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The Hurwitz action in complex reflection groups

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Abstract. We study the reflection factorizations of an arbitrary element in the complex reflection groups $G(m, p, n)$ under the Hurwitz action. Using combinatorial and graph theoretical techniques, we present an if-and-only-if statement of when two factorizations are Hurwitz equivalent. Then we prove a formula that counts the number of Hurwitz orbits of an arbitrary element.

Keywords: reflection factorizations, complex reflection groups, Hurwitz action

1 Introduction

Given a group G with a generating set T closed under conjugacy, the *Hurwitz action* is a natural action of the n-strand braid group $\mathcal{B}_n$ on T^n. This action was introduced in the late 19th century by Hurwitz in the case that $G = \mathfrak{S}_n$ is the symmetric group and T is the set of transpositions in G, as part of his study [6] of covering surfaces of the sphere with given monodromy. More recently, the Hurwitz action played an important role in Bessis' proof [5] of the $K(\pi, 1)$ property for complements of complex reflection arrangements, this time in the case that G is a complex reflection group and T is its subset of reflections; in this setting, the transitivity of the Hurwitz action on certain tuples of reflections is a key part of the *Coxeter–Catalan combinatorics* associated to the reflection group G (see, e.g., [13]).

In the recent paper [2], the authors characterized those elements w in a real reflection group W (i.e., a finite Coxeter group) for which the Hurwitz action is transitive when restricted to minimum-length reflection factorizations of w. In particular, they showed that these elements are precisely the *quasi-Coxeter elements* for certain special subgroups, that is, the elements whose factorizations all generate the subgroup in question.

In the present paper, we extend the work of [2] to the complex realm. Our main result (Theorem 3.2) gives an explicit formula for the number of Hurwitz orbits of minimum-length factorizations of an arbitrary element g in the complex reflection group $G = G(m, p, n)$. As a consequence, we characterize the elements for which the action is transitive (Cor. 3.3). As a second consequence, we give an elegant "inverse" result: we

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show that two minimum-length reflection factorizations of an element $g \in G$ belong to the same Hurwitz orbit if and only if they generate the same subgroup of G (Cor. 3.4).

Our approach is combinatorial, taking advantage of the natural graph-theoretic structure of reflections in $G(m, p, n)$, and building on earlier work of Kluitmann [8] and Ben-Itzhak–Teicher [3] in the symmetric group.

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2 Background and notation

2.1 Conventions

Throughout this abstract, m, p, and n will represent positive integers with $p \mid m$. Since p divides m, the cyclic group $\mathbb{Z}/m\mathbb{Z}$, whose elements are equivalence classes of integers modulo m, has a unique subgroup $p\mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/(m/p)\mathbb{Z}$ of order m/p: it consists of those equivalence classes whose elements are divisible by p. We may write $k \equiv 0 \pmod{p}$ to indicate that an element k in $\mathbb{Z}/m\mathbb{Z}$ belongs to this subgroup, and $k \equiv 0 \pmod{m}$ to indicate that k is the identity in $\mathbb{Z}/m\mathbb{Z}$. As in the previous sentence, we do not distinguish notationally between the integer k and its equivalence class modulo m; this should cause no confusion in practice.

Given a collection of $k_1, k_2, \ldots, k_n$ of elements of $\mathbb{Z}/m\mathbb{Z}$, there is some minimal subgroup $k\mathbb{Z}/m\mathbb{Z}$ that contains all of them. If we take $k_1, \ldots, k_n$ to be any representatives of their equivalence classes, then the smallest positive representative k of $k\mathbb{Z}$ is $k = \gcd(m, k_1, \ldots, k_n)$. All greatest common divisors that appear in this paper (particularly, as in Definition 3.1) will be meant in this sense.

2.2 Complex reflection groups

Given a finite-dimensional complex vector space V, a *complex reflection* is a linear transformation $t : V \to V$ whose fixed space $\ker(t - 1)$ is a hyperplane (i.e., has codimension 1), and a finite subgroup G of $GL(V)$ is called a *complex reflection group* if G is generated by its subset R of complex reflections. Complex reflection groups were classified by Shephard and Todd [11]: every complex reflection group is a direct product of irreducibles, and every irreducible is either of the form

$$G(m,p,n) = \left\{ \begin{array}{l} n \times n \text{ monomial matrices whose nonzero entries are} \\ m\text{th roots of unity with product a } \frac{m}{p}\text{th root of unity} \end{array} \right\}$$

for positive integers m, p, n with $p \mid m$ or is one of 34 exceptional examples.

