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On a rank-unimodality conjecture of Morier-Genoud and Ovsienko

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Abstract. Let $\alpha = (a, b, \ldots)$ be a composition. Consider the associated poset $F(\alpha)$, called a fence, whose covering relations are

$$x_1 \lhd x_2 \lhd \ldots \lhd x_{a+1} \rhd x_{a+2} \rhd \ldots \rhd x_{a+b+1} \lhd x_{a+b+2} \lhd \ldots .$$

We study the associated distributive lattice $L(\alpha)$ consisting of all lower order ideals of $F(\alpha)$. These lattices are important in the theory of cluster algebras and their rank generating functions can be used to define q-analogues of rational numbers. In particular, we make progress on a recent conjecture of Morier-Genoud and Ovsienko that $L(\alpha)$ is rank unimodal. We show that if one of the parts of α is greater than the sum of the others, then the conjecture is true. We conjecture that $L(\alpha)$ enjoys the stronger properties of having a nested chain decomposition and having a rank sequence which is either top or bottom interlacing, the latter being a recently defined property of sequences. We verify that these properties hold for compositions with at most three parts and for what we call d-divided posets, generalizing work of Claussen and simplifying a construction of Gansner.

Résumé. Soit $\alpha = (a, b, \ldots)$ une composition. Considérons l'ensemble partiellement ordonné associé $F(\alpha)$, appelé une clôture, dont les rélations de couverture sont

$$x_1 \lhd x_2 \lhd \ldots \lhd x_{a+1} \rhd x_{a+2} \rhd \ldots \rhd x_{a+b+1} \lhd x_{a+b+2} \lhd \ldots .$$

Nous étudions le treillis distributif associé $L(\alpha)$ composé de tous les idéaux inférieurs de $F(\alpha)$. Ces treillis sont importants en la théorie d'algèbres ammassées et leurs fonctions génératrices de rang peuvent être utilisées pour définir des q-analogues des nombres rationnels. En particulier, nous progressons sur une conjecture récente de Morier-Genoud et Ovsienko que $L(\alpha)$ est rang unimodal. Nous vérifions la conjecture quand une des parties de la composition est plus grande que la somme des autres. Nous conjecturons que $L(\alpha)$ a les propriétés plus puissantes d'avoir une décompoisition en chaînes imbriquées et d'avoir une séquence de rangs qui entrelace soit en haut soit en bas, ce dernier étant une propriété récemment définie. Nous vérifions que ces propriétés tiennent pour les compositions avec au plus trois parties en généralisant le travail de Claussen.

Keywords: heaviness, interlacing, distributive lattice, fence, nested chain decomposition, rank unimodal

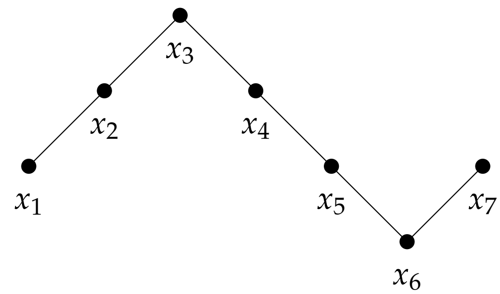

Figure 1: The fence $F(2, 3, 1)$

1 Basic definitions and background

This extended abstract is a summary of the results in the paper of the same name and with the same authors [12].

We will be studying the conjectured rank unimodality of certain distributive lattices. We begin by defining the posets from which they arise. Our terminology for partially ordered sets and other structures will follow Sagan's combinatorics text [16]. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ be a *composition of* $n-1$, that is, a sequence of positive integers summing to $n-1$. To simplify notation we will sometimes write $\alpha = (a, b, c, \ldots)$. For each α we have a corresponding *fence poset*, $F = F(\alpha)$, with elements $x_1, x_2, \ldots, x_n$ and covering relations

$$x_1 \lhd x_2 \lhd \ldots \lhd x_{a+1} \rhd x_{a+2} \rhd \ldots \rhd x_{a+b+1} \lhd x_{a+b+2} \lhd \ldots \lhd x_{a+b+c+1} \rhd x_{a+b+c+2} \rhd \ldots \quad (1.1)$$

where $\unlhd$ is the order relation in F. The Hasse diagram of the fence $F(2, 3, 1)$ is shown in Figure 1. We will call the maximal chains of F *segments* so that the ith part of α is equal to the length of the ith segment of F. Because of this convention, the sum of the parts of α is one less than $\#F$, the cardinality of F.

