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Shard polytopes

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Abstract. For any lattice congruence of the weak order on permutations, N. Reading proved that glueing together the cones of the braid fan that belong to the same congruence class defines a complete fan, called quotient fan, and V. Pilaud and F. Santos showed that it is the normal fan of a polytope, called quotientope. We provide an alternative simpler approach based on Minkowski sums of elementary polytopes, called shard polytopes, which have remarkable combinatorial and geometric properties. In contrast to the original construction of quotientopes, our approach extends to type B.

Résumé. Pour toute congruence de treillis de l'ordre faible sur les permutations, N. Reading a montré que recoller ensemble les cônes de l'éventail de tresses qui appartiennent à une même classe de congruence définit un éventail complet, appelé éventail quotient, et V. Pilaud et F. Santos ont montré que cet éventail quotient est l'éventail normal d'un polytope, appelé quotientope. Nous présentons une approche alternative basée sur des sommes de Minkowski de polytopes élémentaires, appelés polytopes de tessons, avec des propriétés combinatoires et géométriques remarquables. Contrairement à la construction originale des quotientopes, notre approche s'étend au type B.

Keywords: weak order, lattice quotients, shards, type cone, deformed permutahedra

1 Introduction

The *weak order* is a fundamental lattice structure on the set $\mathfrak{S}_n$ of permutations of $[n]$, defined by inclusion of inversion sets: $\sigma \le \tau$ if and only if $\mathrm{inv}(\sigma) \subseteq \mathrm{inv}(\tau)$ where $\mathrm{inv}(\sigma) := \{(\sigma(i), \sigma(j)) \mid i < j \text{ and } \sigma(i) > \sigma(j)\}$. See Figure 1 for illustrations when $n = 4$. Its Hasse diagram can be seen geometrically as:

- the dual graph of the *braid fan* $\mathcal{F}_n$, defined by the hyperplanes $\{x \in \mathbb{R}^n \mid x_i = x_j\}$ for all $1 \le i < j \le n$, directed from the region $x_1 < \cdots < x_n$ to the opposite one,
- or the graph of the *permutahedron* $\mathsf{Perm}_n := \mathrm{conv}\{(\sigma^{-1}(1), \ldots, \sigma^{-1}(n)) \mid \sigma \in \mathfrak{S}_n\}$, oriented in the linear direction $\gamma := (-n+1, -n+3, \ldots, n-3, n-1)$.

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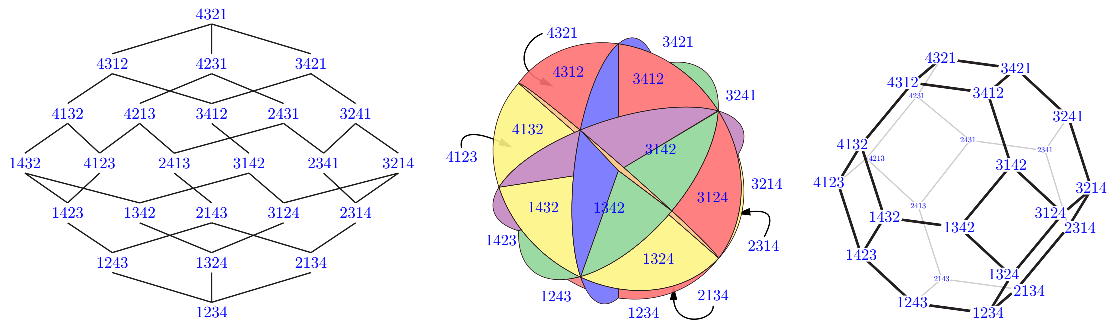

Figure 1: The Hasse diagram of the weak order on $\mathfrak{S}_4$ (left) can be seen as the dual graph of the braid fan $\mathcal{F}_4$ (middle) or as the graph of the permutahedron Perm_4 (right).

Here, we discuss similar geometric realizations for lattice quotients of the weak order. A *lattice congruence* is an equivalence relation $\equiv$ that respects meets and joins ($x \equiv x'$ and $y \equiv y'$ implies $x \wedge y \equiv x' \wedge y'$ and $x \vee y \equiv x' \vee y'$). It defines a *lattice quotient* $\mathfrak{S}_n/\equiv$ on its equivalence classes. The prototype is the classical *Tamari lattice* on binary trees with n nodes [18], seen as the quotient of the weak order on $\mathfrak{S}_n$ by the *sylvester congruence*. Its Hasse diagram is the graph of the classical *associahedron* Asso_n which admits elegant descriptions by vertices [6], by facets [17], or as a Minkowski sum of the faces of the standard simplex [12]. See Figure 2. In general, for any lattice congruence $\equiv$ of the weak order, the Hasse diagram of the lattice quotient $\mathfrak{S}_n/\equiv$ can be seen geometrically as:

- the dual graph of the *quotient fan* $\mathcal{F}_\equiv$ of [14], obtained by glueing together the chambers of the braid fan $\mathcal{F}_n$ that belong to the same congruence class of $\equiv$,
- or the graph of a *quotientope* of [11], oriented in the direction γ defined above.

The quotientopes of [11] were obtained by a careful but slightly obscur choice of right-hand sides defining an inequality normal to each ray of the braid fan. We present here an alternative approach based on Minkowski sums of elementary polytopes.

