Definition 1.1. For any skew standard tableau T with weakly decreasing row sizes from top to bottom, let $T^{(k)}$ denote the result of attaching $(k-1)$ shifted copies of T to the right of T so that the result is a skew standard tableau. Let n denote the size of T and k be a positive integer. Let Rect denote Schützenberger's rectification operator [10]. We say *T stabilizes* at k if each element of $[(k-1)n+1, kn]$ lies in the same row in $\mathrm{Rect}(S^{(k)})$ as it does in $S^{(k)}$. Let $\mathrm{stab}(S)$ denote the minimum value at which S stabilizes.

Remark 1.2. Having weakly decreasing row sizes from top to bottom is required for $T^{(k)}$ to be a skew standard tableau for all $k \geq 1$. If this is not the case, $T^{(k)}$ need not be standard. Consider Figure 1, where $T^{(2)}$ is not a skew tableau both because of its shape

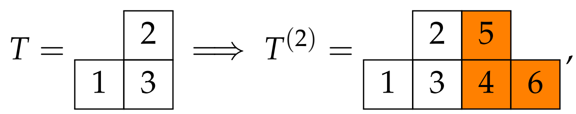


Figure 1: The number of boxes in each row has to be weakly decreasing from top to bottom, or else $T^{(k)}$ need not be standard.

and the third column not being increasing.

Example 1.3. In Figure 2, 9 does not lie in the same row in $T^{(3)}$ and $\mathrm{Rect}(T^{(3)})$, but $15, 16, \ldots, 21$ do, so $\mathrm{stab}(T) = 3$.

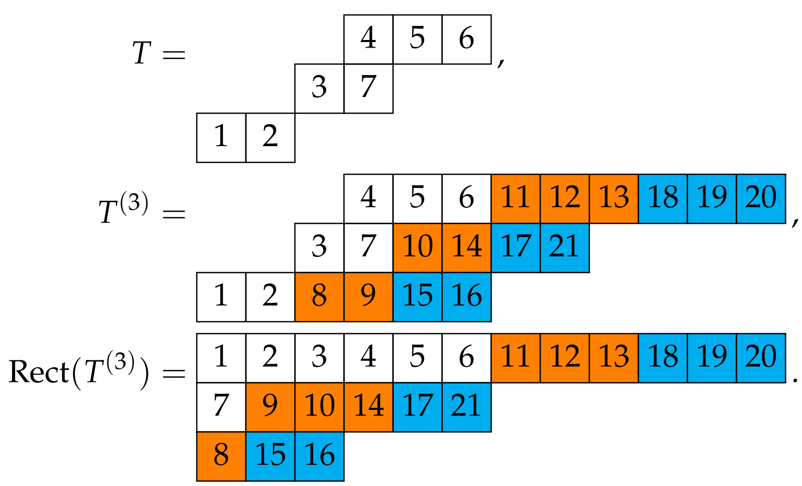


Figure 2: Example of a skew tableau T, the tableau $T^{(3)}$ constructed from T, and its rectification.

Ahlbach proved that once a tableaux stabilizes at a value, it continues to stabilize for higher values, [1, Lemma 3.9]. In the special case where the rows of T have the same size, he derived a formula for the shape of $\mathrm{Rect}(T^{(k)})$ for $k \geq r-1$, [1, Theorem 1.6], and used it to show that any tableaux with r rows of the same size stabilizes at r, [1, Theorem 1.4]. He conjectured that the same bound still holds when the rows have weakly decreasing sizes but was only able to prove a bound of $\max(1, 2r-2)$. The main purpose of this paper is to prove this conjecture.

Theorem 1.4. For any skew standard tableaux T with r rows and weakly decreasing row sizes from top to bottom,

$$\mathrm{stab}(T) \leq r.$$

The proof of the same-size-rows case of Theorem 1.4 in [1] relied on a formula for the shape of the stabilized tableau. Unfortunately, an analogous formula for the general case would have to involve new terms not present in the previous shape formula, and we have not found such a formula.

Yet, we generalize the lattice path argument in the proof of Lemma 4.2 in [1] to prove Theorem 1.4. Our argument relies on Greene's Theorem [3] characterizing the shape of the insertion tableaux of words coming from the RSK-correspondence [9], in terms of their increasing subsequences and a careful analysis of the increasing subsequences of reading words of $T^{(k)}$ for skew standard tableaux T.