Given any poset P, its set of (lower) order ideals forms a distributive lattice $L(P)$. We will shorten $L(F(\alpha))$ to $L(\alpha)$ and use similar abbreviations with other notation. The complement of an order ideal of P is an order ideal of P^*, the poset dual of P. And if $\alpha = (\alpha_1, \ldots, \alpha_s)$ has an odd number of segments then $F(\alpha^r) \cong F(\alpha)^*$ where $\alpha^r = (\alpha_s, \ldots, \alpha_2, \alpha_1)$ is the reversal of α and $\cong$ is a poset isomorphism. Combining these two observations and translating to the corresponding lattices we have the following result which we record for future use.

Lemma 1.1. *For any* $\alpha = (\alpha_1, \ldots, \alpha_s)$ *with s odd we have*

$$L(\alpha) \cong L(\alpha^r)^*. \qquad \square$$

The lattices $L(\alpha)$ will be our principal objects of study. They are important objects in the theory of cluster algebras. In particular, one can view $F(\alpha)$ as a quiver formed from the Dynkin diagram of type A by replacing each cover $x \lessdot y$ with an arrow from x to y. Then $L(\alpha)$ can be used to compute a mutation in a corresponding cluster algebra on a surface. In fact, there are (at least) six different descriptions of $L(\alpha)$ or its dual which are useful for this computation. These are in terms of perfect matchings on snake graphs [15], perfect matchings of angles [23, 24], T-paths [19, 18, 20], lattice paths on snake graphs [15], lattice paths of angles [8], or S-paths [8].

In order to introduce the conjecture on which we will focus, we need some definitions related to sequences and their generating functions. We say that a sequence of nonnegative real numbers $a_0, a_1, \ldots, a_n$ is *unimodal* if there is some index m such that

$$a_0 \leq a_1 \leq \ldots \leq a_m \geq a_{m+1} \geq \ldots \geq a_n.$$

Unimodal sequences arise frequently in combinatorics, algebra, and geometry; see the survey articles of Stanley [22], Brenti [7] or Brändén [5]. We will say that the generating function $f(q) = \sum_k a_k q^k$ has a property such as unimodality if its coefficient sequence does.

Now suppose that P is a finite poset. We call P *ranked* if, for all $x \in P$, the length of any saturated chain from a minimal element to x is invariant. This length is called the *rank of* x and denoted $\operatorname{rk} x$. We also define the *rank of* P, $\operatorname{rk} P$, to be the maximum of $\operatorname{rk} x$ over all $x \in P$. The *kth rank* of P is the set

$$R_j(P) = \{x \in P \mid \operatorname{rk} x = j\}$$

and we let $r_j(P) = \#R_j(P)$. Any finite distributive lattice is ranked by the cardinality of each element viewed as an order ideal of the corresponding poset of join irreducibles. We say that P is *rank unimodal* if the sequence $r_0(P), r_1(P), \ldots, r_n(P)$ is unimodal where $n = \operatorname{rk} P$. We will similarly prepend “rank" to other properties of sequences when applied to the rank sequence of a poset. Our main object of study is the following conjecture of Morier-Genoud and Ovsienko.

Conjecture 1.2 ([13]). *For any α, the lattice $L(\alpha)$ is rank unimodal.*

We note that Morier-Genoud and Ovsienko used the rank generating functions for the $L(\alpha)$ to define q-analogues for rational numbers. Interestingly, special cases of this conjecture had already been proven even before it was stated because the problem is so natural in its own right. Gansner [9] proved Conjecture 1.2 for certain dual fences which we call d-divided. Munarini and Zagaglia [14] gave a different proof of the conjecture for 2-divided fences which are those with $\alpha = (1, 1, \ldots, 1)$. Since the conjecture was posed, Claussen [8] has shown that it is true for all fences with at most four segments. One of our main results is that Conjecture 1.2 holds if one of the segments is sufficiently long.

Theorem 1.3. *Suppose $\alpha = (\alpha_1, \ldots, \alpha_s)$ and there is an index t such that*

$$\alpha_t > \sum_{i \neq t} \alpha_i.$$

Then $L(\alpha)$ is rank unimodal.