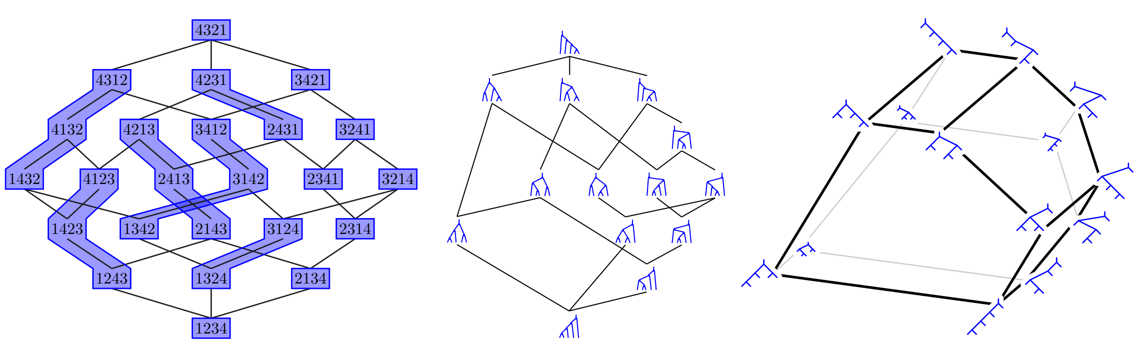

Figure 2: The quotient of the weak order by the sylvester congruence (left) is the Tamari lattice (middle), and its quotient fan is the normal fan of the associahedron (right).

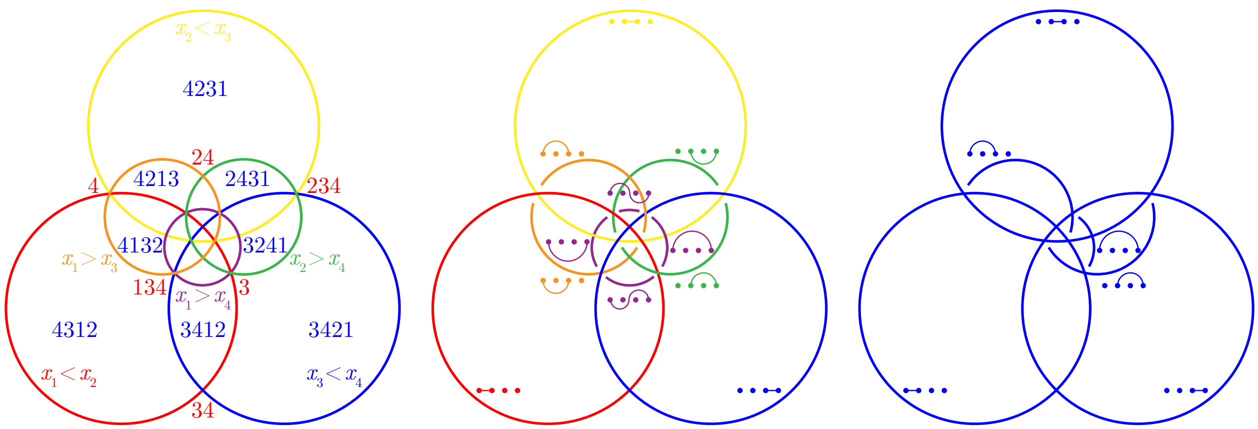

Figure 3: The stereographic projection of the braid fan $\mathcal{F}_4$ from the pole 4321 (left), the corresponding shards (middle), the quotient fan by the sylvester congruence (right).

Our construction is based on arcs and shards. An *arc* is a quadruple $\alpha := (a, b, A, B)$ where $1 \le a < b \le n$ and $A \sqcup B =]a, b[$. It is represented by a curve wiggling around points on the horizontal axis, joining a to b while passing above the points of A and below the points of B. The set of all arcs is denoted by $\mathcal{A}_n$. Each lattice congruence $\equiv$ of $\mathfrak{S}_n$ corresponds to an upper ideal $\mathcal{A}_\equiv$ of the *forcing order* on arcs, defined by $(a, b, A, B) \prec (a', b', A', B')$ if $a \le a' < b' \le b$ and $A' \subseteq A$ and $B' \subseteq B$. For instance, the sylvester congruence defining the Tamari lattice corresponds to the ideal of up arcs $\{(a, b,]a, b[, \varnothing) \mid 1 \le a < b \le n\}$. Geometrically, an arc $\alpha := (a, b, A, B)$ defines a *shard* $\mathsf{S}(\alpha)$, given by the piece of the hyperplane $\boldsymbol{x}_a = \boldsymbol{x}_b$ satisfying the inequalities $\boldsymbol{x}_{a'} \le \boldsymbol{x}_a = \boldsymbol{x}_b \le \boldsymbol{x}_{b'}$ for all $a' \in A$ and $b' \in B$. The union of the walls of the quotient fan $\mathcal{F}_\equiv$ is the union of shards $\mathsf{S}(\alpha)$ over all arcs of $\mathcal{A}_\equiv$. See Figure 3.

The central idea of our work is to realize the quotient fan $\mathcal{F}_\equiv$ as a Minkowski sum where each summand is responsible for some shards of $\mathcal{A}_\equiv$ to appear in the normal fan. To illustrate this idea, let us start with a simple construction. For any arc α, denote by $\mathcal{A}_\alpha$ the arc ideal generated by α. The corresponding congruence $\equiv_\alpha$ is a *Cambrian congruence* [15], and the quotient fan $\mathcal{F}_\alpha$ is the normal fan of the *α-associahedron* Asso_α [4].

Theorem 1.1. *Consider an arbitrary congruence $\equiv$ of the weak order, and let $\alpha_1, \dots, \alpha_p$ denote the arcs generating the ideal $\mathcal{A}_\equiv$. Then the quotient fan $\mathcal{F}_\equiv$ is*

- *the common refinement of the Cambrian fans $\mathcal{F}_{\alpha_1}, \dots, \mathcal{F}_{\alpha_p}$, and*
- *the normal fan of the Minkowski sum of the associahedra $\mathsf{Asso}_{\alpha_1}, \dots, \mathsf{Asso}_{\alpha_p}$.*

This construction was already used for certain specific quotients but never exploited for arbitrary lattice congruences. In contrast to the intricate construction of [11], Theorem 1.1 has the advantage to transfer the geometric difficulty into the construction of the α-associahedron Asso_α, already done in [4]. Here, each α_i-associahedron Asso_{α_i} is responsible for the shards of the ideal $\mathcal{A}_{\alpha_i}$ to appear in the normal fan of the Minkowski sum.

