Séminaire Lotharingien de Combinatoire **85B** (2021)
Article #20, 13 pp.

Proceedings of the 33rd Conference on Formal Power Series and Algebraic Combinatorics (Ramat Gan)

On Combinatorial Models for Affine Crystals

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Abstract. We biject two combinatorial models for tensor products of (single-column) Kirillov–Reshetikhin crystals of any classical type $A-D$: the quantum alcove model and the tableau model. This allows for the translation of calculations in the former model (of the energy function, the combinatorial R-matrix, the key statistics relevant to Demazure crystals, etc.) to the latter, which is simpler.

Résumé. Nous établissons une bijection entre deux modèles combinatoires pour les produits tensoriels des cristaux de Kirillov–Reshetikhin (indexés par colonnes) de tout type classique $A-D$: le modèle des alcôves quantique et les tableaux. Cela nous permet de traduire des calculs du premier modèle (la fonction d'energie, la matrice R combinatoire, les clefs pour les cristaux de Demazure) en second, qui est plus simple.

Keywords: Kirillov–Reshetikhin crystal, Kashiwara–Nakashima column, alcove model.

1 Introduction

Kashiwara's *crystals* are colored directed graphs encoding the structure of certain bases, called *crystal bases*, for representations of quantum groups, as the quantum parameter goes to zero [2]. The first author and Postnikov realized the highest weight crystals of symmetrizable Kac–Moody algebras in terms of the so-called *alcove model* [9]. This is a combinatorial model whose objects (indexing the vertices of the crystal graph) are saturated chains in the Bruhat order of the corresponding Weyl group W. Later, the first author and Lubovsky introduced a more general model, called the *quantum alcove model* [6]; its objects are paths in the *quantum Bruhat graph* on W, denoted $\mathrm{QBG}(W)$. It was shown in [8] that the quantum alcove model uniformly describes (tensor products of) single-column *Kirillov–Reshetikhin (KR) crystals* for all the untwisted affine Lie algebras.

In classical types, there are (type-specific) models for KR crystals based on fillings of Young diagrams [2]. While they are simpler, they have less easily accessible information; so it is generally hard to use them in specific computations: of the energy function (which induces a grading on KR crystals), the combinatorial R-matrix (the unique affine crystal isomorphism interchanging tensor factors), the key statistics (relevant to Demazure

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crystals), etc. On the other hand, these computations were carried out in the quantum alcove model in [8, 7], respectively. Thus, our goal is to translate them to the tableau models, via an explicit bijection between the two models.

Such bijections were given in types A, C in [5, 6], and here we construct them in types B, D. The map from the quantum alcove model to the tableau model is a "forgetful map," while the inverse map is highly nontrivial in types B, D. We construct it by successively applying four algorithms; the first three (called "split," "extend," "reorder") operate on column fillings, while the last one, which is a greedy-type algorithm, constructs the needed paths in $\mathrm{QBG}(W)$. The last two algorithms are based on a subtle pattern avoidance concept ("blocked off" configurations), stemming from the complexity of $\mathrm{QBG}(W)$ in types B, D. The "extend" algorithm addresses another complexity in these types, related to the splitting of the KR (affine) crystals when viewed as classical crystals.

2 Background

2.1 Root systems

Let $\mathfrak{g}$ be a complex simple Lie algebra, and $\mathfrak{h}$ a Cartan subalgebra. Let $\Phi \subset \mathfrak{h}^*$ be the corresponding irreducible *root system*, $\mathfrak{h}^*_{\mathbb{R}}$ the real span of the roots, and $\Phi^+ \subset \Phi$ the set of positive roots. As usual, we denote $\rho := \frac{1}{2}(\sum_{\alpha\in\Phi^+} \alpha)$. Let $\alpha_i \in \Phi^+$ be the *simple roots*, for i in an indexing set I. We denote $\langle\cdot,\cdot\rangle$ the nondegenerate scalar product on $\mathfrak{h}^*_{\mathbb{R}}$ induced by the Killing form. Given a root α, we consider the corresponding *coroot* $\alpha^\vee$ and reflection s_α. The *weight lattice* Λ is generated by the *fundamental weights* ω_i, for $i \in I$, which form the dual basis to the simple coroots. Let Λ^+ be the set of *dominant weights*.

Let W be the corresponding *Weyl group*, whose Coxeter generators are denoted, as usual, by $s_i := s_{\alpha_i}$. The length function on W is denoted by $\ell(\cdot)$. The *Bruhat order* on W is defined by its covers $w \lessdot ws_\alpha$, for $\ell(ws_\alpha) = \ell(w) + 1$, where $\alpha \in \Phi^+$.

Given $\alpha \in \Phi$ and $k \in \mathbb{Z}$, we denote by $s_{\alpha,k}$ the reflection in the affine hyperplane $H_{\alpha,k} := \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} : \langle\lambda,\alpha^\vee\rangle = k\}$. These reflections generate the *affine Weyl group* W_{aff} for the *dual root system* $\Phi^\vee$. The hyperplanes $H_{\alpha,k}$ divide the vector space $\mathfrak{h}^*_{\mathbb{R}}$ into open regions, called *alcoves*. The *fundamental alcove* is denoted by $A_\circ$.

