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The first higher Stasheff–Tamari orders are quotients of the higher Bruhat orders

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Abstract. We study a map from the higher Bruhat orders to the first higher Stasheff–Tamari orders and show that it is a quotient map of posets, resolving a conjecture of Dimakis and Müller-Hoissen. The result reveals that the first higher Stasheff–Tamari orders are connected with KP solitons and polygon equations. Our work also involves further development of the theory of quotients of posets.

Keywords: Cyclic polytopes, cyclic zonotopes, higher Bruhat orders, higher Stasheff–Tamari orders, triangulations, cubillages

1 Introduction

Two of the best known and most widely studied partially ordered sets in mathematics are the Tamari lattice and the weak Bruhat order on the symmetric group. These posets both possess higher-dimensional versions, namely the first higher Stasheff–Tamari orders $\mathcal{S}(n, \delta)$ [5, 10] and the higher Bruhat orders $\mathcal{B}(n, \delta + 1)$ [12]. These also have the structure of higher categories, as described by Kapranov and Voevodsky in [10].

The relation between the Tamari lattice and the weak Bruhat order has been of significant interest. There is a classical surjection from the latter to the former, which can be realised as a map from permutations to binary trees. Kapranov and Voevodsky extended this map to a map $f\colon \mathcal{B}(n, \delta) \to \mathcal{S}(n + 2, \delta + 1)$ [10], which they conjectured was a surjection. This remains an open problem despite some detailed study [15, 18].

In the paper [20], we consider a different map from the higher Bruhat orders to the first higher Stasheff–Tamari orders. The elements of the higher Bruhat orders are cubillages of cyclic zonotopes and the elements of the higher Stasheff–Tamari orders are triangulations of cyclic polytopes. We study the map

$$g\colon \mathcal{B}(n, \delta + 1) \to \mathcal{S}(n, \delta),$$

given by taking the vertex figure of a cubillage at the bottom vertex. Other cross-sections of zonotopal subdivisions were considered in [6, 14]. The dual of this map was first

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considered in [18], and the map itself was considered in [2, Appendix B]. We provide a new proof that this map is a surjection, which was shown in [16], and go further by showing that the map g is full. We call order-preserving maps which are both surjective and full *quotient maps of posets*. Indeed, in this paper we develop further the theory of quotients of posets, as studied in [9, 17].

Theorem A (Corollary 6.3)**.** *The map* $g\colon \mathcal{B}(n, \delta+1) \to \mathcal{S}(n, \delta)$ *is a quotient map of posets.*

Part of the motivation for considering the map g stems from the paper [3], where Dimakis and Müller-Hoissen define a quotient of the higher Bruhat orders called the "higher Tamari orders". These orders describe how a class of solutions to the KP equation evolve. Dimakis and Müller-Hoissen conjecture the higher Tamari orders to coincide with the first higher Stasheff–Tamari orders. We prove this conjecture using Theorem A by showing that the higher Tamari orders are given by the image of the map g, as first noted in [2, Appendix B]. This unites two far-reaching sets of combinatorics: the first higher Stasheff–Tamari orders, with their connections to representation theory of algebras [19], and the higher Tamari orders, which describe classes of KP solitons [3] and from which arise the polygon equations [4].

Theorem B (Corollary 6.4)**.** *The higher Tamari orders and the first higher Stasheff–Tamari orders coincide.*

Our approach is to use the description of cubillages of cyclic zonotopes in terms of separated collections established in [7] and studied extensively in [2]. This description allows us to construct pre-images under the map g, which is instrumental in the proof of Theorem A.

The present paper is an extended abstract of [20]. In Section 2 we provide background on the higher Bruhat orders and first higher Stasheff–Tamari orders. In Section 3 we give three different interpretations of the map g. We outline our framework of quotient posets in Section 4. Finally, in Sections 5 and 6, we respectively prove that g is surjective and full. Due to space constraints, we omit some proofs from [20], and only provide sketches for the others.

2 Background

2.1 Higher Bruhat orders

2.1.1 Cubillages

We first give the geometric description of the higher Bruhat orders due to [10, 18]. Consider the *Veronese curve*

$$\begin{aligned}\xi : \mathbb{R} &\to \mathbb{R}^{\delta+1}\\ t &\mapsto \xi_t = (1, t, \dots, t^{\delta}),\end{aligned}$$

and let $\{t_1, \dots, t_n\} \subset \mathbb{R}$ with $t_1 < \cdots < t_n$ and $n \geqslant \delta + 1$. The *cyclic zonotope* $Z(n, \delta + 1)$ is defined to be the Minkowski sum of the line segments $\overline{\mathbf{0}\xi_{t_1}}, \dots, \overline{\mathbf{0}\xi_{t_n}}$. The properties of $Z(n, \delta + 1)$ do not depend on the exact choice of $\{t_1, \dots, t_n\} \subset \mathbb{R}$. Hence, for ease we choose $t_i = i$. For $k \geqslant l$ we have a canonical projection map

$$\begin{aligned}\pi_{k,l} \colon \mathbb{R}^k &\to \mathbb{R}^l\\ (x_1, \dots, x_k) &\mapsto (x_1, \dots, x_l)\end{aligned}$$

which maps $Z(n, k) \to Z(n, l)$.