We will also be interested in various strengthenings of Conjecture 1.2. To state them, we will need to define other properties of sequences and posets. Say that the sequence $a_0, a_1, \ldots, a_n$ is *symmetric* if $a_k = a_{n-k}$ for $k < n/2$. Symmetric unimodal sequences are common, for example a row of Pascal's triangle or the coefficients of a q-binomial coefficient. Even when one does not have symmetry, there may be some relation between a_k and a_{n-k}. Call the sequence *top heavy* (respectively, *bottom heavy*) if $a_k \le a_{n-k}$ (respectively, $a_k \ge a_{n-k}$) for $k < n/2$. As an illustration, a special case of a result of Björner and Ekedahl [4] states that the rank sequence for Bruhat order on a crystallographic Coxeter group is top heavy. More recently, a new property of sequences has been identified which implies both unimodality and heaviness. Call the sequence *top interlacing* if

$$a_0 \le a_n \le a_1 \le a_{n-1} \le \ldots \le a_{\lceil n/2 \rceil} \tag{1.2}$$

where $\lceil \cdot \rceil$ is the ceiling function. Similarly, the sequence is *bottom interlacing* if

$$a_n \le a_0 \le a_{n-1} \le a_1 \le \ldots \le a_{\lfloor n/2 \rfloor}$$

with $\lfloor \cdot \rfloor$ being the floor function. See [1, 2, 3, 6, 17, 21] for research related to this concept. We note that in the literature (1.2) has been called "alternately increasing". However, we prefer our terminology both because "alternating" usually refers to a sequence satisfying $a_0 < a_1 > a_2 < a_3 > \ldots$, and since (1.2) implies that the first half of the sequence and the reverse of the second half interlace in the usual sense of the term. We propose the following strengthening of Conjecture 1.2. In it, we refer to the rank sequence

$$r(\alpha) = (r_0(\alpha), r_1(\alpha), \ldots, r_n(\alpha)) \tag{1.3}$$

where $r_j(\alpha) = r_j(L(\alpha))$ and $n = \#F(\alpha)$.

Conjecture 1.4. *Suppose $\alpha = (\alpha_1, \ldots, \alpha_s)$.*

(a) If $s = 1$ then $r(\alpha) = (1, 1, \ldots, 1)$ is symmetric.

(b) If s is even, then $r(\alpha)$ is bottom interlacing.

(c) Suppose $s \ge 3$ is odd and let $\alpha' = (\alpha_2, \ldots, \alpha_{s-1})$.

- *(i) If $\alpha_1 > \alpha_s$ then $r(\alpha)$ is bottom interlacing.*
- *(ii) If $\alpha_1 < \alpha_s$ then $r(\alpha)$ is top interlacing.*
- *(iii) If $\alpha_1 = \alpha_s$ then $r(\alpha)$ is symmetric, bottom interlacing, or top interlacing depending on whether $r(\alpha')$ is symmetric, top interlacing, or bottom interlacing, respectively.*

Statement (a) in this conjecture is trivial, but is needed as a base case. We have verified this conjecture by computer for up to 5 segments of lengths at most 10, and for 6 segments having lengths at most 5. We have been able to prove the conjecture for various fences, including those with at most three segments and the d-divided posets, by showing that the corresponding lattices satisfy an even stronger condition which we now describe.

Let P be a ranked poset with $\operatorname{rk} P = n$. Also, let C be a saturated x–y chain in P. The *center* and *interval* of C are the rational number

$$\operatorname{cen} C = \frac{\operatorname{rk} x + \operatorname{rk} y}{2}$$

and interval of integers

$$[C] = [\operatorname{rk} x, \operatorname{rk} y],$$

respectively. A *chain decomposition* or *CD* of P is a set partition of P into saturated chains. In a *symmetric chain decomposition* or *SCD*, every chain C in the partition must satisfy $\operatorname{cen} C = n/2$. Equivalently, if C is an x–y chain of the partition then $\operatorname{rk} y = n - \operatorname{rk} x$. If P admits an SCD then its rank sequence is symmetric and unimodal. In fact, P even enjoys the strong Sperner property which says that, for all $k \geq 1$, the maximum cardinality of a union of k antichains is just the sum of the k largest ranks. See the survey article of Greene and Kleitman [11] for more information about chain decompositions and the Sperner property.