In section 2, we show that the family of increasing sequences can be encoded by a family of lattice paths. In section 3, we go over various operations on a family of lattice paths that weakly increase the maximum combined length of corresponding increasing subsequences. In section 4, we prove the main result using the tools developed in the previous sections.

2 Longest Increasing Subsequences

In this section we will go over Greene's theorem and set up the language of the matrix and lattice paths we will use. For a more detailed introduction and applications of longest increasing sequences, we recommend the reader to [8]. Before we introduce the essential tools, we will briefly explain why we need them.

2.1 Greene's theorem

Definition 2.1. (Reading Word) The *reading word* of a tableau is the word obtained by concatenating the rows from bottom to top. For a non-skew tableau T, let $\mathrm{sh}(T)$ denote its shape.

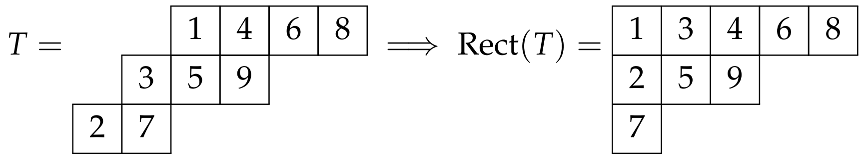


Figure 3: The reading word of T is 273591468. The tableau to the right is its rectification, and $\mathrm{sh}(\mathrm{Rect}(T)) = (5, 3, 1)$.

Theorem 2.2 (Greene's Theorem, [3])**.** Let π be the reading word of a (skew) standard Young tableau T and let $\ell_k(\pi)$ denote the maximum combined length of k disjoint increasing subsequences of π. Then,

$$\ell_k(\pi) = \mathrm{sh}(\mathrm{Rect}(T))_1 + \cdots + \mathrm{sh}(\mathrm{Rect}(T))_k.$$

2.2 Lattice paths

Suppose T is a skew standard tableau with weakly decreasing row sizes from top to bottom. Let n denote the size of T, and r the number of rows of T. For each positive integer j, let $T + (j-1)n$ be obtained from T by shifting the entries up by $(j-1)n$. We define $T^{(q)}$ to be obtained by concatenating $T, T+n, \ldots, T+(q-1)n$ together from left to right. For each positive integer q, we create an r-by-q matrix $M = M(q, T)$ from T: each entry $M_{i,j}$ is a word set as the $(r-j+1)$-th row of T, with all entries shifted up by $(i-1)n$.

Remark 2.3. We will label coordinates Cartesian-style rather than matrix-style since that is more natural for lattice paths.

Example 2.4. With T as in Figure 3, we have

$$M(3,T) = \begin{array}{ccc} 1468 & 1468 & 1468 \\ 359 & 359 & 359 \\ 27 & 27 & 27 \end{array}$$

The orange and cyan colors indicate that the entries are shifted up by 9 and 18 respectively. So 3 is actually $3 + 9 = 12$ whereas 3 is actually $3 + 18 = 21$. We have $M_{1,1} = 27$ and $M_{3,2} = 359$.

Definition 2.5. For any sequences A, B, let $A \preceq B$ denote that A is a subsequence of B. For a sequence A and a set I we use $A\,|_I$ to denote the restriction of A to I. For example, if $A = 7164532$, then

$$A\,|_{\{1,2,3,4\}} = 1432 \preceq A.$$

A ***lattice path*** within M is a sequence of entries of M that move adjacently right or up at each step. Given a lattice path S, we use $S|_{i,*}$ to denote the subpath of S by restricting ourselves to column i of M. Similarly we use $S|_{*,j}$ to denote the subpath of S by restricting ourselves to row j of M.

Definition 2.6. For possibly overlapping words $A_1, \ldots, A_k$, let

$$\ell(A_1, \ldots, A_k) = \text{the maximum combined length of disjoint increasing subsequences of } A_1, \ldots, A_k \text{ respectively.}$$

We say that a collection of disjoint increasing sequences $S_1 \preceq A_1, \ldots, S_k \preceq A_k$ ***exhibits*** $\ell(A_1, \ldots, A_k)$ if $|S_1| + \cdots + |S_k| = \ell(A_1, \ldots, A_k)$.

Lemma 2.7. Any increasing subsequence of the reading word of $T^{(k)}$ is a subsequence of a lattice path within M.