A *cubillage* $\mathcal{Q}$ of $Z(n, \delta + 1)$ is a subcomplex of $Z(n, n)$ such that $\pi_{n,\delta+1} \colon \mathcal{Q} \to Z(n, \delta + 1)$ is a bijection. Note that $\mathcal{Q}$ therefore contains faces of $Z(n, n)$ of dimension at most $\delta + 1$. The elements of the *higher Bruhat poset* $\mathcal{B}(n, \delta + 1)$ consist of cubillages of $Z(n, \delta + 1)$. Cubillages are often called *fine zonotopal tilings*, see [7, 8].

The *lower facets* of a polytope in $\mathbb{R}^k$ are the facets which can be seen from a very large negative k-th coordinate; the *upper facets* of a polytope in $\mathbb{R}^k$ are those which can be seen from a very large positive k-th coordinate. The covering relations of $\mathcal{B}(n, \delta + 1)$ are given by pairs of cubillages $\mathcal{Q} \lessdot \mathcal{Q}'$ where there is a $(\delta + 2)$-face Γ of $Z(n, n)$ such that $\pi_{n,\delta+2}(\mathcal{Q})$ and $\pi_{n,\delta+2}(\mathcal{Q}')$ differ only in that $\pi_{n,\delta+2}(\mathcal{Q})$ contains the lower facets of $\pi_{n,\delta+2}(\Gamma)$ and $\pi_{n,\delta+2}(\mathcal{Q}')$ contains the upper facets of $\pi_{n,\delta+2}(\Gamma)$. Here we say that $\mathcal{Q}'$ is an *increasing flip* of $\mathcal{Q}$. In terms of the interpretation of $\mathcal{B}(n, \delta + 1)$ as a higher category in [10], increasing flips constitute the 1-morphisms, with higher-dimensional morphisms given by higher-dimensional faces of $Z(n, n)$.

2.1.2 Separated collections

Now we explain how one may characterise cubillages as separated collections of subsets, as shown in [7] and studied in [2, 1]. The subsets $E \subseteq [n] := \{1, \dots, n\}$ are naturally identified with the corresponding points $\xi_E := \sum_{e \in E} \xi_e$ in $Z(n, n)$, where $\xi_{\varnothing} = \mathbf{0}$. This represents each vertex of a cubillage $\mathcal{Q}$ as a subset of $[n]$. For a cubillage $\mathcal{Q}$ of $Z(n, \delta + 1)$, the collection of subsets corresponding to its vertices is called the *spectrum* of $\mathcal{Q}$ and is denoted by $\mathrm{Sp}(\mathcal{Q})$.

Let n be a positive integer and let $\delta \in [n-1]$. Given two sets $A, B \subseteq [n]$, we say that A *δ-interweaves* B if there exist $i_{\delta+1}, i_{\delta-1}, \ldots \in B \setminus A$ and $i_{\delta}, i_{\delta-2}, \ldots \in A \setminus B$ such that

$$i_0 < i_1 < \cdots < i_{\delta+1}.$$

If either A δ-interweaves B or B δ-interweaves A, then we say that A and B are *δ-interweaving*. If A and B are not δ-interweaving then we say that A and B are *δ-separated*. We call a collection $\mathcal{A} \subset 2^{[n]}$ δ-separated if it is pairwise δ-separated.

It follows from [7, Theorem 2.7] that the correspondence $\mathcal{Q} \mapsto \mathrm{Sp}(\mathcal{Q})$ gives a bijection between the set of cubillages on $Z(n, \delta+1)$ and the set of δ-separated collections of maximal size in $2^{[n]}$. The maximal size of a δ-separated collection in $2^{[n]}$ is $\Sigma_{i=0}^{\delta+1}\binom{n}{i}$, so $\#\mathrm{Sp}(\mathcal{Q}) = \Sigma_{i=0}^{\delta+1}\binom{n}{i}$ for any cubillage $\mathcal{Q}$ of $Z(n, \delta+1)$. Hence, one may define the higher Bruhat orders $\mathcal{B}(n, \delta+1)$ to consist of δ-separated collections in $2^{[n]}$ of maximal size. As we show in [20], the covering relations are then given by pairs of maximal-size separated collections $\mathcal{C} \lessdot \mathcal{C}'$ such that $\mathcal{C}' = (\mathcal{C} \setminus \{A\}) \cup \{B\}$, where A δ-interweaves B.