For every m, p, n, there is a natural projection map $\pi : G(m,p,n) \twoheadrightarrow G(1,1,n) = \mathfrak{S}_n$ defined as follows: for $g \in G(m,p,n)$, the matrix $\pi(g)$ is the result of replacing every root of unity in the matrix of g with 1. The resulting permutation is called the *underlying permutation* of g. It will be convenient to use the following, more compact, notation for elements of $G(m,p,n)$: one writes $g = [u; (a_1, \ldots, a_n)]$ where $u = \pi(g)$ and $a_j \in \mathbb{Z}/m\mathbb{Z}$ is the exponent of $\exp(2\pi i/m)$ in the nonzero entry of the jth column of g. This notation reveals that $G(m,1,n)$ has the structure of a *wreath product* $G(m,1,n) \cong (\mathbb{Z}/m\mathbb{Z}) \wr \mathfrak{S}_n$, with multiplication given by $[u; (a_1, \ldots, a_n)] \cdot [v; (b_1, \ldots, b_n)] = \big[uv; (a_{v(1)} + b_1, \ldots, a_{v(n)} + b_n)\big]$.

Given an element $g = [u; (a_1, \ldots, a_n)]$ of $G(m,p,n)$, the value a_j is called the *weight* of j. Further, for any subset $S \subset \{1, \ldots, n\}$, we define $\sum_{j \in S} a_j$ to be the *weight* of S. This notion will be particularly relevant when S is the set of entries of a cycle of g, or when $S = \{1, \ldots, n\}$ and $a_1 + \ldots + a_n$ is the weight of g. In this language, an element of $G(m,1,n)$ belongs to $G(m,p,n)$ if and only if its weight is a multiple of p.

When $p < m$, the group $G(m,p,n)$ contains two types of complex reflections: for $a \in \{0, 1, \ldots, m-1\}$, the *transposition-like reflections*

$$[(i\,j); a] \stackrel{\text{def}}{=\!=} [(i\,j); (0, \ldots, 0, a, 0, \ldots, 0, -a, 0, \ldots, 0)] \tag{2.1}$$

(with i having weight a and j having weight $-a$) of weight 0 and order 2, and for $b \in \{p, 2p, \ldots, m-p\}$ the *diagonal reflections* [id; $(0, \ldots, 0, b, 0, \ldots, 0)$] of weight b and various orders; the group $G(m,m,n)$ contains only the transposition-like reflections.

2.3 Shi's formula for reflection length

Fix a complex reflection group G with reflections R. Since G is a reflection group, every element g of G can be written as a product of reflections. If $f = (t_1, \ldots, t_\ell)$ is a tuple of reflections such that $g = t_1 \cdots t_\ell$, we say that f is a *(reflection) factorization* of g. We say that a reflection factorization F of g is *shortest*, *minimum*, or *of minimum length* if there is no reflection factorization of g using fewer reflections, and we define the *reflection length* $\ell_R(g)$ of g to be the length

$$\ell_R(g) = \min\{\ell : g = t_1 t_1 \cdots t_\ell \text{ for some } t_i \in R\}$$

of the shortest factorizations. Throughout the paper, we use the word "factorizations" as a shorthand for "reflection factorizations of minimum length".

In [12], Shi gave a combinatorial formula for reflection length in the group $G(m,p,n)$ that we now describe. For an element $g \in G(m,p,n)$, let $\mathrm{cyc}(g)$ be the number of cycles in $\pi(g)$. A *cycle partition* Π of g is a set partition of the set $\{C_1, C_2, \ldots, C_{\mathrm{cyc}(g)}\}$ of cycles of g such that for every part in Π, the corresponding cycle weights sum to 0 (mod p). (Such partitions always exist because the weight of g is 0 (mod p).) For example, the element $g = [\mathrm{id}; (2,2,2,2)] \in G(4,4,4)$ has four cycle partitions:

$$\mathrm{Par}(g) = \Big\{ [\![(1)(2) \mid (3)(4)]\!], [\![(1)(3) \mid (2)(4)]\!], [\![(1)(4) \mid (2)(3)]\!], [\![(1)(2)(3)(4)]\!] \Big\}.$$

Observe that the set of cycle partitions depends on the choice of the group containing g: if we view this element g as an element of $G(4,2,4)$ then $\mathrm{Par}(g)$ consists of all fifteen set partitions of the four cycles.