Example 2.8. With T as in Figure 3, we have

$\mathrm{Rect}(T^{(3)}) =$

1	3	4	6	8	10	13	15	17	19	22	24	26
2	5	9	12	14	18	21	23	27				
7	11	16	20	25								

with shape $\lambda = (13, 9, 5)$ and

$$M(3,T) = \begin{array}{ccc} 1468 & 1468 & 1468 \\ 359 & 359 & 359 \\ 27 & 27 & 27 \end{array}.$$

A longest increasing subsequence of $M(3,T)$ is 2356814681468, which has size 13 and is a subsequence of the lattice path in Figure 4. Note this agrees with Greene's Theorem

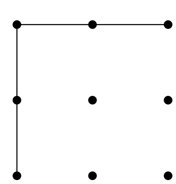


Figure 4: The lattice path 273591468146814681468 containing 2356814681468.

since $\lambda_1 = 13$. Moreover, two disjoint increasing subsequences of $M(3,T)$ with the longest combined length are 146814681468 and 2359359359, have total size 22 and are subsequences of the lattice paths in Figure 5. Again, this agrees with Greene's Theorem since $\lambda_1 + \lambda_2 = 13 + 9 = 22$.

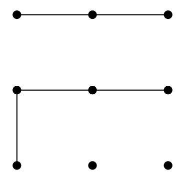
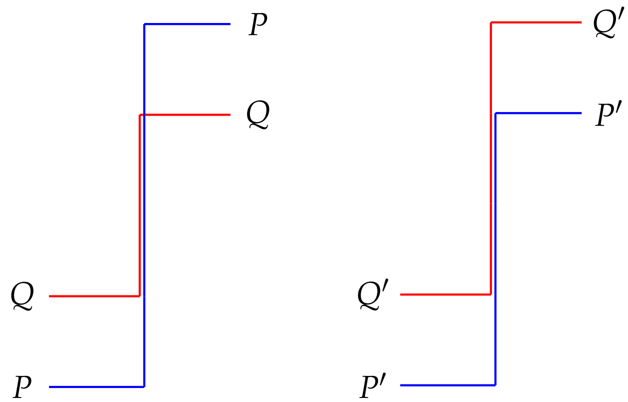


Figure 5: The minimal lattice paths containing 146814681468 and 2359359359.



(a) Crossing paths. (b) After switching.

Figure 6: The top-down switching process.

3 Lattice Path Tools

In this section, we will introduce some transformations on the family of lattice paths so that the maximum combined length of increasing subsequences they contain weakly increases.

The first tool, top-down switching, will allow us to modify the paths so that they do not cross (intersections can happen, but they will be non-transversal), and thus can be ordered in a top-down fashion. The second tool, left-shifting, will allow us to shift portions of vertical segments of a path to the left as long as no new intersections appear. The last two tools, rectangular and reverse rectangular flip, will allow us to split paths with shared horizontal segments, while maintaining a top-down order.

3.1 Top-Down Switching

Consider two lattice paths P(blue) and Q(red) within the matrix M that cross in a way that they switch from Q being on top to P being on top, as in the left figure of Figure 6.

Consider the result of switching the labels on the paths after P and Q diverge, giving new paths P' (blue) and Q' (red) as shown in the right figure of Figure 6. After this top-down switch, we have a clear upper and lower path.

An example of top-down switching is given in going from rightmost figure of Figure 11 to leftmost figure of Figure 12. Each application of top-down switching eliminates a transverse crossing from our set of paths. Thus, after applying top-down switching as many times as possible in any order, there are no longer any transverse crossings in the resulting family of paths. Then we can order the family of paths from top to bottom. Hence, top-down switching lets us adjust the paths so they come in a top-down order $P_1, \ldots, P_k$ where P_i lies weakly above P_{i+1} for all i.

3.2 Left-shifting

Definition 3.1. A ***left-shift*** of a lattice path is an operation where part of the column of a path, following a horizontal segment of length at least two, is shifted one column to the left and creates no new intersections, as in Figure 7.

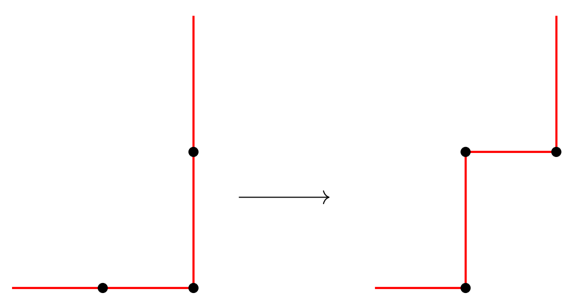


Figure 7: Left-shifting applied to a lattice path

We are introducing left-shifting because it again weakly increases the maximum combined length of increasing subsequences contained in the lattice paths.