For $A \subseteq [n]$, if $\pi_{n,\delta+1}(\xi_A)$ is a boundary vertex of the zonotope $Z(n, \delta+1)$, then ξ_A is a vertex of every cubillage of $Z(n, \delta+1)$, and hence in every δ-separated collection in $2^{[n]}$ of maximal size. Hence the subsets of interest are those whose corresponding points lie in the interior of the zonotope $Z(n, \delta+1)$. We call these *internal* vertices of the cubillage and define the internal spectrum $\mathrm{ISp}(\mathcal{Q})$ of $\mathcal{Q}$ to consist of the subsets corresponding to internal vertices of $\mathcal{Q}$. We have that $\#\mathrm{ISp}(\mathcal{Q}) = \binom{n-1}{\delta+1}$ for a cubillage $\mathcal{Q}$ of $Z(n, \delta+1)$, as we explain in [20].

2.1.3 Admissible orders

The original definition of the higher Bruhat orders from [12] is as follows. Given $A \in \binom{[n]}{\delta+2}$, the set

$$P(A) = \left\{ B \;\middle|\; B \in \binom{[n]}{\delta+1}, B \subset A \right\}$$

is called the *packet* of A. The set $P(A)$ is naturally ordered by the *lexicographic order*, where $P(A) \setminus a_i < P(A) \setminus a_j$ if and only if $j < i$. An ordering α of $\binom{[n]}{\delta+1}$ is *admissible* if the elements of any packet appear in lexicographic or reverse-lexicographic order under α. Two orderings α and α' are *equivalent* if they differ by a sequence of interchanges of pairs of adjacent elements that do not lie in a common packet. We use $[\alpha]$ to denote the equivalence class of α. The *inversion set* $\mathrm{inv}(\alpha)$ of an admissible ordering α is the set of all $(\delta+2)$-subsets of $[n]$ whose packets appear in reverse-lexicographic order in α. The higher Bruhat poset $\mathcal{B}(n, \delta+1)$ is the partial order on equivalence classes of admissible orders of $\binom{[n]}{\delta+1}$, with covering relations given by $[\alpha] \lessdot [\alpha']$ for $\mathrm{inv}(\alpha') = \mathrm{inv}(\alpha) \cup \{A\}$ where $A \in \binom{[n]}{\delta+2} \setminus \mathrm{inv}(\alpha)$.

2.2 Higher Stasheff–Tamari orders

In this section we give the definition of the first higher Stasheff–Tamari orders. These were originally defined by Kapranov and Voevodsky under the name the *higher Stasheff orders* [10]. Edelman and Reiner then introduced the *first* and *second higher Stasheff–Tamari orders* in [5]. Thomas later proved that the first higher Stasheff–Tamari orders were the same as the higher Stasheff orders [18, Proposition 3.3].

The *moment curve* is defined by $p_t = (t, t^2, \ldots, t^\delta) \subset \mathbb{R}^\delta$ for $t \in \mathbb{R}$. Choose $t_1, \ldots, t_n \in \mathbb{R}$ such that $t_1 < t_2 < \cdots < t_n$, where $n \geqslant \delta + 1$. The *cyclic polytope* $C(n, \delta)$ is defined to be the convex hull $\operatorname{conv}(p_{t_1}, \ldots, p_{t_n})$. The properties of the cyclic polytope do not depend on the exact choice of $\{t_1, \ldots, t_n\} \subset \mathbb{R}$. Hence, for ease we choose $t_i = i$. For a subset $A = \{a_0, \ldots, a_k\} \subseteq [n]$, we use $|A|$ to denote its geometric realisation $\operatorname{conv}(p_{a_0}, \ldots, p_{a_k})$.

A *triangulation* of the cyclic polytope $C(n, \delta)$ is a subcomplex $\mathcal{T}$ of $C(n, n-1)$ such that $\pi_{n-1,\delta} \colon \mathcal{T} \to C(n, \delta)$ is a bijection. After [10, 18], we define the *first higher Stasheff–Tamari poset* $\mathcal{S}(n, \delta)$ as follows. The elements of $\mathcal{S}(n, \delta)$ are triangulations of $C(n, \delta)$. The covering relations of $\mathcal{S}(n, \delta)$ are given by pairs of triangulations $\mathcal{T} \lessdot \mathcal{T}'$ where there is a $(\delta + 1)$-face Σ of $C(n, n-1)$ such that $\pi_{n-1,\delta+1}(\mathcal{T})$ and $\pi_{n-1,\delta+1}(\mathcal{T}')$ differ only in that $\pi_{n-1,\delta+1}(\mathcal{T})$ contains the lower facets of $\pi_{n-1,\delta+1}(\Sigma)$ and $\pi_{n-1,\delta+1}(\mathcal{T}')$ contains the upper facets of $\pi_{n-1,\delta+1}(\Sigma)$. Here we say that $\mathcal{T}'$ is an *increasing flip* of $\mathcal{T}$. The interpretation of $\mathcal{S}(n, \delta)$ as a higher category in [10] is similar to $\mathcal{B}(n, \delta + 1)$: the 1-morphisms are increasing flips, with the higher-dimensional morphisms given by higher-dimensional faces of $C(n, n-1)$.