To deal with the case when the rank sequence is not symmetric, consider a *nested chain composition*, NCD, which is a CD where any two of its chains C, D satisfy either $[C] \subseteq [D]$ or $[D] \subseteq [C]$. If P admits an NCD then it is rank unimodal and still has the strong Sperner property. We will be particularly concerned with a special type of NCD. Call a CD *top centered* if every chain C in the partition satisfies $\operatorname{cen} C = n/2$ or $\operatorname{cen} C = (n+1)/2$. It follows easily that this is an NCD and the rank sequence of P is top interlacing. Similarly, a CD is *bottom centered* if its chains satisfy $\operatorname{cen} C = n/2$ or $(n-1)/2$. Again, this is an NCD and the rank sequence is now bottom interlacing. Note also that if a poset has an NCD and its rank sequence is top or bottom interlacing then the NCD must be top or bottom centered, respectively. This can be proven inductively using the observation that an NCD must contain a chain from a minimum rank element to a maximum rank element. This leads to the strongest of our conjectures so far.

Conjecture 1.5. *For any α, the lattice $L(\alpha)$ admits a CD which is either symmetric, top centered, or bottom centered consistent with Conjecture 1.4.*

We sketch proofs of a number of special cases of this conjecture in the sequel. In particular, when the fence has at most three segments we have the following refinement of Claussen's result on the rank unimodality of the $L(\alpha)$.

Theorem 1.6. *If α has at most three parts then Conjecture 1.5 is true.*

The rest of this abstract is structured as follows. In the next section we will sketch a proof of Theorem 1.3 in the case that the long segment is the first or the last. We will do this using a recursion which will also permit us to replace the strict inequality with a weak one for these particular segments. In Section 3 we indicate how to complete the proof of Theorem 1.3. We will also describe an inductive procedure for proving Conjecture 1.5 when a long segment exists, given that it holds for an appropriate base case. The following section will be devoted to giving a construction to prove Theorem 1.6.

2 Long initial or final segments

In this section we outline a proof of a stronger version of Theorem 1.3 where the long segment is either the first or the last. This will be based on a recursion for the rank generating function

$$r(q;\alpha) = \sum_{j\geq 0} r_j(\alpha) q^j$$

where the $r_j(\alpha)$ are given by (1.3). The method of proof involves considering the ideals of $F(\alpha)$ which do or do not contain an element x, which we will call *toggling on x*. Also, for the recursions to make sense, we must permit compositions $(\alpha_1, \ldots, \alpha_s)$ where $\alpha_s = 0$. But in this case we just define

$$F(\alpha_1, \ldots, \alpha_{s-1}, 0) = F(\alpha_1, \ldots, \alpha_{s-1}).$$

Lemma 2.1. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$. Then for s odd*

$$r(q;\alpha) = r(q;\alpha_1, \ldots, \alpha_{s-1}, \alpha_s - 1) + q^{\alpha_s+1} \cdot r(q;\alpha_1, \ldots, \alpha_{s-2}, \alpha_{s-1} - 1)$$

and for s even

$$r(q;\alpha) = r(q;\alpha_1, \ldots, \alpha_{s-2}, \alpha_{s-1} - 1) + q \cdot r(q;\alpha_1, \ldots, \alpha_{s-1}, \alpha_s - 1).$$

In order to make use of this lemma, we will have to consider the indices where the coefficients of a polynomial achieve their maximum. Given $f = \sum_k a_k q^k$ we define the *set of maxima indices* as

$$\mathrm{mi}(f) = \{k \mid a_k = m\}$$

where m is the maximum value of a coefficient of f. Note that if f is unimodal then $\mathrm{mi}(f)$ will be an interval of integers. The next result is easy to prove.

Lemma 2.2. *Let f, g be unimodal polynomials and suppose that $\mathrm{mi}(f) \cap \mathrm{mi}(g) \neq \emptyset$. Then $f + g$ is unimodal and $\mathrm{mi}(f+g) = \mathrm{mi}(f) \cap \mathrm{mi}(g)$.*

The two preceding results can be used to prove a stronger version of Theorem 1.3 for the first segment.

Theorem 2.3. *If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ satisfies*

$$\alpha_1 \geq \alpha_2 + \alpha_3 + \cdots + \alpha_s$$

then $r(q;\alpha)$ is unimodal with

$$\mathrm{mi}(r(q;\alpha)) = [\alpha_2 + \alpha_3 + \cdots + \alpha_s,\ \alpha_1].$$

We have the same result for the last segment.

Theorem 2.4. *If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ satisfies*

$$\alpha_s \geq \alpha_1 + \alpha_2 + \cdots + \alpha_{s-1}$$

then $r(q;\alpha)$ is unimodal.

3 Arbitrary long segments

Theorem 1 can be proved by using induction on the length of the long segment and the following lemma which locates the ranks of maximum size in the lattice coming from a poset with a long segment.