We already mentioned that the classical associahedron decomposes as the Minkowski sum of faces of the standard simplex [12]. In general, the α-associahedron can be decomposed further into Minkowski sums of more elementary pieces. The main idea of our work is to push this idea forward, looking for the most possible elementary pieces.

Theorem 1.2. *For each arc α, there exists a Minkowski indecomposable polytope $\mathsf{SP}(\alpha)$, called shard polytope, such that the union of the walls of the normal fan of the shard polytope $\mathsf{SP}(\alpha)$ contains the shard $\mathsf{S}(\alpha)$ and is contained in the union of the shards $\mathsf{S}(\alpha')$ for all arcs α' forcing α.*

We describe these polytopes by a simple combinatorial construction in Section 2.1, and as matroid polytopes of series-parallel graphs in Section 2.2.

As a consequence of their normal fan property, we can use shard polytopes to construct polytopal realizations of quotient fans, in a similar way as we used α-associahedra in Theorem 1.1. Each shard polytope $\mathsf{SP}(\alpha)$ is now responsible for the shard $\mathsf{S}(\alpha)$ to appear in the normal fan, and the remaining of the normal fan of $\mathsf{SP}(\alpha)$ does not mess up the picture since it is contained in the union of the shards $\mathsf{S}(\alpha')$ for all arcs α' forcing α.

Theorem 1.3. *For any lattice congruence $\equiv$ of the weak order and positive coefficients $\boldsymbol{s}_\alpha \in \mathbb{R}^{\mathcal{A}_\equiv}_{>0}$, the quotient fan $\mathcal{F}_\equiv$ is the normal fan of the Minkowski sum $\mathsf{SP}(\mathcal{A}_\equiv) := \sum_{\alpha \in \mathcal{A}_\equiv} \boldsymbol{s}_\alpha \, \mathsf{SP}(\alpha)$.*

Already setting the coefficients $\boldsymbol{s}_\alpha = 1$, this construction recovers (up to translation) relevant realizations of some specific quotient fans mentioned above: the classical associahedron of [17, 6, 12] for the sylvester congruence and the α-associahedron of [4] for the α-Cambrian congruence. More generally any quotientope of [11] is a Minkowski sum of dilated shard polytopes (up to translation). More details are given in Section 2.3.

In fact, if we allow for Minkowski sums and differences, the family of shard polytopes provides a relevant Minkowski basis of the space of *deformed permutahedra* of [12, 13] (or "generalized permutahedra", those polytopes whose normal fan coarsens the braid fan).

Theorem 1.4. *Up to translation, any deformed permutahedron has a unique decomposition as a Minkowski combination $\mathsf{DP}_\mathsf{s}(\boldsymbol{s}) := \sum_{\alpha \in \mathcal{A}_n} \boldsymbol{s}_\alpha \, \mathsf{SP}(\alpha)$ of shard polytopes, with $\boldsymbol{s}_\alpha \in \mathbb{R}$ for $\alpha \in \mathcal{A}_n$.*

This statement and the exchange matrix with the classical basis of faces of the standard simplex [1] are presented in Section 3.2, and used to the compute the (mixed) volumes of shard polytopes as reported in Section 3.3.

Finally, our long term objective is to extend the construction of shard polytopes to lattices of regions of hyperplane arrangements beyond the braid arrangement (see [16] for an introduction to the topic). We achieve the first step in this perspective by constructing shard polytopes for the type B Coxeter group in Section 4. They provide elementary pieces for the first construction of quotientopes for all lattice quotients of the type B weak order, and the first natural Minkowski basis for type B deformed permutahedra.

Many details and all proofs are omitted in this extended abstract for space reason, but a complete treatment can be found in the long version of this work [10].

2 Shard polytopes

2.1 Definition and basic properties

Definition 2.1. For an arc $\alpha := (a, b, A, B)$, we define

- an *α-alternating matching* as a (possibly empty) sequence $M = \{a_1, b_1, \dots, a_k, b_k\}$ with $a \le a_1 < b_1 < \dots < a_k < b_k \le b$ and $a_i \in \{a\} \cup A$ while $b_i \in B \cup \{b\}$ for $i \in [k]$,
- the *characteristic vector* of the α-alternating matching M as $\chi(M) = \sum_{i \in [k]} \boldsymbol{e}_{a_i} - \boldsymbol{e}_{b_i}$,
- an *α-fall* (resp. *α-rise*) as a position $j \in [a, b[$ such that $j \in \{a\} \cup A$ and $j + 1 \in B \cup \{b\}$ (resp. such that $j \in \{a\} \cup B$ and $j + 1 \in A \cup \{b\}$).

Proposition 2.2. *The shard polytope* $\mathsf{SP}(\alpha)$ *of an arc α is the polytope defined equivalently as*

- *the convex hull of the characteristic vectors of all α-alternating matchings,*
- *the subset of the hyperplane* $\mathbb{H} := \{\boldsymbol{x} \in \mathbb{R}^n \mid \sum_{i \in [n]} \boldsymbol{x}_i = 0\}$ *defined by*
  - $\boldsymbol{x}_i = 0$ *for any* $i \in [n] \smallsetminus [a, b]$, $\boldsymbol{x}_{a'} \ge 0$ *for any* $a' \in A$, *and* $\boldsymbol{x}_{b'} \le 0$ *for any* $b' \in B$,
  - $\sum_{i \le f} \boldsymbol{x}_i \le 1$ *for any α-fall f and* $\sum_{i \le r} \boldsymbol{x}_i \ge 0$ *for any α-rise r.*

Figure 4 shows some shard polytopes and illustrates the following elementary properties of their vertices, edges, faces and facets, and their behavior by central symmetry.