3 Interpretations of the map

We study the map

$$g \colon \mathcal{B}(n, \delta + 1) \to \mathcal{S}(n, \delta)$$

which has the following three interpretations.

(1) *Cubillages.* Every cubillage $\mathcal{Q}$ of $Z(n, \delta + 1)$ induces a triangulation of $C(n, \delta)$, given by taking the vertex figure of $Z(n, n)$ at $\xi_\varnothing$. This can be seen from the fact that the vertex figure at $\xi_\varnothing$ is given by the intersection of $Z(n, n)$ with the affine hyperplane

$$H_n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = 1\},$$

which gives the cyclic polytope $C(n, n-1)$. Such cross-sections of cubillages were also considered in [6, 8, 14]. The fact that $\pi_{n,\delta+1} \colon \mathcal{Q} \to Z(n, \delta + 1)$ is a bijection gives that $\pi_{n,\delta+1} \colon \mathcal{Q} \cap H_n \to Z(n, \delta + 1) \cap H_{\delta+1} \cong C(n, \delta)$ is a bijection, so that $\mathcal{Q} \cap H_n$ is indeed a triangulation of $C(n, \delta)$. Our principal definition of the map g is thus that $g(\mathcal{Q}) = \mathcal{Q} \cap H_n$.

One can show that the map g is order-preserving, as was noted in [2, Appendix B]. Suppose an increasing flip of a cubillage $\mathcal{Q}$ is induced by a $(\delta + 2)$-face Γ of $Z(n, n)$. Then

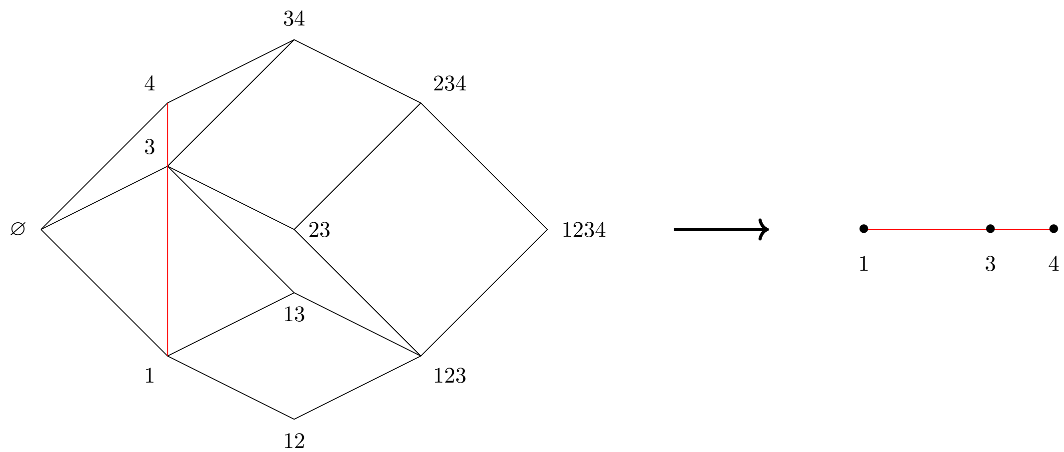

Figure 1: Applying the map g to a cubillage of $Z(4,2)$ to obtain a triangulation of $C(4,1)$.

if $\xi_\varnothing$ is not a vertex of Γ, the increasing flip will not affect $g(\mathcal{Q})$. If $\xi_\varnothing$ is a vertex of Γ, then $\Gamma \cap H_n$ is a simplex, and the increasing flip replaces the lower facets of $\pi_{n,\delta+2}(\Gamma \cap H_n)$ with its upper facets, giving an increasing flip of $g(\mathcal{Q})$.

Examples of the map g being applied to cubillages of $Z(4,2)$ and $Z(4,3)$ are shown in Figures 1 and 2. Note that we illustrate cubillages of $Z(n,\delta+1)$ by their projections to $\delta+1$ dimensions and triangulations of $C(n,\delta)$ by their projections to δ dimensions.