An example of left-shifting is given in going from leftmost figure to the middle figure in Figure 12. Beware that when we left-shift, we are not allowed to introduce a new intersection: hence the furthest we can shift a particular path to the left is bounded by paths above.

3.3 Rectangular flip and reverse rectangular flip.

A rectangular flip is given in Figure 8. Note that the red and blue paths in the left figure have a common horizontal segment at the start. After using the rectangular flip there, the blue path now is one height higher than its original position at the segment. An example of rectangular flip used in a family of paths is illustrated in going from leftmost figure to the middle figure in Figure 11.

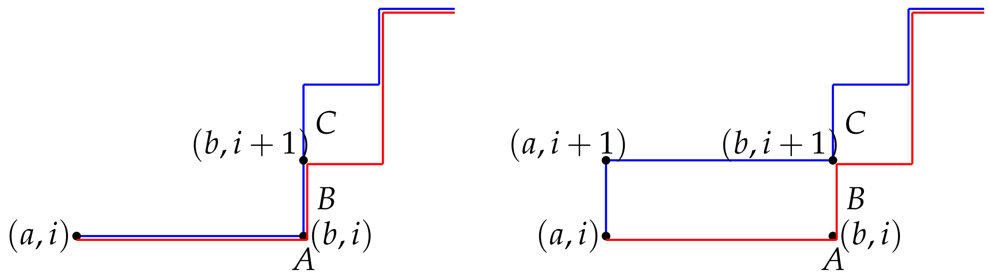


Figure 8: A rectangular flip on the blue path with respect to the red path

A reverse rectangular flip is given in Figure 9. Initially, the red and blue paths in the left figure have a common horizontal segment at the end. After using the reverse rectangular flip there, the blue path now is one height lower than its original position at the segment. An example of a reverse rectangular flip used in a family of paths is illustrated in going from middle figure to the rightmost figure in Figure 13.

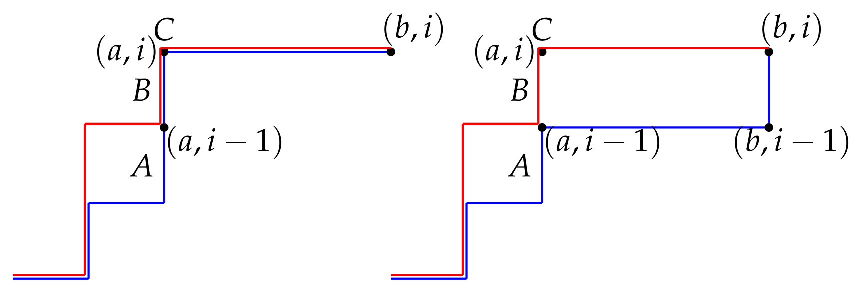


Figure 9: A reverse rectangular flip on the blue path with respect to the red path

4 The main result

We have developed several tools regarding how the lattice paths encoding the longest sequences behave and how families of paths can be adjusted while weakly increasing the combined length they represent. In this section we provide an example of how we use these tools to transform a family of paths while weakly increasing the combined length of increasing sequences they encode, proving Theorem 1.4.

Lemma 4.1. There exists a length-maximizing family of k lattice paths $P_1, \ldots, P_k$ where for each i, the path P_i starts at the point $(1, k+1-i)$ in the matrix.

Hence we may assume that each path P_i starts at the point $(1, k-i+1)$. Beware that this does not imply that each P_i has the horizontal segment $(1, k-i+1) - (2, k-i+1)$.