Proposition 2.3. *For any arc* $\alpha := (a, b, A, B)$,

- *the shard polytope* $\mathsf{SP}(\alpha)$ *has dimension* $b - a$,
- *the vertices of* $\mathsf{SP}(\alpha)$ *are precisely all characteristic vectors of α-alternating matchings,*
- *two α-alternating matchings M, M' form an edge of* $\mathsf{SP}(\alpha)$ *if and only if* $|M \triangle M'| = 2$,
- *any face of* $\mathsf{SP}(\alpha)$ *is a Cartesian product of shard polytopes,*
- *the facets of* $\mathsf{SP}(\alpha)$ *are precisely defined by the inequalities of Proposition 2.2.*

Proposition 2.4. *Consider the central symmetries on arcs* $\theta(a, b, A, B) = (\bar{b}, \bar{a}, \bar{B}, \bar{A})$ *and on vectors* $\Theta(\boldsymbol{e}_i) = -\boldsymbol{e}_{\bar{i}}$ *where* $\bar{i} := n + 1 - i$. *Then* $\mathsf{SP}(\theta(\alpha)) = \Theta(\mathsf{SP}(\alpha))$ *for any arc α.*

Finally, the central property of shard polytopes is the following.

Proposition 2.5. *For any arc α, the union of the walls of the normal fan of the shard polytope* $\mathsf{SP}(\alpha)$ *contains the shard* $\mathsf{S}(\alpha)$ *and is contained in the union of the shards* $\mathsf{S}(\alpha')$ *for α' forcing α.*

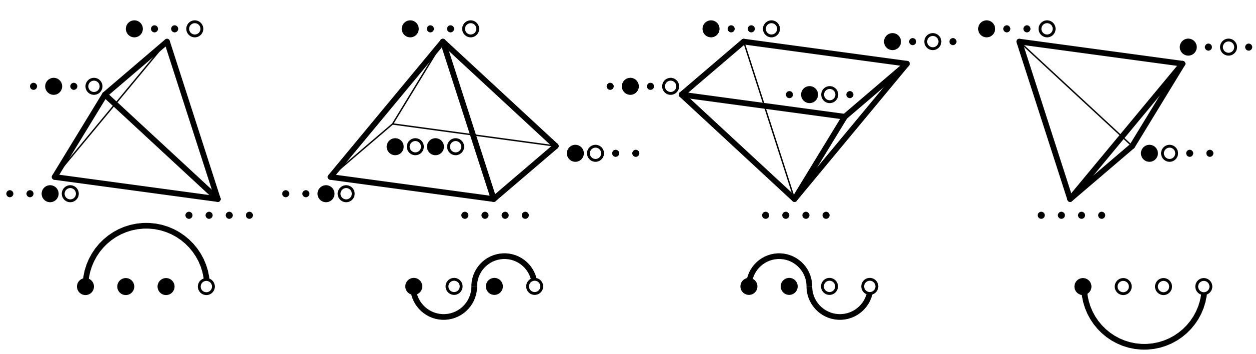

Figure 4: Some shard polytopes with $n = 4$ (by convention $\bullet = 1$, $\cdot = 0$ and $\circ = -1$).

2.2 Shard polytopes as matroid polytopes

Let M be a matroid on the ground set $[n]$ (see [8] for an introduction to matroid theory). Its *matroid polytope* $\mathsf{P}_M \subset \mathbb{R}^n$ is the convex hull of the characteristic vectors of its bases. The following characterization gives a geometric axiomatization of matroids, which provides directly the proof that shard polytopes are actually matroid polytopes.

Theorem 2.6 ([3, Thm. 4.1]). *A polytope is a matroid polytope if and only if all its vertices have $0/1$ coordinates and all its edges are translations of some vectors $\boldsymbol{e}_i - \boldsymbol{e}_j$ with $i \neq j$.*

Corollary 2.7. *The translated shard polytope $\overrightarrow{\mathsf{SP}}(\alpha) := \mathsf{SP}(\alpha) + \mathbf{1}_{B\cup\{b\}}$ is a matroid polytope for any arc $\alpha := (a, b, A, B)$.*

We can give a precise description of these matroids, which are actually certain connected series-parallel graphic matroids. Let us recall some terminology. A graph is *series-parallel* if it can be obtained from a single edge with distinct endpoints via the operations of series extension (replacing an edge by a path of length 2) and parallel extension (replacing an edge by two parallel edges with the same endpoints). The *(cycle) matroid* of a connected graph $G := (V, E)$ is the matroid on E whose bases are the edge sets of spanning trees of G. A matroid is *graphic* if it is the cycle matroid of a graph, and *series-parallel* if it is the cycle matroid of a series-parallel graph, see [8, Sect. 5.4].

Definition 2.8. For an arc $\alpha := (a, b, A, B) \in \mathcal{A}_n$, let $\{a\} \cup A = \{a = a_1 < \cdots < a_{|A|+1}\}$ and $B \cup \{b\} = \{b_1 < \cdots < b_{|B|+1} = b\}$, and set $b_0 := a - 1$ for convenience. Define the *shard graph* Γ_α to be the (multi-)graph with vertex set $[0, |B| + 1]$ and

- for each $1 \leq i \leq |A| + 1$, an edge labeled a_i joining vertex k to vertex $|B| + 1$, where $0 \leq k \leq |B|$ is such that $b_k < a_i < b_{k+1}$,
- for each $1 \leq j \leq |B| + 1$, an edge labeled b_j joining vertex $j - 1$ to vertex j,
- for each $k \in [n] \smallsetminus [a, b]$, a loop labeled by k on vertex $|B| + 1$.

The *shard matroid* of the arc α is the cycle matroid M_α of Γ_α, whose ground set is $[n]$.