(2) *Separated collections.* A triangulation of $C(n,\delta)$ is determined by its internal $\lfloor \delta/2 \rfloor$-simplices. One may define $g(\mathcal{Q})$ to be the triangulation $\mathcal{T}$ of $C(n,\delta)$ with its internal $\lfloor \delta/2 \rfloor$-simplices given by $\mathrm{ISp}(\mathcal{Q}) \cap \binom{[n]}{\lfloor \delta/2 \rfloor + 1}$. This can be verified in Figures 1 and 2. In Figure 1, the internal spectrum of the cubillage is $\{3, 13, 23\}$, so applying g gives the triangulation of $C(4,1)$ with $|3|$ as its only internal 0-simplex. In Figure 2, the internal spectrum of the cubillage is $\{13\}$, so applying g gives the triangulation of $C(4,2)$ with $|13|$ as its only internal 1-simplex. We use this definition to prove that g is surjective and full.

(3) *Admissible orders.* The following notions were used to define the higher Tamari orders in [3]. Let α be an admissible order of $\binom{n}{\delta+1}$ and $I \in \binom{n}{\delta+1}$. Given $k \in [n] \setminus I$, we say that I is *invisible in* $P(I \cup \{k\})$ if either

- $I \cup \{k\} \notin \mathrm{inv}(\alpha)$ and $\#\{i \in I \mid i > k\}$ is odd, or
- $I \cup \{k\} \in \mathrm{inv}(\alpha)$ and $\#\{i \in I \mid i > k\}$ is even.

We say that I is *invisible in* α if there is a $k \in [n] \setminus I$ such that I is invisible in $P(I \cup \{k\})$. Otherwise, we say that I is *visible in* α. Given an admissible order α of $\binom{[n]}{\delta+1}$, we use $V(\alpha)$ to denote the elements of $\binom{[n]}{\delta+1}$ which are visible in α.

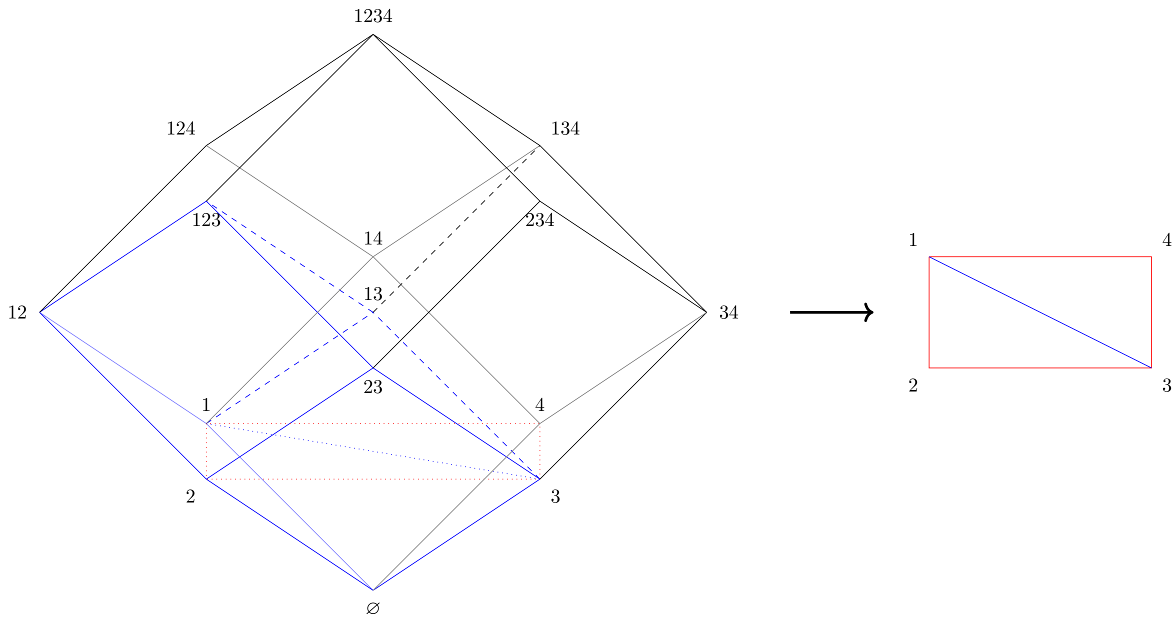

Figure 2: Applying the map g to a cubillage of $Z(4,3)$ to obtain a triangulation of $C(4,2)$.