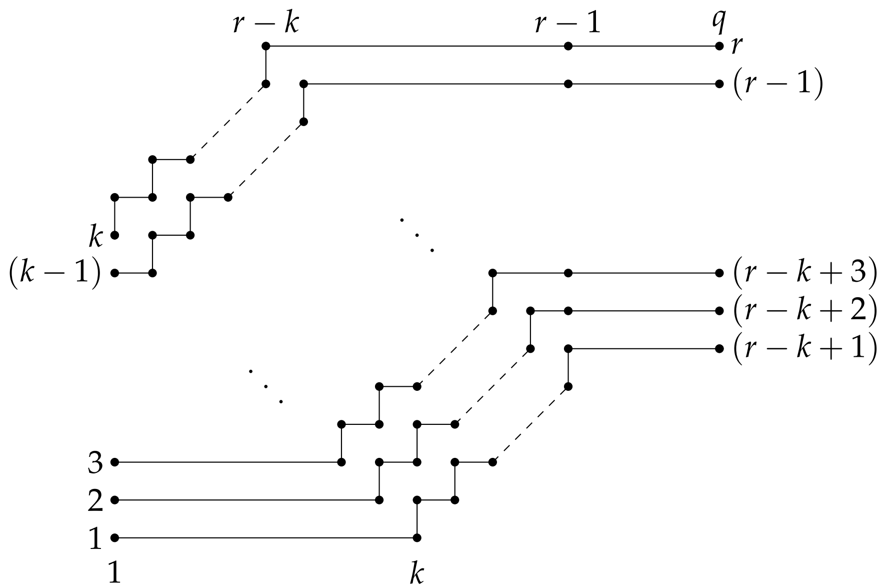


Figure 10: The lower boundary paths for k paths: $L_1, \ldots, L_k$ (from top to bottom). The numbers on the left (and right) indicate the row indices and the numbers on the top and bottom indicate the column indices.

Figure 10 shows $L_1, \ldots, L_k$. These paths appear in [1, Lemma 4.2] as well. If all rows of T have the same size, this family of paths is a length-maximizing family [1, Lemma 4.2]. If the rows or T have different sizes, this may no longer be the case, and we have to allow our family of paths to be weakly above these in respective top to bottom order.

Lemma 4.2. There exists a length-maximizing family of paths $P_1, \ldots, P_k$ such that P_i is weakly above L_i for all i, and each P_i contains the horizontal segment $(r-k+i-1, r-i+1) - (q, r-i+1)$.

Figures 11 through 13 illustrates the the whole transformation process using the tools we have developed so far. From using the horizontal segments of the family we produced and Green's theorem, 1.4 follows:

The main idea of our proof was to narrow our search for a length-maximizing family of lattice paths to those satisfying a certain property - namely lying above the bounding paths and ending in the same horizontal segments at the bounding paths. Is there is an efficient way to construct a particular family of length-maximizing paths given the tableau?

Open Problem 4.3. Given a skew tableau T, is there a simple method to find a length-maximizing family of k-paths in $M(q, T)$ for each k and q?

We have found an upper bound on the stabilization index of a tableau. It seems a far-fetched goal at this moment to obtain this index without using Greene's theorem or

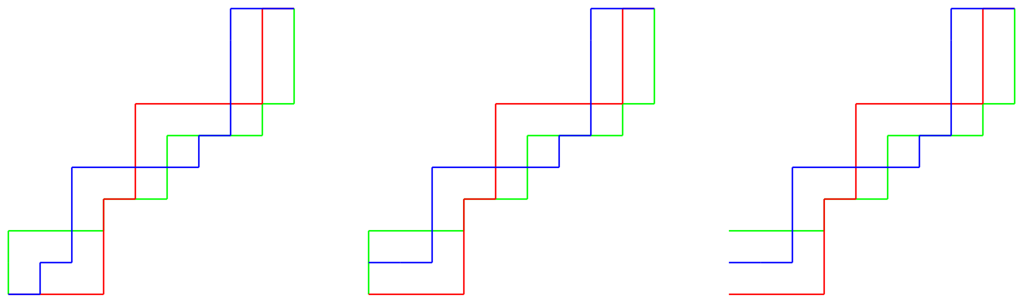


Figure 11: Running example: Start with a family of paths. Apply rectangular flip(s) to get a family satisfying Lemma 4.1. After this we may ignore the vertical segments in the first column to get a family satisfying the conditions of Lemma 4.1.