Proposition 2.9. *The graph Γ_α stripped of loops is a 2-connected series-parallel graph.*

Proposition 2.10. *The matroid polytope of the shard matroid M_α is the translated shard polytope $\overrightarrow{\mathsf{SP}}(\alpha) := \mathsf{SP}(\alpha) + \mathbf{1}_{B\cup\{b\}}$.*

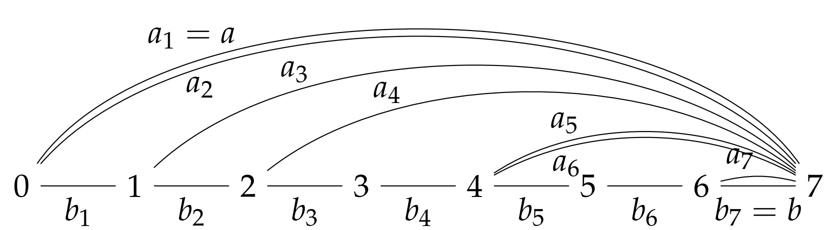

Figure 5: The graph Γ_α for the arc $\alpha := (1, 14, \{2, 4, 6, 9, 10, 13\}, \{3, 5, 7, 8, 11, 12\})$.

2.3 Quotientopes from shard polytopes

We now construct polytopal realizations of quotient fans in the same spirit as in Theorem 1.1, but using shard polytopes rather than associahedra as elementary summands. The following statements immediately follow from Propositions 2.4 and 2.5.

Proposition 2.11. *For any lattice congruence $\equiv$ of the weak order, the quotient fan $\mathcal{F}_{\equiv}$ is the normal fan of the Minkowski sum $\mathsf{SP}(\mathcal{A}_{\equiv}) := \sum_{\alpha \in \mathcal{A}_{\equiv}} \mathsf{SP}(\alpha)$.*

Proposition 2.12. *If an arc ideal $\mathcal{A}$ is centrally symmetric, then $\mathsf{SP}(\mathcal{A}) = \Theta(\mathsf{SP}(\mathcal{A}))$.*

Observe that the quotient fan $\mathcal{F}_{\equiv}$ is actually the normal fan of any Minkowski sum $\sum_{\alpha \in \mathcal{A}_{\equiv}} \boldsymbol{s}_\alpha \, \mathsf{SP}(\alpha)$ with $\boldsymbol{s}_\alpha > 0$ for any $\alpha \in \mathcal{A}_{\equiv}$, see Theorem 1.3. We stick with coefficients $\boldsymbol{s}_\alpha = 1$ as this convention recovers the original constructions of [6, 4] as described in Examples 2.14 and 2.15. The following four examples are illustrated in Figure 6.

Example 2.13. For basic arcs, the $(i, i+1, \varnothing, \varnothing)$-alternating matchings are $\varnothing$ and $\{i, i+1\}$, thus the shard polytope $\mathsf{SP}(i, i+1, \varnothing, \varnothing)$ is just the segment $[0, \boldsymbol{e}_i - \boldsymbol{e}_{i+1}]$. The Minkowski sum $\mathsf{SP}(\{(i, i+1, \varnothing, \varnothing) \mid i \in [n-1]\})$ is thus the parallelotope $\sum_{i \in [n-1]} [0, \boldsymbol{e}_i - \boldsymbol{e}_{i+1}]$.

Example 2.14. For up arcs, the $(a, b,]a, b[, \varnothing)$-alternating matchings are $\varnothing$ and $\{i, b\}$ for $a \le i < b$, thus the shard polytope $\mathsf{SP}(a, b,]a, b[, \varnothing)$ is the translate of the standard simplex $\triangle_{[a,b]}$ by the vector $-\boldsymbol{e}_b$. The Minkowski sum $\mathsf{SP}(\{(a, b,]a, b[, \varnothing) \mid 1 \le a < b \le n\})$ is thus the translate by the vector $-\sum_{i \in [n]} \boldsymbol{e}_i$ of the classical associahedron of [17, 6, 12].

Example 2.15. For the α-Cambrian congruence, the Minkowski sum $\mathsf{SP}(\mathcal{A}_\alpha)$ is actually the translate by the vector $-\sum_{i \in [a,b]} \boldsymbol{e}_i$ of the α-associahedron Asso_α of [4].

Example 2.16. For the ideal of all arcs $\mathcal{A}_n$, the Minkowski sum of all shard polytopes gives a realization of the braid fan $\mathcal{F}_n$. Although it is not the convex hull of all permutations of a given point as the classical permutahedron Perm_n, the resulting polytope has clearly centrally symmetry by Proposition 2.12.

Besides these specific examples, the following statement shows that our construction strictly contains that of [11].

Proposition 2.17. *Any quotientope of [11] is a Minkowski sum of dilated shard polytopes.*

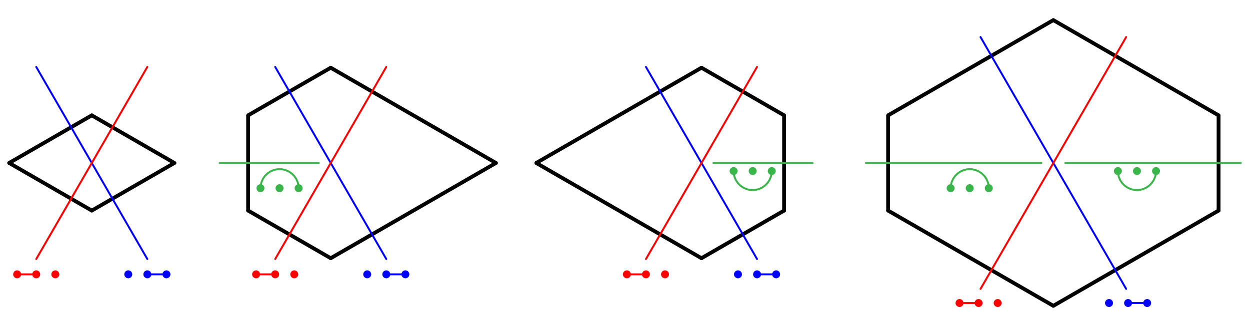

Figure 6: Minkowski sums $\mathsf{SP}(\mathcal{A})$ for all arc ideals $\mathcal{A} \subseteq \mathcal{A}_3$ containing the basic arcs.