Given an admissible order α of $\binom{[n]}{\delta+1}$, we write $\mathcal{Q}_\alpha$ for the corresponding cubillage of $Z(n,\delta+1)$. As was noted in [2, Appendix B], one may define $g(\mathcal{Q}_\alpha)$ to be the triangulation with

$$\{|A| \mid A \in V(\alpha)\}$$

as its set of δ-simplices. This follows from the description from [18] of the cubillage given by an admissible order, provided one swaps a sign convention for δ odd.

4 Quotient posets

We wish to show that g realises $\mathcal{S}(n,\delta)$ as a quotient of $\mathcal{B}(n,\delta+1)$, but first we need to make this statement precise. Given a poset $(X, \leqslant)$ subject to an equivalence relation $\sim$, the *quotient* $(X/{\sim}, R)$ is defined to be the set of $\sim$-equivalence classes $[x]$ of X, with the binary relation R defined by $[x]R[y]$ if and only if there exist $x' \in [x]$ and $y' \in [y]$ such that $x' \leqslant y'$.

The quotient of a poset is in general only a reflexive binary relation, not a partial order. The relation R is not necessarily anti-symmetric or transitive. In the literature, various different sufficient conditions have been imposed on the equivalence relation

$\sim$ to ensure that $X/{\sim}$ is a poset [9, 17]. Here we consider instead the necessary and sufficient conditions on $\sim$ for $(X/{\sim}, R)$ to be a poset, which are as follows.

Proposition 4.1. *The quotient $X/{\sim}$ is a poset if and only if*

1. *if there exist $x_1 \sim x$ and $y_1 \sim y$ such that $x_1 \leqslant y_1$, and $x_2 \sim x$ and $y_2 \sim y$ such that $x_2 \geqslant y_2$, then $x \sim y$, and*
2. *given $x, y, z \in X$ such that there exist $x_1 \sim x$ and $y_1 \sim y$ such that $x_1 \leqslant y_1$, and $y_2 \sim y$ and $z_2 \sim z$ such that $y_2 \leqslant z_2$, then there exist $x_3 \sim x$ and $z_3 \sim z$ such that $x_3 \leqslant z_3$.*

If both condition (1) and condition (2) hold, then we write $\leqslant$ instead of R, to acknowledge that the relation gives us a partial order. In this case, we say that $\sim$ is a *weak order congruence* on the poset X, and we have a canonical order-preserving map

$$\begin{aligned} X &\to X/{\sim} \\ x &\mapsto [x]. \end{aligned}$$

Indeed, for any order-preserving map of posets $f\colon X \to Y$, one can consider the equivalence relation on X defined by $x \sim x'$ if and only if $f(x) = f(x')$. We then define the *image* of f to be the quotient $f(X) = X/{\sim}$. We identify the $\sim$-equivalence class $[x]$ of X with the element $f(x) \in Y$, so that $f(X) \subseteq Y$, and the quotient relation on $f(X)$ is a subrelation of the partial order on Y. If the equivalence relation $\sim$ on X is a weak order congruence, so that the image $f(X)$ is a well-defined poset, then we say that the map f is *photogenic*. We say that $f\colon X \to Y$ is *full* if whenever $f(x_1) \leqslant f(x_2)$ in Y, there exist $x_1', x_2' \in X$ such that $x_1' \leqslant x_2'$, with $f(x_1') = f(x_1)$ and $f(x_2') = f(x_2)$.

Proposition 4.2. *Let X and Y be posets, with $f\colon X \to Y$ an order-preserving map. Then the relation on $f(X)$ is anti-symmetric. If f is full, then the relation on $f(X)$ is transitive, and so f is photogenic. Moreover, $f(X) = Y$ as posets if and only if X is full and surjective.*

Hence, if an order-preserving map f is full and surjective, then we say that f is a *quotient map of posets*. With this framework in mind, the *higher Tamari order* $T(n, \delta + 1)$ [3] is defined to be the image of the map $g\colon \mathcal{B}(n, \delta+1) \to \mathcal{S}(n, \delta)$. Note that it is not evident that $T(n, \delta + 1)$ is a well-defined poset, since it is not clear that the map g is photogenic. However, in Section 6 we shall sketch how to prove that g is full, which implies that g is photogenic by Proposition 4.2, since we already know that g is order-preserving. In Section 5, we shall sketch how to prove that g is surjective, which entails that g is a quotient map of posets, and so that $T(n, \delta + 1) \cong \mathcal{S}(n, \delta)$.

The relation between the higher Tamari orders and KP solitons established in [3] amounts to the fact that certain soliton solutions to the KP equation, including their higher evolution structure, may be described as the Poincaré duals of regular triangulations of cyclic polytopes. See also [8, 11].