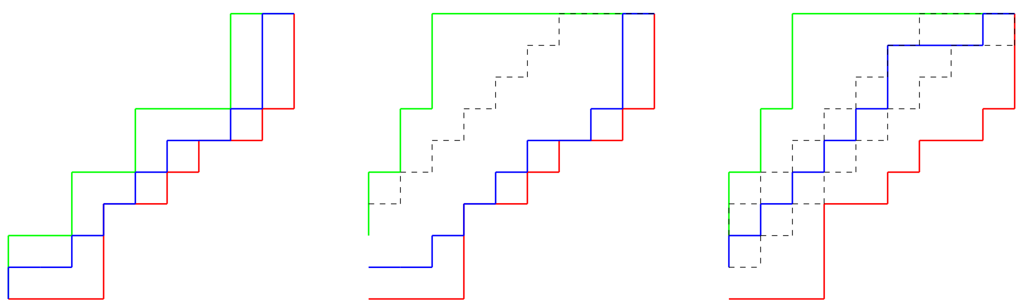


Figure 12: Running example: Apply top-down switching to get a family of paths that do not cross. Next apply left-shift on P_1 to get a path that stays weakly above the bounding path L_1, drawn in black dotted lines. Do the same for P_2 afterwards.

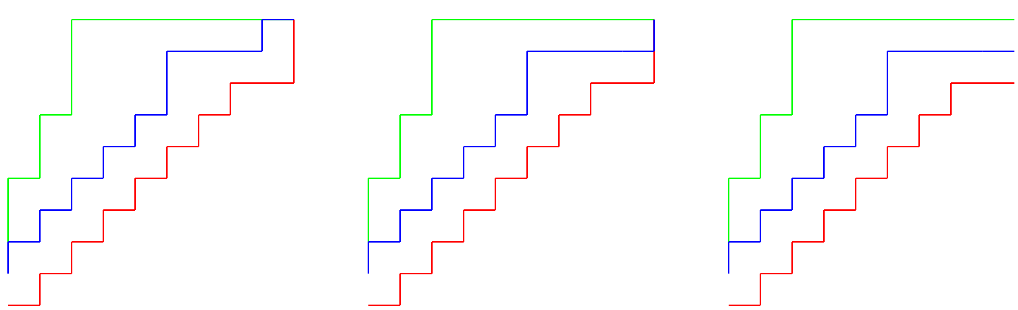


Figure 13: Running example: Apply left-shift on P_3. Lastly, apply reverse rectangular flip(s) to get a family satisfying Lemma 4.2.

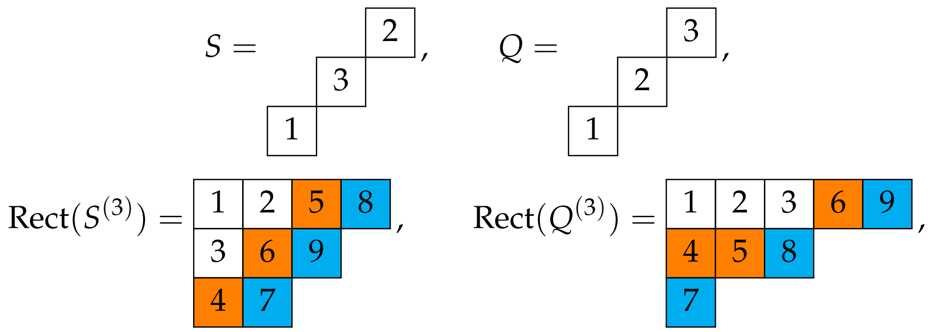


Figure 14: Tableaux $S = T_{132}$ and $Q = T_{123}$. We can see that the stabilization index of 132 is 2 whereas the stabilization index of 123 is 3.

constructing the rectification. A reasonable subclass of tableaux to restrict attention to would be tableaux constructed from permutations as in Figure 14. Given a permutation w, we can construct the skew tableau T_w which has one entry per row and w as its reading word. Then we can define $\operatorname{stab}(w)$ as $\operatorname{stab}(T_w)$, which give us a permutation statistic!

In Chapter 8 of [1], Ahlbach introduced the stabilization index as a permutation statistic. He showed that $\operatorname{stab}(w)$ is bounded strictly below by the ascent statistic, [1, Lemma 8.4], showed that $\operatorname{stab}(w)$ depends only the recording tableau $Q(w)$, [1, Lemma 8.3], characterized the permutations with stab 1, [1, Lemma 8.5], and characterized the permutations with stab 2, [1, Theorem 8.7]. But there are still many open questions! A full characterization of $\operatorname{stab}(w)$ in terms of $Q(w)$ would be ideal.

Open Problem 4.4 ([1])**.** For a permutation w, is there a way to find $\operatorname{stab}(w)$ directly from the permutation or its recording tableau (that is, without constructing the rectification or using Greene's theorem)? What is the relationship between $\operatorname{stab}(w)$ and $Q(w)$?

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