Given a partition Π of an element $g \in G(m,p,n)$, let $|\Pi|$ denote the number of parts of Π and let $v_m(\Pi)$ denote the number of parts of Π of weight 0 (mod m) (not just 0 (mod p)). Shi defines the *value* $v(\Pi)$ of a cycle partition Π to be $v(\Pi) = |\Pi| + v_m(\Pi)$. A partition is *maximum* if its value is the largest among the values of all possible cycle partitions of g (relative to the given m, p), and we denote by $\mathrm{Par}_{\max}(g)$ the set of maximum cycle partitions of g. For example, with $g = [\mathrm{id}; (2,2,2,2)] \in G(4,4,4)$ as above, the three partitions into two parts have value 4, while the partition in one part has value 2, and so $\mathrm{Par}_{\max}(g) = \Big\{ [\![(1)(2) \mid (3)(4)]\!], [\![(1)(3) \mid (2)(4)]\!], [\![(1)(4) \mid (2)(3)]\!] \Big\}$.

Theorem 2.1 (Shi [12])**.** *Given an element $g \in G(m,p,n)$ with reflections R, we have*

$$\ell_R(g) = n + \mathrm{cyc}(g) - v(\Pi)$$

where $\mathrm{cyc}(g)$ *is the number of cycles of g and $\Pi \in \mathrm{Par}_{\max}(g)$ is a maximum cycle partition of g.*

Using Shi's formula, we see that the element $g = [\mathrm{id}; (2,2,2,2)] \in G(4,4,4)$ has reflection length $\ell_R(g) = 4 + 4 - (2+2) = 4$. In general, computing $\mathrm{Par}_{\max}(g)$ is computationally challenging: there is a standard reduction from the SubsetSum problem, so it is NP-hard.

2.4 The Hurwitz action

The *Hurwitz move* σ_i on a tuple $(t_1, \ldots, t_n)$ of elements of a group G is the operation $\sigma_i(\ldots, t_i, t_{i+1}, \ldots) = (\ldots, t_{i+1}, t_{i+1}^{-1} t_i t_{i+1}, \ldots)$. It is easy to check that $\sigma_1, \ldots, \sigma_{n-1}$ satisfy the braid relations, and consequently give rise to the *Hurwitz action* of the braid group on n strands. This action was first studied by Hurwitz [6] in the context of the symmetric group $\mathfrak{S}_n$. The question of its orbit structure on reflection factorizations in Coxeter groups and complex reflection groups has been an area of substantial interest in the past two decades, especially in cases when the action is transitive, i.e., when there is a single Hurwitz orbit (see, e.g., [1, 2, 3, 5, 7, 10, 14]). The following result of Kluitmann [8] on the symmetric group will be of particular use for us.

Theorem 2.2 (Kluitmann [8])**.** *The set of all transposition factorizations $(t_1, t_2, \ldots, t_k)$ of an element $w \in \mathfrak{S}_n$ such that $\langle t_1, t_2, \ldots, t_k \rangle = \mathfrak{S}_n$ forms a single orbit under the Hurwitz action.*

3 The Main Theorem

To state our main results, we need one further piece of terminology. (The reader may wish to recall our convention for greatest common divisors from Section 2.1.)

Definition 3.1. Let $g \in G(m, p, n)$, let Π be a cycle partition of g, and let B be a block in Π. Suppose that the weights of the cycles in B are $(k_1, k_2, \ldots, k_{|B|})$. Define

$$r(B) = \gcd(m, k_1, k_2, \ldots, k_{|B|-1}, k_{|B|}).$$

Theorem 3.2. *Given an element $g \in G(m, p, n)$, the number of Hurwitz orbits of its shortest factorizations is given by*

$$\sum_{\Pi \in \mathrm{Par}_{\max}(g)} \prod_{B \in \Pi} (r(B))^{|B|-1}.$$

For example, the element $g = [\mathrm{id}; (2, 2, 2, 2)] \in G(4, 4, 4)$ discussed above has 12 Hurwitz orbits of shortest reflection factorizations, including $4 = 2^{2-1} \times 2^{2-1}$ from each of its three maximum cycle partitions.

Corollary 3.3. *Let $g \in G(m, p, n)$. The shortest factorizations of g form a single orbit under the Hurwitz action if and only if $|\mathrm{Par}_{\max}(g)| = 1$ and either $|B| = 1$ or $r(B) = 1$ for every block B in $\Pi \in \mathrm{Par}_{\max}(g)$. In particular, if g has a single cycle, then g is Hurwitz transitive. Further, when $p = 1$, g is Hurwitz transitive if and only if no two cycles of g have nonzero weights that sum to $0 \pmod m$.*

Corollary 3.4. *Two shortest factorizations f_1 and f_2 of an element g in $G = G(m, p, n)$ are Hurwitz-equivalent if and only if they generate the same subgroup of G.*

The remainder of this extended abstract is devoted to the proof of these results. We begin by defining a standard form for factorizations, and show that every factorization is Hurwitz-equivalent to a factorization in standard form.