Lemma 3.1. *Let $\alpha = (\alpha_1, \ldots, \alpha_s)$ and $n = \#F(\alpha)$. Suppose that for some t we have*

$$\alpha_t > \sum_{i \neq t} \alpha_i. \tag{3.1}$$

Then the maximum size of a rank of $L = L(\alpha)$ is $\ell = \#L(F')$ where F' is the poset obtained by removing the elements of segment t from $F = F(\alpha)$. And this maximum occurs at ranks $m+1$ through $n-m-1$ where $m = \#F'$.

We now give an inductive method for proving that Conjecture 1.5 holds. To do so, we must first investigate the finer structure of $L(\alpha)$ where α has a long segment. For any ranked poset P, let

$$P_k = \{x \in P \mid \mathrm{rk}\, x \leq k\}$$

and

$$P^k = \{x \in P \mid \mathrm{rk}\, x \geq k\}.$$

Lemma 3.2. *Let $\alpha = (\alpha_1, \ldots, \alpha_s)$, $F = F(\alpha)$, and $L = L(\alpha)$. Also let $n = \#F$ and $m = \#F'$ where F' is as in Lemma 3.1. Suppose that for some t we have*

$$\alpha_t > \sum_{i \neq t} \alpha_i.$$

Let $G = F(\beta)$ and $M = L(\beta)$ where

$$\beta = (\alpha_1, \ldots, \alpha_{t-1}, \alpha_t + 1, \alpha_{t+1}, \ldots, \alpha_s).$$

Then we have isomorphisms

$$L_{n-m-1} \cong M_{n-m-1} \quad \text{and} \quad L^{n-m} \cong M^{n-m+1}.$$

We now have everything in place to state our inductive criterion for checking whether a poset with a long segment satisfies Conjecture 1.5.

Theorem 3.3. *Assume the hypotheses and notation of Lemma 3.2. If L has an NCD then so does M. Furthermore, if the NCD of L is symmetric, top centered, or bottom centered then the NCD of M has the same property.*

Using the previous lemma and induction on the length of the long segment, we immediately get the following result. Note that showing one lattice has an NCD of a certain form immediately gives an infinite family of lattices with NCDs of the same form.

Corollary 3.4. *Let $\alpha = (\alpha_1, \ldots, \alpha_s)$ where*

$$\alpha_t = 1 + \sum_{i \neq t} \alpha_i \tag{3.2}$$

for some t. If $L = L(\alpha)$ has an NCD then so does $M = L(\beta)$ where

$$\beta = (\alpha_1, \ldots, \alpha_{t-1}, \alpha_t + a, \alpha_{t+1}, \ldots, \alpha_s)$$

for any $a \geq 0$. Furthermore, if the NCD of L is symmetric, top centered, or bottom centered then the NCD of M has the same property.

Now one can use a computer to verify the following.

Corollary 3.5. *Let $F = F(\alpha_1, \ldots, \alpha_s)$ where*

$$\alpha_t > \sum_{i \neq t} \alpha_i$$

where

$$\sum_{i \neq t} \alpha_i \leq 5.$$

Then $L = L(\alpha)$ satisfies Conjecture 1.5.

4 At most three segments

This section is devoted to sketching a proof of Theorem 1.6. The result is trivial if the fence $F(\alpha)$ has one segment and fairly easy if it has two. But for three, we will have to use a modified version of the famous Greene–Kleitman symmetric chain decomposition of the Boolean algebra of all subsets of a finite set [10].

In particular, we will use the idea of a Greene–Kleitman core. To define this object, let $w = w_1 w_2 \ldots w_n$ be a sequence (or word) of zeros and ones. The *Greene–Kleitman (GK) core of w*, $\mathrm{GK}(w)$, is a set of pairs of indices formed as follows. If $w_i = 0$ and $w_{i+1} = 1$ then $(i, i+1) \in \mathrm{GK}(w)$. We continue to add pairs (i,j) to $\mathrm{GK}(w)$ as long as $w_i = 0$ and $w_j = 1$, where $i < j$ and all the elements between w_i and w_j are already in pairs of the core. For example, if

$$w = 110001011000111$$

then

$$\mathrm{GK}(w) = * * * \widehat{0\widehat{01}\widehat{01}1} * \widehat{0\widehat{01}1} *$$

where elements not in the GK core have been replaced by stars, and pairs in the core are indicated by the hats. Writing out the pairs themselves gives

$$\mathrm{GK}(w) = \{(4,9),\ (5,6),\ (7,8),\ (11,14),\ (12,13)\}.$$

We will refer to the elements in the pairs of $\mathrm{GK}(w)$ as *matched*.