5 Surjectivity of the map

We now outline our proof that the map g is a surjection in [20]. This was first shown in [16] using the different language of *lifting subdivisions*. Our strategy is to explicitly show that g is a surjection when δ is even, and then to use this to deduce the case where δ is odd. Given a triangulation $\mathcal{T}$ of $C(n, 2d)$, we will construct a cubillage $\mathcal{Q}_{\mathcal{T}}$ of $Z(n, 2d+1)$ such that $g(\mathcal{Q}_{\mathcal{T}}) = \mathcal{T}$. We will define $\mathcal{Q}_{\mathcal{T}}$ by specifying its internal spectrum.

For $I \subseteq [n]$, we write $I = J \sqcup J'$ if $I = J \cup J'$ and there are no $j \in J, j' \in J'$ such that j, j' are cyclically consecutive. Given a union of cyclic intervals $I = [i_0, i'_0] \sqcup \cdots \sqcup [i_l, i'_l]$, we use the notation $\widehat{I} = \{i_0, \ldots, i_l\}$. We claim that the collection of subsets

$$U(\mathcal{T}) = \left\{ I \subset [n] \mid |\widehat{I}| \text{ is a } d'\text{-simplex of } \mathcal{T} \text{ for } d' \geqslant d \right\}$$

defines the internal spectrum of a cubillage on $Z(n, 2d+1)$. This is similar to the construction in [13, Theorem 3.8]. We begin by showing that $U(\mathcal{T})$ is $2d$-separated, for which we need the following lemma. This generalises one direction of [13, Lemma 3.7], though the proof in *op. cit.* requires only minor changes.

Lemma 5.1. *Let I, J be two subsets of $[n]$. Then I δ-interweaves J only if there exist subsets $X \subseteq \widehat{I}$ and $Y \subseteq \widehat{J}$ such that $\#X = \lfloor \delta/2 \rfloor$ and $\#Y = \lceil \delta/2 \rceil$, and X δ-interweaves Y.*

Lemma 5.2. *The collection $U(\mathcal{T})$ is $2d$-separated.*

Proof (sketch). This follows by combining Lemma 5.1 with the fact that for $I, J \in U(\mathcal{T})$, we have that $|\widehat{I}|$ and $|\widehat{J}|$ are simplices in the same triangulation. Hence $\pi_{n-1,2d}(|\widehat{I}|)$ and $\pi_{n-1,2d}(|\widehat{J}|)$ cannot intersect each other in the interior of $C(n, 2d)$. This means that there do not exist subsets $X \subseteq \widehat{I}$ and $Y \subseteq \widehat{J}$ as described in Lemma 5.1. Hence I and J cannot be $2d$-interweaving. □

We must now show that $\#U(\mathcal{T}) = \binom{n-1}{2d+1}$. We use induction for this, showing that the size of $U(\mathcal{T})$ is preserved by increasing flips of $\mathcal{T}$, which requires the following lemma.

Lemma 5.3. *Let $|S| = |a_0, b_0, a_1, \ldots, a_d, b_d|$ be a $(2d+1)$-simplex inducing an increasing flip of a triangulation $\mathcal{T}$ of $C(n, 2d)$, and let $A = \{a_0, \ldots, a_d\}, B = \{b_0, \ldots, b_d\}$. Then the following two sets have the same cardinality:*

$$\mathcal{I}_l(S, n) = \left\{ I \in 2^{[n]} \mid A \subseteq \widehat{I} \subsetneq S \right\},$$
$$\mathcal{I}_u(S, n) = \left\{ I \in 2^{[n]} \mid B \subseteq \widehat{I} \subsetneq S \right\}.$$

This allows us to prove that our $2d$-separated collection $U(\mathcal{T})$ is the right size to be the internal spectrum of a cubillage.

Lemma 5.4. *Given a triangulation $\mathcal{T}$ of $C(n, 2d)$, we have that $\#U(\mathcal{T}) = \binom{n-1}{2d+1}$.*

Proof (sketch). This follows from induction on increasing flips. One can verify the claim explicitly for the minimal triangulation in $\mathcal{S}(n,\delta)$. Suppose that $\mathcal{T}'$ is an increasing flip of $\mathcal{T}$, with $|S|$ is the simplex inducing the increasing flip. Then, by Gale's Evenness Criterion, we have that $U(\mathcal{T}') = (U(\mathcal{T}) \setminus \mathcal{I}_l(S,n)) \cup \mathcal{I}_u(S,n)$. Lemma 5.3 then establishes that $\#U(\mathcal{T}') = \#U(\mathcal{T})$, as required. □

Theorem 5.5. *The map $g\colon \mathcal{B}(n,\delta+1) \to \mathcal{S}(n,\delta)$ is a surjection for even δ.*