3.1 Standard forms

In this section, we define a standard form for factorizations in $G(m, p, n)$ and show that every shortest factorization is Hurwitz-equivalent to a factorization in standard form. It is easiest to describe the standard forms in terms of a graph object associated to a factorization.

Definition 3.5. Given a factorization $f = (t_1, t_2, \ldots, t_\ell)$ of an element $g = t_1 \cdots t_\ell \in G(m, p, n)$, the *factorization graph* of f is the graph $\Gamma_f = (V, E)$ on (labeled) vertex set $V = \{1, \ldots, n\}$ with (labeled) edges $E = \{e_1, \ldots, e_\ell\}$ defined as follows: if t_k has underlying permutation $(i\, j)$ then e_k joins vertices i and j, while if t_k is diagonal with nonzero weight in position i then e_k is a loop at vertex i.

Fix an element g in $G(m, p, n)$ and a factorization $f = (t_1, \ldots, t_\ell)$ of g. The connected components of the graph Γ_f form a set partition of $\{1, \ldots, n\}$. In fact, this set partition corresponds to a cycle partition of g. We denote by Π_f the cycle partition induced in this way by the factorization f. If B is a part in Π_f, we denote by $|B|$ the number of cycles in B and by $\Gamma_f|_B$ the connected component of Γ_f that corresponds to B.

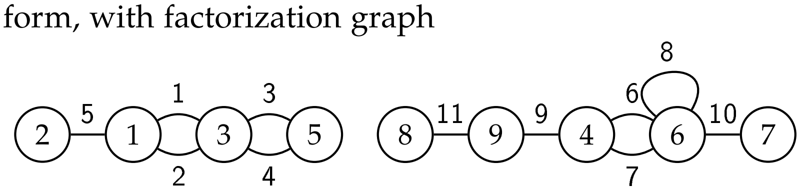
We are now prepared for the key definition of this section.

Definition 3.6. Given a shortest factorization f of an element $g \in G(m, p, n)$, with factorization graph Γ_f and cycle partition Π_f. Let B_1, B_2, $\ldots$, $B_{|\Pi_f|}$ be the parts of Π_f, and for each part $B_i \in \Pi_f$, say that $C_{i,1}$, $\ldots$, $C_{i,|B_i|}$ are the cycles of g contained in B_i, and that for each j, $v_{i,j}$ is the smallest element of $\{1, \ldots, n\}$ permuted by $C_{i,j}$. Without loss of generality, assume that the indices of the B_i and $C_{i,j}$ have been chosen so that $v_{i,1} < v_{i,2} < \ldots < v_{i,|B_i|}$ and $v_{1,1} < v_{2,1} < \ldots < v_{|\Pi_f|,1}$. We say that f is in *standard form* if the following conditions are met: (1) if $i' < i$ then every edge in $\Gamma_f|_{B_{i'}}$ has smaller label than all edges in $\Gamma_f|_{B_i}$; (2) for each i, the first $2|B_i| - 2$ factors $t_1, \ldots, t_{2|B_i|-2}$ in f that correspond to edges in $\Gamma_f|_{B_i}$ have underlying transpositions $\pi(t_1) = \pi(t_2) = (v_{i,1}\, v_{i,2})$, $\pi(t_3) = \pi(t_4) = (v_{i,2}\, v_{i,3})$, $\ldots$, $\pi(t_{2|B_i|-3}) = \pi(t_{2|B_i|-2}) = (v_{i,|B_i|-1} v_{i,|B_i|})$; and (3) for each i, if there is a loop in $\Gamma_f|_{B_i}$ then it is the edge labeled $2|B_i| - 1$ and acts on vertex $v_{i,|B_i|}$.

For example, the factorization $[(13); 0] \cdot [(13); 1] \cdot [(35); 1] \cdot [(35); 3] \cdot [(12); 0] \cdot [(46); 0] \cdot [(46); 4] \cdot [\mathrm{id}; (0, 0, 0, 0, 0, 3, 0, 0, 0)] \cdot [(49); 1] \cdot [(67); 2] \cdot [(89); 3]$ of the element

$$[(12)(3)(498)(5)(67); (0, 1, 1, 1, 7, 2, 6, 6, 6)] \in G(9, 3, 9)$$

is in standard form, with factorization graph



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Lemma 3.7. *For any $g \in G(m, p, n)$ and any minimum-length factorization f of g, there is a standard form factorization of g that is Hurwitz-equivalent to f.*