3 Minkowski geometry of shard polytopes

3.1 Type cones and shard polytopes

A *weak Minkowski summand* of a polytope P is a polytope Q satisfying the following equivalent conditions:

- there are a real $\lambda \geq 0$ and a polytope R such that $\mathsf{Q} + \mathsf{R} = \lambda \mathsf{P}$,
- the normal fan of Q coarsens the normal fan of P,
- Q is obtained from P by parallelly translating its facets without passing vertices.

The set of weak Minkowski summands of a polytope P has the structure of a polyhedral cone, which is sometimes called (closed) *type cone* [7] or *deformation cone* [12, 13] of the polytope P. This cone has dimension equal to the number N of facets of P, but it has a lineality space of dimension equal to the dimension n of P (corresponding to translations), so its intrinsic dimension is $N - n$. An important property is that Minkowski sums of weak Minkowski summands translate to positive combinations in the type cone. Thus, the rays of the type cone represent *Minkowski indecomposable* summands of P.

The weak Minkowski summands of the classical permutahedron Perm_n form a particularly interesting family, studied under the name *generalized permutahedra* in [12, 13]. Here, we prefer to use the more explicit name *deformed permutahedra*. They are usually described by submodular inequalities [12] or as Minkowski sums and differences of faces $\triangle_J := \operatorname{conv}\{e_j \mid j \in J\}$ of the standard simplex $\triangle_{[n]}$ [1] as follows.

Proposition 3.1 ([12])**.** *Any deformed permutahedron can be represented as*

$$\mathsf{DP_z}(z) := \left\{ x \in \mathbb{R}^n \;\middle|\; \langle \mathbf{1} \mid x \rangle = z_{[n]} \text{ and } \langle \mathbf{1}_R \mid x \rangle \geq z_R \text{ for all } \varnothing \neq R \subsetneq [n] \right\}$$

for some $z \in \mathbb{R}^{2^{[n]}}$ with $z_\varnothing = 0$ and $z_R + z_S \leq z_{R\cup S} + z_{R\cap S}$ for all $R, S \in 2^{[n]}$. Moreover, this representation is unique if all inequalities $\langle \mathbf{1}_R \mid x \rangle \geq z_R$ are tight (as always implicitly assumed).

Proposition 3.2 ([1])**.** *Any deformed permutahedron has a unique decomposition as a Minkowski combination $\mathsf{DP_y}(y) := \sum_{J\subseteq[n]} y_J \triangle_J$ of faces of the standard simplex, with $y_J \in \mathbb{R}$ for $J \subseteq [n]$.*

Proposition 3.3 ([12, 1])**.** *The parameters y and z in Propositions 3.1 and 3.2 are related by*

$$z_R = \sum_{J\subseteq R} y_J \qquad \text{and} \qquad y_J = \sum_{R\subseteq J} (-1)^{|J\smallsetminus R|} z_R.$$

Example 3.4. Type cones are high dimensional objects difficult to visualize. We can however see the type cone of the 2-dimensional permutahedron Perm_3 by intersecting it with a hyperplane. The resulting polytope is a triangular bipyramid illustrated in Figure 7. We have located in the type polytope the shard polytopes of the four arcs of $\mathcal{A}_3$ together with different polytopes considered along this extended abstract. Note that the four shard polytopes are all vertices of the type polytope (see Proposition 3.5) and form an affine basis of the space (see Proposition 3.9).

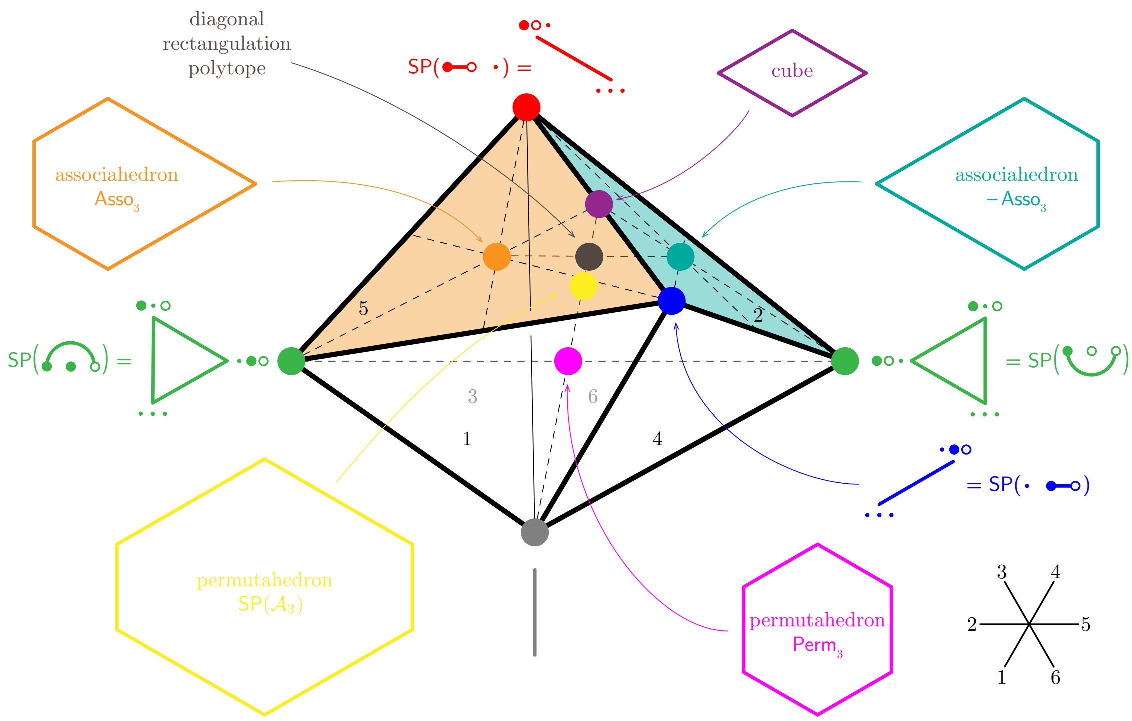

Figure 7: The type polytope of the permutahedron Perm_3.