Proof (sketch). Given a triangulation $\mathcal{T} \in \mathcal{S}(n,\delta)$, it follows from Lemma 5.2 and Lemma 5.3 that $U(\mathcal{T})$ is the internal spectrum of a cubillage $\mathcal{Q}_{\mathcal{T}}$. We have that $|A|$ an internal d-simplex of $\mathcal{T}$ if and only if $A \in U(\mathcal{T})$ and $\#A = d+1$. The interpretation of the map g in terms of separated collections then implies that $g(\mathcal{Q}_{\mathcal{T}}) = \mathcal{T}$. □

Theorem 5.6. *The map $g\colon \mathcal{B}(n,\delta+1) \to \mathcal{S}(n,\delta)$ is a surjection for odd δ.*

Proof (sketch). Given a triangulation $\mathcal{T}$ of $C(n,2d+1)$, one may consider the triangulation $\hat{\mathcal{T}}$ of $C(n+1,2d+2)$ defined in [15]. Theorem 5.5 provides us with a cubillage $\mathcal{Q}_{\hat{\mathcal{T}}}$ of $Z(n+1,2d+3)$ such that $g(\mathcal{Q}_{\hat{\mathcal{T}}}) = \hat{\mathcal{T}}$. Applying [1, Lemma 5.2], we obtain a cubillage $\mathcal{Q}_{\mathcal{T}}$ of $Z(n,2d+2)$ such that $g(\mathcal{Q}_{\mathcal{T}}) = \mathcal{T}$. □

6 Fullness of the map

We now outine how we show in [20] that the map g is full, and hence is a quotient map of posets. We follow the approach of Section 5, whereby we work explicitly for even-dimensional triangulations, and then use this to show the result for odd dimensions. Indeed, we show that for triangulations $\mathcal{T},\mathcal{T}'$ of $C(n,2d)$ with $\mathcal{T} \leqslant \mathcal{T}'$, we have that $\mathcal{Q}_{\mathcal{T}} \leqslant \mathcal{Q}_{\mathcal{T}'}$, where these are the cubillages constructed in Section 5. For this, it suffices to show that if $\mathcal{T} \lessdot \mathcal{T}'$, then $\mathcal{Q}_{\mathcal{T}} < \mathcal{Q}_{\mathcal{T}'}$.

Theorem 6.1. *Given triangulations $\mathcal{T},\mathcal{T}'$ of $C(n,2d)$ such that $\mathcal{T} \lessdot \mathcal{T}'$, we have that $\mathcal{Q}_{\mathcal{T}} < \mathcal{Q}_{\mathcal{T}'}$.*

Proof (sketch). As in the proof of Lemma 5.3, we are required to replace $\mathcal{I}_l(S,n)$ with $\mathcal{I}_u(S,n)$, where $|S|$ induces the increasing flip from $\mathcal{T}$ to $\mathcal{T}'$. We prove this by showing that one may gradually exchange the elements of $\mathcal{I}_l(S,n)$ for $\mathcal{I}_u(S,n)$ while preserving $2d$-separatedness of the collections. This uses the description of the higher Bruhat orders from Section 2.1.2. □

Theorem 6.2. *Given triangulations $\mathcal{T},\mathcal{T}'$ of $C(n,2d+1)$ such that $\mathcal{T} \lessdot \mathcal{T}'$, we have that $\mathcal{Q}_{\mathcal{T}} < \mathcal{Q}_{\mathcal{T}'}$.*

Proof (sketch). We use a similar approach to Theorem 5.6, and consider the triangulations $\hat{\mathcal{T}}$ and $\hat{\mathcal{T}}'$ of $C(n+1,2d+2)$. By [15], we have $\hat{\mathcal{T}}' < \hat{\mathcal{T}}$, and so $\mathcal{Q}_{\hat{\mathcal{T}}'} < \mathcal{Q}_{\hat{\mathcal{T}}}$ by Theorem 6.1. We can then use the techniques of the proof of [1, Lemma 5.2] to show that this implies that $\mathcal{Q}_{\mathcal{T}} < \mathcal{Q}_{\mathcal{T}'}$. □

Putting together the fact that g is a surjection in even (Theorem 5.5) and odd (Theorem 5.6) dimensions, and the fact that g is full in even (Theorem 6.1) and odd (Theorem 6.2) dimensions, we obtain the following.

Corollary 6.3. *The map* $g\colon \mathcal{B}(n, \delta+1) \to \mathcal{S}(n, \delta)$ *is a quotient map of posets.*

Hence, we obtain that the higher Tamari orders are indeed the same posets as the first higher Stasheff–Tamari orders, noting Proposition 4.2 and the definition of the higher Tamari orders as the image of the map g.

Corollary 6.4. *The higher Tamari order* $T(n, \delta+1)$ *is isomorphic to the first higher Stasheff–Tamari order* $\mathcal{S}(n, \delta)$*.*

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