Proof sketch. The factorization f induces a cycle partition Π_f. It is straightforward to write down a standard form factorization f' of g such that $\Pi_{f'} = \Pi_f$. One can check using Shi's formula (Theorem 2.1) that every connected component in Γ_f must have the same number of edges as the corresponding component in $\Gamma_{f'}$. By Kluitmann's theorem (Theorem 2.2), the projected factorizations $\pi(f)$ and $\pi(f')$ are Hurwitz-equivalent, i.e., there is some braid β such that $\beta(\pi(f)) = \pi(f')$. Then $\beta(f)$ is the desired factorization. $\square$

3.2 Hurwitz paths between standard form factorizations

In this section, we construct explicit sequences of Hurwitz moves connecting certain standard form factorizations to each other. The main result of the section is Lemma 3.12.

Definition 3.8. For $n \geq 1$, we say that a factorization in $G(m,p,n)$ is a *doubled path* if it is of the form

$$\Big([(12);a_1],\ [(12);b_1],\ [(23);a_2],\ [(23);b_2],\ldots,[(n-1\ n);a_{n-1}],\ [(n-1\ n);b_{n-1}]\Big)$$

or

$$\Big([(12);a_1],\ [(12);b_1],\ldots,[(n-1\ n);a_{n-1}],\ [(n-1\ n);b_{n-1}],\ [\mathrm{id};(0,\ldots,0,d)]\Big).$$

By multiplying out, it's easy to see that every doubled path in $G(m,p,n)$ is a factorization of a diagonal element $g = [\mathrm{id};(k_1,\ldots,k_n)] \in G(m,p,n)$ of weight 0 (if no diagonal element is present) or d, and that for each i one has $b_i = a_i + k_1 + k_2 + \ldots + k_i$. Consequently, given the product g, the a_i determine the entire factorization. This suggests the following definition.

Definition 3.9. Let $f = \Big([(12);a_1],[(12);b_1],[(23);a_2],[(23);b_2],\ldots\Big)$ be a doubled path, factoring an element $g = [\mathrm{id};(k_1,\ldots,k_n)] \in G(m,p,n)$. Define the *pair weight* of the i-th pair of factors (i.e., with underlying transposition $(i\,i+1)$) to be a_i and the corresponding *pair difference* to be the difference $d_i \stackrel{\mathrm{def}}{=} b_i - a_i = k_1 + \cdots + k_i$.

The next result gives a sufficient condition for two doubled paths to belong to the same Hurwitz orbit.

Proposition 3.10. *Suppose $f_1 = \Big([(12);a_1],[(12);b_1],\ldots\Big)$ and $f_2 = \Big([(12);a_1'],[(12);b_1'],\ldots\Big)$ are two doubled paths factoring the same element g of weight d in $G(m,p,n)$. If there exists an $n \times (n-1)$ $\mathbb{Z}$-matrix $M = (m_{ij})$ such that $m_{ij} = m_{ji}$ for $i,j \leq n-1$ and*

$$\begin{pmatrix} a_1 & \cdots & a_{n-1} \end{pmatrix} + \begin{pmatrix} d_1 & \cdots & d_{n-1} & d \end{pmatrix} \cdot M \equiv \begin{pmatrix} a_1' & \cdots & a_{n-1}' \end{pmatrix} \pmod{m} \qquad (3.1)$$

(with congruence taken coordinate-wise), then there exists a Hurwitz path from f_1 to f_2.

Proof sketch. We define two families of operations that can be applied to doubled paths: For any pair of indices i,j with $1 \leq i < j \leq n-1$, define $\tau_{i,j}$ to be the following sequence of Hurwitz moves:

$$\tau_{i,j} \stackrel{\mathrm{def}}{=} \sigma_{2j-2}\sigma_{2j-1}\sigma_{2j-3}^{-1}\sigma_{2j-2}^{-1}\cdots\sigma_{2i+2}\sigma_{2i+3}\sigma_{2i+1}^{-1}\sigma_{2i+2}^{-1} \circ \sigma_{2i}^{-1}\sigma_{2i+1}^{-1}\sigma_{2i-1}^{-1}\sigma_{2i}^{-1} \circ$$
$$\sigma_{2i}^{-1}\sigma_{2i+1}^{-1}\sigma_{2i-1}^{-1}\sigma_{2i}^{-1} \circ \sigma_{2i+2}\sigma_{2i+1}\sigma_{2i+3}^{-1}\sigma_{2i+2}^{-1}\cdots\sigma_{2j-2}\sigma_{2j-3}\sigma_{2j-1}^{-1}\sigma_{2j-2}^{-1},$$