The main point of this section is the following statement. It is shown in [10] either from a direct Minkowski indecomposability criterion, or from the description of shard polytopes as matroid polytopes of 2-connected series-parallel graphs (see Section 2.2).

Proposition 3.5. *For any arc α, the shard polytope $\mathsf{SP}(\alpha)$ is Minkowski indecomposable.*

Thus, shard polytopes correspond to certain rays of the submodular cone. However, not all indecomposable deformed permutahedra are shard polytopes.

Theorem 3.6. *For any arc $\alpha \in \mathcal{A}_n$, the shard polytopes of the arcs forcing α are precisely (representatives of) the rays of the type cone of the α-associahedron.*

We get the following result as a direct consequence of the simpliciality of the type cones of associahedra and the description of its rays [9].

Corollary 3.7. *Any polytope whose normal fan is the α-Cambrian fan $\mathcal{F}_\alpha$ has a unique decomposition (up to translation) as a Minkowski sum of dilated shard polytopes $\mathsf{SP}(\alpha')$ for α' forcing α.*

Remark 3.8. Theorem 3.6 connects shard polytopes to other interpretations of the rays of the type cone of the Cambrian fans: as Newton polytopes of F-polynomials of cluster variables of acyclic type A cluster algebras [2], and as brick polytope summands of certain sorting networks [5]. We skip all precise definitions here as these interpretations are not needed in the rest of our discussion. We are not aware that our vertex and facet descriptions from Proposition 2.2 have been observed earlier for these polytopes.

3.2 Minkowski basis of shard polytopes

It turns out that shard polytope also provide a relevant Minkowski basis for the space of deformed permutahedra, similar to faces of the standard simplex (see Proposition 3.2).

Proposition 3.9. *Up to translations, any deformed permutahedron has a unique decomposition as a Minkowski combination* $\mathsf{DP}_\mathsf{s}(\boldsymbol{s}) := \sum_{\alpha \in \mathcal{A}_n} \boldsymbol{s}_\alpha \, \mathsf{SP}(\alpha)$ *of shard polytopes, with* $\boldsymbol{s}_\alpha \in \mathbb{R}$ *for* $\alpha \in \mathcal{A}_n$.

We thus have three parametrizations of the space of deformed permutahedra: as Minkowski combinations of shard polytopes $\mathsf{DP}_\mathsf{s}(\boldsymbol{s}) := \sum_{\alpha \in \mathcal{A}} \boldsymbol{s}_\alpha \, \overrightarrow{\mathsf{SP}}(\alpha)$ (see Proposition 3.9), as Minkowski combinations of faces of the standard simplex $\mathsf{DP}_\mathsf{y}(\boldsymbol{y}) := \sum_{J \subseteq [n]} \boldsymbol{y}_J \triangle_J$ (see Proposition 3.2), or from their right hand sides as $\mathsf{DP}_\mathsf{z}(\boldsymbol{z})$ (see Proposition 3.1). The exchange matrices between the parameters $\boldsymbol{s}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ are given by explicit combinatorial formulas. Next, we describe the connection between the parameters $\boldsymbol{s}$ and $\boldsymbol{y}$, which can be combined with Proposition 3.3 to get the connection between the parameters $\boldsymbol{s}$ and $\boldsymbol{z}$. Note that we only consider simplices $\triangle_J$ with $|J| \geq 2$, as we work up to translations.

Proposition 3.10. *The parameters* $\boldsymbol{s}$ *and* $\boldsymbol{y}$ *in Propositions 3.2 and 3.9 are related by*

$$\boldsymbol{s}_\alpha = \sum_{J \triangleright (A \cup \{a,b\})} (-1)^{|\{a,b\} \cap \{\min J, \max J\}|} \, \boldsymbol{z}_R \qquad \text{and} \qquad \boldsymbol{y}_J = \sum_{\substack{\alpha = (a,b,A,B) \\ (A \cup \{a,b\}) \triangleright J}} (-1)^{|J \cap (B \cup \{a,b\})|} \, \boldsymbol{s}_\alpha,$$

where we write $I \triangleright J$ *when* $\{\min J, \max J\} \subseteq \,]\min I, \max I[\, \triangle I$ *and* $]\min J, \max J[\, \cap I \subseteq J$.

3.3 Mixed volumes of shard polytopes

The *mixed volume* is the unique function $\mathrm{Vol}(-, \cdots, -)$ on n-tuples of polytopes such that $\mathrm{Vol}(y_1 \mathsf{P}_1 + \cdots + y_m \mathsf{P}_m) = \sum_{(i_1, \dots, i_n) \in \binom{[m]}{n}} \mathrm{Vol}(\mathsf{P}_{i_1}, \dots, \mathsf{P}_{i_n}) \, y_{i_1} \dots y_{i_n}$ for any collection of $m \geq n$ polytopes $\mathsf{P}_1, \dots, \mathsf{P}_m$ and any real numbers $y_1, \dots, y_m$ such that $y_1 \mathsf{P}_1 + \cdots + y_m \mathsf{P}_m$ is a polytope. Note that $\mathrm{Vol}(\mathsf{P}, \dots, \mathsf{P}) = \mathrm{Vol}(\mathsf{P})$ and that mixed volumes are multilinear. Via Proposition 3.10, we can thus compute (mixed) volumes of shard polytopes using mixed volumes of simplices already computed by A. Postnikov in [12].