The *quantum Bruhat graph* $\mathrm{QBG}(W)$ on W is defined by adding downward edges, denoted $w \triangleleft ws_\alpha$, to the covers of the Bruhat order, i.e., its (labeled) edges are:

$$w \xrightarrow{\alpha} ws_\alpha \;\text{ if }\; w \lessdot ws_\alpha \;\text{ or }\; \ell(ws_\alpha) = \ell(w) - 2\langle\rho,\alpha^\vee\rangle + 1\,, \quad \text{where } \alpha \in \Phi^+\,.$$

2.2 Kirillov–Reshetikhin (KR) crystals

Given a simple or an affine Lie algebra $\mathfrak{g}$, a $\mathfrak{g}$-*crystal* is a nonempty set B along with maps $e_i, f_i : B \to B \cup \{\mathbf{0}\}$ for $i \in I$ (where $\mathbf{0} \notin B$ and I indexes the simple roots corresponding to $\mathfrak{g}$), and $wt : B \to \Lambda$. We require that $b' = f_i(b)$ if and only if $b = e_i(b')$. The maps e_i and f_i are called crystal operators, and are represented as arrows $b \to b' = f_i(b)$; thus they endow B with the structure of a colored directed graph. Given two $\mathfrak{g}$-crystals B_1 and B_2, their tensor product $B_1 \otimes B_2$ is defined on the Cartesian product of the two sets of vertices by a specific rule [2]. The *highest weight crystal* $B(\lambda)$, for $\lambda \in \Lambda^+$, is a certain crystal with a unique element v_λ such that $e_i(v_\lambda) = \mathbf{0}$, for all $i \in I$, and $wt(v_\lambda) = \lambda$. It encodes the structure of the crystal basis of the irreducible representation with highest weight λ of the quantum group $U_q(\mathfrak{g})$ as q goes to 0 [2].

A *Kirillov–Reshetikhin (KR) crystal* is a finite crystal $B^{r,s}$ for an affine algebra, associated to a rectangle of height r and length s [2]. We now describe the KR crystals $B^{r,1}$ for type $A_{n-1}^{(1)}$ (where $r \in \{1, 2, \ldots, n-1\}$), as well as for types $B_n^{(1)}$, $C_n^{(1)}$, and $D_n^{(1)}$ (where $r \in \{1, 2, \ldots, n\}$). As a classical crystal (i.e., with arrows f_0 removed), in types A_{n-1} and C_n, we have that $B^{r,1}$ is isomorphic to the corresponding highest weight crystal $B(\omega_r)$. By contrast, in types B_n and D_n, we have that $B^{r,1}$ becomes isomorphic to the disjoint union $B(\omega_r) \sqcup B(\omega_{r-2}) \sqcup B(\omega_{r-4}) \sqcup \ldots$.

In classical types, the fundamental crystal $B(\omega_k)$ is realized in terms of *Kashiwara–Nakashima (KN) columns* of height k [2]. These are fillings of the column with entries $\{1 < 2 < \ldots < n\}$ in type A_{n-1}, and entries $\{1 < \ldots < n < 0 < \overline{n} < \ldots < \overline{1}\}$ in types B_n, C_n, and D_n (see the exceptions below), such that the following conditions hold.

1. The entries are strictly increasing from the top to bottom with the exception that:
   (a) the letter 0 only appears in type B_n and can be repeated;
   (b) the letters n and $\overline{n}$ in type D_n are incomparable, and thus can alternate.
2. If both letters i and $\bar{\imath}$ appear in the column, while i is in the a-th box from the top and $\bar{\imath}$ is in the b-th box from the bottom, then $a + b \leq i$.

2.3 The quantum alcove model

Fix a dominant weight λ in a finite root system. The quantum alcove model depends on the choice of a sequence of positive roots $\Gamma := (\beta_1, \ldots, \beta_m)$, called a λ-chain [9]. This encodes a shortest sequence of pairwise adjacent alcoves connecting the fundamental alcove $A_\circ$ to $A_\circ - \lambda$; on another hand, the latter sequence corresponds to a reduced decomposition of the unique affine Weyl group element sending $A_\circ$ to $A_\circ - \lambda$. Any choice of such a reduced decomposition (or, equivalenly, of a mentioned alcove path) is allowed. Let $r_i := s_{\beta_i}$.

Definition 2.1 ([5]). *A subset $J = \{j_1 < j_2 < \ldots < j_s\} \subseteq [m] := \{1, \ldots, m\}$ (possibly empty) is an admissible subset (of* folding positions*) if we have the following path in* QBG(W)*:*

$$1 \xrightarrow{\beta_{j_1}} r_{j_1} \xrightarrow{\beta_{j_2}} r_{j_1} r_{j_2} \xrightarrow{\beta_{j_3}} \ldots \xrightarrow{\beta_{j_s}} r_{j_1} r_{j_2} \ldots r_{j_s}.$$

Let $\mathcal{A}(\lambda) = \mathcal{A}(\Gamma)$ be the collection of all admissible subsets (cf. Example 3.4).