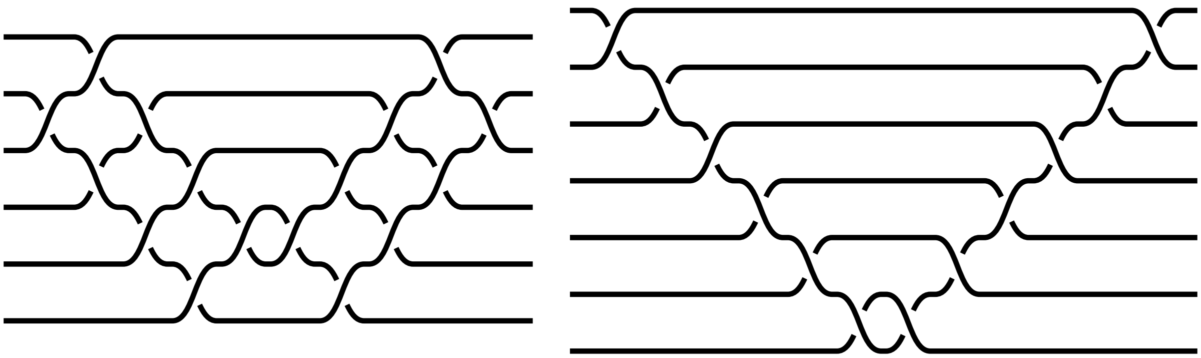

Figure 1: The braids corresponding to the operations $\tau_{1,3}$ and γ_1 in the case $n = 4$.

and for any index $i \in \{1, \ldots, n-1\}$, define γ_i to be the following sequence of Hurwitz moves:

$$\gamma_i \overset{\text{def}}{=\!=} \sigma_{2n-2}\sigma_{2n-3}^{-1}\sigma_{2n-4}\sigma_{2n-5}^{-1} \cdots \sigma_{2i+2}\sigma_{2i+1}^{-1} \circ \sigma_{2i}\sigma_{2i-1}\sigma_{2i-1}\sigma_{2i} \circ \\ \sigma_{2i+1}\sigma_{2i+2}^{-1} \cdots \sigma_{2n-5}\sigma_{2n-4}^{-1}\sigma_{2n-3}\sigma_{2n-2}^{-1}.$$

The associated braids are illustrated in Figure 1.

By an inductive computation, one shows that if f is a doubled path in $G(m, p, n)$ with pair weights $(a_1, \ldots, a_{n-1})$ and pair differences $(d_1, \ldots, d_{n-1})$, then $\tau_{i,j}(f)$ is a doubled path with the same pair weights and pair differences as f, except that the ith pair weight of $\tau_{i,j}(f)$ is $a_i + d_j$ and the jth pair weight of $\tau_{i,j}(f)$ is $a_j + d_i$. Similarly, if f includes a diagonal factor of weight d, then $\gamma_i(f)$ is again a doubled path, and $\gamma_i(f)$ has the same pair weights and pair differences as f, except that the i-th pair weight of $\gamma_i(f)$ is $a_i + d$. Finally, it is easy to check that (under the same hypotheses) $\sigma_{2i-1}(f)$ is a doubled path, with the same pair differences and pair weights as f, except that the i-th pair weight of $\sigma_{2i-1}(f)$ is $a_i + d_i$.

Consequently, if (3.1) holds, applying the operations $\sigma_{2i-1}^{m_{ii}}$ for $1 \leq i \leq n-1$, $\tau_{i,j}^{m_{ij}}$ for $1 \leq i < j \leq n-1$, and $\gamma_i^{m_{nj}}$ for $1 \leq j \leq n-1$, to f_1 in any order produces f_2, as needed. □

By a moderately complicated argument in elementary number theory, it is possible to rephrase the previous proposition as follows.

Corollary 3.11. *Follow the notation of Proposition 3.10 and define $r(B)$ as in Definition 3.1. If $a_i \equiv a'_i \pmod{r(B)}$ for $i = 1, \ldots, n-1$, then there exists a matrix M as in Proposition 3.10.*

By definition, every connected component $\Gamma_f|_B$ in the factorization graph of a standard form factorization f includes a doubled path of length $2|B|-1$ or $2|B|-2$. This allows us to restate the previous result in terms of standard form factorizations.