Lemma 3.11 ([12])**.** *The mixed volume of the faces* $\triangle_{J_1}, \dots, \triangle_{J_{n-1}}$ *of the standard simplex is*

$$\mathrm{Vol}(\triangle_{J_1}, \dots, \triangle_{J_{n-1}}) = \begin{cases} \dfrac{1}{(n-1)!} & \text{if } J_1, \dots, J_{n-1} \text{ satisfy the dragon marriage condition of [12]:} \\ & |J_{i_1} \cup \cdots \cup J_{i_k}| \geq k+1 \text{ for any distinct } i_1, \dots, i_k \in [n-1] \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.12. *For any arcs* $\alpha_1, \dots, \alpha_{n-1} \in \mathcal{A}_n$*, the mixed volume of* $\mathsf{SP}(\alpha_1), \dots, \mathsf{SP}(\alpha_{n-1})$ *is*

$$\mathrm{Vol}(\mathsf{SP}(\alpha_1), \dots, \mathsf{SP}(\alpha_{n-1})) = \frac{1}{(n-1)!} \sum_{J_1, \dots, J_{n-1}} (-1)^{\sum_{i \in [n-1]} |J_i \cap (B_i \cup \{a_i, b_i\})|}$$

summing over all $J_1, \dots, J_{n-1}$ *with* $|J_i| \geq 2$ *and* $(A_i \cup \{a_i, b_i\}) \triangleright J_i$ *for all* $i \in [n-1]$*, and such that* $J_1, \dots, J_{n-1}$ *satisfy the dragon marriage condition.*

4 Type B shard polytopes

Based on Propositions 2.4 and 2.12, we extend shard polytopes to the type B_n Coxeter group. It is the group of permutations σ of $[\pm n] = \{-n, \dots, -1\} \cup \{1, \dots, n\}$ such that $\sigma(-i) = -\sigma(i)$ for all $i \in [\pm n]$. The following are the analogues of arcs.

Definition 4.1. A *B-arc* on $[\pm n]$ is either a centrally symmetric A-arc on $[\pm n]$ or a centrally symmetric and noncrossing pair of A-arcs on $[\pm n]$ with disjoint endpoints.

As in type A, there is a forcing order on B-arcs, and the lattice congruences of the type B_n weak order correspond to the upper ideals in forcing order. Geometrically, the type B_n arrangement is defined by the hyperplanes $\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x}_a = \boldsymbol{x}_b\}$ for $a < b \in [\pm n]$, with the convention that $\boldsymbol{x}_{-i} := -\boldsymbol{x}_i$. Each B-arc $\beta := (-\alpha, \alpha)$ with $\alpha := (a, b, A, B)$ corresponds to a shard $\mathsf{S}(\beta)$ defined as the piece of the hyperplane $\boldsymbol{x}_a = \boldsymbol{x}_b$ satisfying the inequalities $\boldsymbol{x}_{a'} \le \boldsymbol{x}_a = \boldsymbol{x}_b \le \boldsymbol{x}_{b'}$ for all $a' \in A$ and $b' \in B$, again with the convention that $\boldsymbol{x}_{-i} = -\boldsymbol{x}_i$. See Figure 8. The following are the analogues of shard polytopes.

Definition 4.2. The *shard polytope* $\mathsf{SP}(\beta)$ of a B-arc $\beta := (-\alpha, \alpha)$ is the convex hull of the characteristic vectors of all α-alternating matchings, with the convention that $\boldsymbol{e}_{-i} = -\boldsymbol{e}_i$.

Again, these polytopes are designed to fulfill the following normal fan property.

Proposition 4.3. *For any B-arc β, the union of the walls of the normal fan of the shard polytope $\mathsf{SP}(\beta)$ contains the shard $\mathsf{S}(\beta)$ and is contained in the union of the shards $\mathsf{S}(\beta')$ for β' forcing β.*

Corollary 4.4. *For any B-arc ideal $\mathcal{A}^{\mathrm{B}} \subseteq \mathcal{A}^{\mathrm{B}}_n$, the quotient fan $\mathcal{F}^{\mathrm{B}}_{\mathcal{A}^{\mathrm{B}}}$ is the normal fan of the Minkowski sum $\mathsf{SP}(\mathcal{A}^{\mathrm{B}}) := \sum_{\beta \in \mathcal{A}^{\mathrm{B}}} \mathsf{SP}(\beta)$ of the shard polytopes $\mathsf{SP}(\beta)$ of all B-arcs $\beta \in \mathcal{A}^{\mathrm{B}}$.*

Finally, we conjecture that type B shard polytopes are Minkowski indecomposable. In any case, we prove that they form a Minkowski basis for type B deformed permutahedra.

Theorem 4.5. *Up to translation, any type B deformed permutahedron has a unique decomposition as a Minkowski sum and difference of dilated type B shard polytopes.*

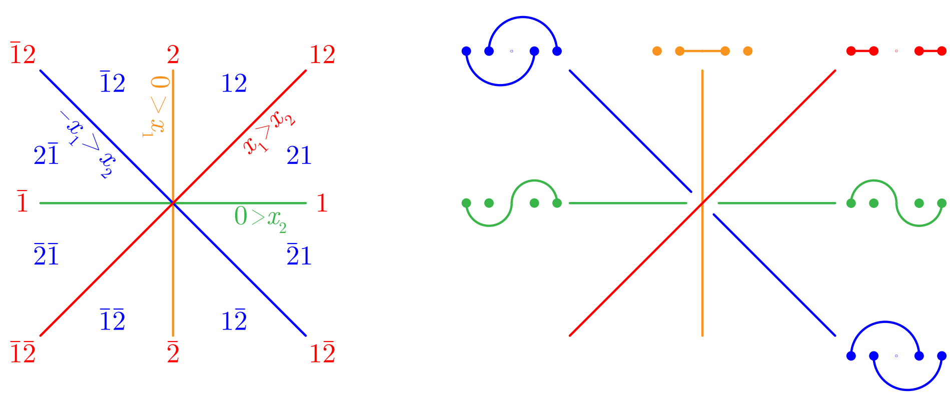

Figure 8: The type B_2 Coxeter fan $\mathcal{F}^{\mathrm{B}}_2$ (left) and the corresponding B-shards (right).

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