To apply this idea to fences, we will have to modify the GK core. And to do that it will be convenient to think of a fence F as a partial order on $[n] = \{1, 2, \ldots, n\}$ where, as usual, $n = \#F$. When doing this, it will be important to distinguish $i \leq j$ which is the usual total order on the integers and $i \trianglelefteq j$ which will be an order relation in F. So consider $F = F(a,b,c)$ as the fence with covering relations

$$b+c+1 \lhd b+c+2 \lhd \ldots \lhd a+b+c+1 \rhd b \rhd b-1 \rhd \ldots \rhd 1 \lhd b+1 \lhd b+2 \lhd \ldots \lhd b+c.$$

In other words label the second segment except for its maximum element with the elements of the interval $[1, b]$ from bottom to top. Then label the elements of the third segment (except its minimum which has already been labeled) bottom to top with $[b+1, b+c]$. Finally label the complete first segment with $[b+c+1, a+b+c+1]$ again from bottom to top. Note that this labeling is a linear extension of F. This labeling for the fence $F(2,3,1)$ is showing in Figure 2.

Now associate with any subset $S \subseteq F$ a word $w = w_S = w_1 \ldots w_n$ where w_i is one or zero depending on whether $i \in S$ or $i \notin S$, respectively, and $n = \#F$. Suppose $I \subseteq F$ is an ideal and $w = w_I$. Since specifying I and specifying w are equivalent, we will often go back and forth without mention.

Given an ideal, I, call an element $f \in F$ *frozen* if it is unmatched in $\mathrm{GK}(w)$ and there exists $(i,j) \in GK(w)$ with $f \rhd i$ or $f \lhd j$. Note that since (i,j) is in the GK core we must

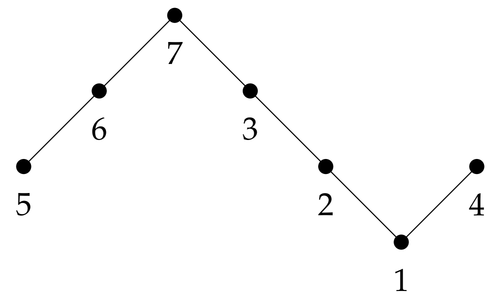

Figure 2: The labeling of $F(2,3,1)$

have $i \notin I$ and $j \in I$. So $f \rhd i$ implies $f \notin I$, since I is an ideal, and $w_f = 0$. Similarly, if $f \lhd j$ then $f \in I$ and $w_f = 1$. Whether an element is frozen or not depends upon the ideal under consideration, but we will make sure I is clear from context. Finally, define the *core* of $w = w_I$ by

$$\operatorname{core} w = \mathrm{GK}(w) \cup \{f \in F \mid f \text{ is frozen}\}.$$

We say that elements of F not in $\operatorname{core} w$ are *free*. For example, suppose $F = F(2,3,1)$ and $I = \{1,4,5\}$. Then we first compute the GK core as indicated by the hats in

$$w = 1\widehat{0\widehat{01}1}00.$$

Since $(3,4) \in \mathrm{GK}(w)$ and $1 \lhd 4$ we have that 1 is frozen in w. Similarly, $(3,4) \in \mathrm{GK}(w)$ and $7 \rhd 3$ implies that 7 is frozen. One can also check that 6 is free so that

$$\operatorname{core} w = 1\widehat{0\widehat{01}1} * 0$$

or

$$\operatorname{core} w = \{1,\ (2,5),\ (3,4),\ 7\}.$$

We now complete the proof of Theorem 1.6 by forming, for each possible $\kappa = \operatorname{core} w_I$ for some ideal I, a saturated chain containing I as follows. We start with $\overline{w}$ which is the word with every element not in $\operatorname{core} w$ equal to zero. We then turn the zeros in $\overline{w}$ to 1s one at a time from left to right until all free positions equal 1 to form a saturated chain C_κ. In our running example, the chain would be

$$C_\kappa = \{1001100,\ 1001110\}.$$

One can prove that in a fence $F(a,b,c)$ with $a > c$ that these form a bottom centered CD. The case $a = c$ is dealt with similarly, and the case $a < c$ by Lemma 1.1.

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