Lemma 3.12. *Suppose that f_1 and f_2 are two standard form factorizations of the element $g \in G(m, p, n)$ and that they induce the same cycle partition Π of g. Suppose furthermore that for each block B of Π, the pair weights $(a_1, \ldots, a_{|B|-1})$ and $(a'_1, \ldots, a'_{|B|-1})$ in the doubled paths in $f_1|_B$ and $f_2|_B$ satisfy $a_i \equiv a'_i \pmod{r(B)}$ for all $i \in \{1, \ldots, |B|-1\}$. Then f_1 and f_2 are Hurwitz-equivalent.*

Proof sketch. We use Kluitmann's theorem (Theorem 2.2) to choose a braid β_1 such that $\Gamma_{\beta_1(f_1)} = \Gamma_{f_2}$ without changing the factors in the doubled paths. For each connected component, Corollary 3.11 gives a second braid β_2 so that the weights of the doubled paths in $\beta_2\beta_1(f_1)|_B$ are the same as those in $f_2|_B$. The weights of the other factors in the component are determined by g. Applying this to each connected component gives a Hurwitz path from f_1 to f_2. □

3.3 When are two standard form factorizations not equivalent?

The main result of this section is Lemma 3.17, which is the converse of Lemma 3.12. As our main tool, we construct an invariant that distinguishes different Hurwitz orbits.

Definition 3.13. Given a factorization f of an element in $G(m, n, p)$, denote by G_f the subgroup of $G(m, n, p)$ generated by the factors in f.

It is easy to see that G_f is preserved by Hurwitz moves. Our first result is completely straightforward.

Proposition 3.14. *If f is a shortest factorization with induced cycle partition Π_f, we have that G_f is the direct product of the restrictions $G_{f|_B}$ to individual blocks B of Π_f.*

Thus, in what follows, it suffices to consider factorizations f for which the factorization graph Γ_f is connected. Also, by Lemma 3.7, it suffices to consider the case of factorizations in standard form.

Proposition 3.15. *Let f be a connected standard form factorization of an element $g \in G(m, p, n)$, and let d be the weight of g. Then $G_f \cong G\left(\frac{m}{r}, \frac{\gcd(m,d)}{r}, n\right)$. More concretely, there is a diagonal element $\delta \in G(m, 1, n)$ such that the conjugation map ϕ_δ defined by $\phi_\delta(g) = \delta g \delta^{-1}$ restricts to an isomorphism $\phi_\delta : G_f \to G\left(\frac{m}{r}, \frac{\gcd(m,d)}{r}, n\right)$.*

Proof sketch. First, consider the following set of factors of f: for each pair of factors in the doubled path with same underlying transposition, include the first corresponding edge, and also include all edges not in the doubled path. These edges form a spanning tree of Γ_f. By iteratively choosing entries of δ to fix one edge of the tree at a time, we can choose δ that simultaneously conjugates each of these edges to a transposition (i.e., to a reflection $[(i\,j); 0]$ of weight 0). We claim this is the desired element.

It is straightforward to check that, after conjugation, every weight in every factor in $\delta f \delta^{-1}$ is a multiple of r, or equivalently every root of unity appearing in their matrices is an $\frac{m}{r}$-th root of unity. Moreover, after conjugation, the diagonal factor will still have weight d. It follows immediately that $\phi_\delta(G_f) \subseteq G\left(\frac{m}{r}, \frac{\gcd(m,d)}{r}, n\right)$.

For the reverse inclusion, we can use the maps $\tau_{i,j}$ and γ_i from Section 3.2 to explicitly construct a set of reflections in $\phi_\delta(G_f)$ that generate $G\left(\frac{m}{r}, \frac{\gcd(m,d)}{r}, n\right)$. $\square$

Since conjugation by a diagonal element fixes diagonal elements, the following consequence is immediate.

Corollary 3.16. *If f_1 and f_2 are two connected standard form factorizations of the same element g, then the diagonal subgroups of G_{f_1} and G_{f_2} are equal (not just isomorphic), and are equal to the diagonal subgroup of $G\left(\frac{m}{r}, \frac{\gcd(m,d)}{r}, n\right)$.*

Lemma 3.17. *Suppose that f_1 and f_2 are two Hurwitz-equivalent standard form factorizations of the element $g \in G(m, p, n)$. Then $\Pi_{f_1} = \Pi_{f_2}$ and for every connected component B, the respective pair weights $(a_1, \ldots, a_{|B|-1})$ and $(a'_1, \ldots, a'_{|B|-1})$ of the doubled paths in $f_1|_B$ and $f_2|_B$ satisfy the condition that $a_i \equiv a'_i \pmod{r(B)}$ for every $i \in [|B|-1]$.*

Proof sketch. We prove the contrapositive: suppose there is a block B for which the pair weights in the two factorizations are not equivalent mod r. Then G_{f_1} contains the cycle $c_1 \stackrel{\text{def}}{=} [(v_1\, v_2); a_1] \cdots [(v_{|B|-1}\, v_{|B|}); a_{|B|-1}]$ (the product of half of the edges in the doubled path) and G_{f_2} contains the cycle $c_2 \stackrel{\text{def}}{=} [(v_1\, v_2); a'_1] \cdots [(v_{|B|-1}\, v_{|B|}); a'_{|B|-1}]$. By multiplying out, using the fact that $a_i \not\equiv a'_i \pmod{r}$ for some i, one sees that $c_1 c_2^{-1}$ is a diagonal element whose weights are not all multiples of r. Such an element does not belong to $G\left(\frac{m}{r}, 1, n\right)$ and so by Corollary 3.16 does not belong to either G_{f_1} or G_{f_2}. Consequently $G_{f_1} \neq G_{f_2}$, and so f_1 and f_2 are not Hurwitz-equivalent. $\square$

3.4 Proof of main results

Proof of Theorem 3.2. By Lemma 3.7, it suffices to study the standard form factorizations of g. Every standard form factorization f may be constructed by first choosing a maximum cycle partition Π, and then for every part $B \in \Pi$ choosing the pair weights $\{a_i\}_{i=1}^{|B|-1}$ of $f|_B$. By Lemmas 3.12 and 3.17, choosing an orbit amounts to independently choosing for each a_i one of the $r(B)$ equivalence classes modulo $r(B)$, and independently repeating this for each part B; this gives $\prod_{B \in \Pi} (r(B))^{|B|-1}$ orbits corresponding to the partition Π. Summing over all possible maximum cycle partitions $\Pi \in \mathrm{Par}_{\max}$ of g gives the total number of Hurwitz orbits. $\square$

Corollary 3.3 is an immediate consequence of Theorem 3.2.

Proof of Corollary 3.4. By Lemma 3.7, it suffices to show that the result is true for standard form factorizations. It follows from the proof of Lemma 3.17 that two standard form factorizations of g generate the same subgroup if and only if they induce the same partition Π and have the same pair weights modulo $r(B)$ for each block $B \in \Pi$. By Lemmas 3.12 and 3.17, this is equivalent to the condition that they generate the same subgroup of G. □

Remark 3.18. There is another natural invariant of the Hurwitz action, namely, the multiset of conjugacy classes (with respect to G, but also (stronger) with respect to the subgroup H that they generate) of the factors. Thus, a surprise consequence of Corollary 3.4 is that among the shortest factorizations of an element g in $G(m, p, n)$, those that generate the subgroup H all have the same multiset of H-conjugacy classes of the factors.

Remark 3.19. We are grateful to Theodosius Douvropoulos for the following example, which shows that Remark 3.18 does not hold in every complex reflection group. The exceptional group $G = G_{16}$ is generated by two reflections a, b subject to the relations $a^5 = b^5 = 1$, $aba = bab$. Both generators have non-unit eigenvalue $\exp(2\pi i/5)$. The element $g \overset{\text{def}}{=\!=} a^2b^3$ is not a reflection (e.g., because it has determinant 1) and so $f_1 = (a^2, b^3)$ is a shortest reflection factorization of g that generates G. It is slightly more work to check that $f_2 = (a^{-1}ba, b^{-2}a^{-1}b^2)$ is another shortest reflection factorization of g that also generates G. However, the factorizations f_1 and f_2 cannot lie in the same Hurwitz orbit because their factors have different conjugacy classes: the reflections in f_1 have determinants $\exp(4\pi i/5)$ and $\exp(6\pi i/5)$, while those in f_2 have determinants $\exp(2\pi i/5)$ and $\exp(8\pi i/5)$.

Remark 3.20. In [4], Berger showed that the three invariants

$$(\text{product } g, \text{generated subgroup } H, \text{multiset of } H\text{-conjugacy classes})$$

distinguish Hurwitz orbits of reflection factorizations of arbitrary lengths in the dihedral group $G(m, m, 2)$. That is, two tuples of reflections in a dihedral group belong to the same Hurwitz orbit if and only if they have the same product, generate the same subgroup, and have the same orbits under conjugacy by the subgroup.

The corresponding result can be shown in the symmetric group essentially by combining the work of Kluitmann [8] and Ben-Itzhak and Teicher [3]. The corresponding result has also been established in some of the small exceptional complex reflection groups (G_4, G_5, G_6, and G_7) in unpublished work of Minnick–Pirillo–Racile–Wang (private communication).

In [9, §5] it was conjectured that the corresponding statement is valid in *any* complex reflection group. Corollary 3.4 may be viewed as further evidence for this conjecture.

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