Theorem 2.2 ([6, 7, 8]). *Let $p := (p_1, \ldots, p_r)$ be a composition and $\lambda := \omega_{p_1} + \ldots + \omega_{p_r}$. The set $\mathcal{A}(\lambda)$, properly endowed with the structure of an affine crystal, is a combinatorial model for the tensor product of KR crystals $B^p := B^{p_1,1} \otimes \ldots \otimes B^{p_r,1}$.*

3 The bijection between the two models in types A_{n-1}, C_n

3.1 The quantum alcove model and filling map in type A_{n-1}

We start with the basic facts about the root system for type A_{n-1}. We identify the space $\mathfrak{h}^*_{\mathbb{R}}$ with the quotient $V := \mathbb{R}^n / \mathbb{R}(1, \ldots, 1)$, where $\mathbb{R}(1, \ldots, 1)$ denotes the subspace spanned by $(1, \ldots, 1)$. Let $\varepsilon_1, \ldots, \varepsilon_n \in V$ be the images of the coordinate vectors in $\mathbb{R}^n$. The root system is $\Phi = \{\alpha_{ij} := \varepsilon_i - \varepsilon_j : i \neq j, 1 \leq i, j \leq n\}$. The simple roots are $\alpha_i = \alpha_{i,i+1}$, for $i = 1, \ldots, n-1$. The weight lattice is $\Lambda = \mathbb{Z}^n / \mathbb{Z}(1, \ldots, 1)$. The fundamental weights are $\omega_i = \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_i$, for $i = 1, \ldots, n-1$. A dominant weight $\lambda = \lambda_1 \varepsilon_1 + \ldots + \lambda_{n-1} \varepsilon_{n-1}$ is identified with the partition $(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq \lambda_n = 0)$ having at most $n-1$ parts. Note that $\rho = (n-1, n-2, \ldots, 0)$. Considering the Young diagram of the dominant weight λ as a concatenation of columns, whose heights are $\lambda'_1, \lambda'_2, \ldots$, corresponds to expressing λ as $\omega_{\lambda'_1} + \omega_{\lambda'_2} + \ldots$ (λ' is the conjugate of λ).

The Weyl group W is the symmetric group S_n, which acts on V by permuting the coordinates $\varepsilon_1, \ldots \varepsilon_n$. Permutations $w \in S_n$ are written in one-line notation $w = w(1) \ldots w(n)$. For simplicity, we use the same notation (i, j), with $1 \leq i < j \leq n$, for the positive root α_{ij} and the reflection $s_{\alpha_{ij}}$, which is the transposition t_{ij} of i and j.

We now consider the specialization of the quantum alcove model to type A_{n-1}. For any $k = 1, \ldots, n-1$, we have the following ω_k-chain, denoted by $\Gamma(k)$ [5]:

$$\begin{array}{llll} (\,(k, k+1), & (k, k+2), & \ldots, & (k, n)\,, \\ & & \ldots & \\ (2, k+1), & (2, k+2), & \ldots, & (2, n)\,, \\ (1, k+1), & (1, k+2), & \ldots, & (1, n)\ )\,. \end{array}$$

We construct a λ-chain $\Gamma = (\beta_1, \beta_2, \ldots, \beta_m)$ as the concatenation $\Gamma := \Gamma_1 \ldots \Gamma_{\lambda_1}$, where $\Gamma_i := \Gamma(\lambda'_i)$. Let $J = \{j_1 < \ldots < j_s\}$ be a set of folding positions in Γ, not necessarily admissible, and let T be the corresponding list of roots of Γ; we will use J and T interchangeably. The factorization of Γ induces a factorization on T as $T = T_1 T_2 \ldots T_{\lambda_1}$.

We denote by $T_1 \dots T_i$ the permutation obtained by multiplying the transpositions in $T_1, \dots, T_i$ considered from left to right. For $w \in W$, written $w = w_1 w_2 \dots w_n$, let $w[i,j] = w_i \dots w_j$. To each J we can associate a filling of a Young diagram λ, as follows.

Definition 3.1. *Let $\pi_i = \pi_i(T) := T_1 \dots T_i$. We define the filling map, which produces a filling of the Young diagram λ, by fill_A(J) = fill_A$(T) := C_1 \dots C_{\lambda_1}$, where $C_i := \pi_i[1, \lambda'_i]$. We define the sorted filling map "sfill_A" to be the composition "sort $\circ$ fill_A," where "sort" reorders increasingly each column of a filling.*

Definition 3.2. *Define a circular order $\prec_i$ on $[n] := \{1, \dots, n\}$ starting at i, by*

$$i \prec_i i+1 \prec_i \dots \prec_i n \prec_i 1 \prec_i \dots \prec_i i-1.$$

It is convenient to think of this order in terms of the numbers $1, \dots, n$ arranged on a circle clockwise. We make the convention that, whenever we write $a \prec b \prec c \prec \dots$, we refer to the circular order $\prec = \prec_a$. Below is a criterion for QBG(W) in type A_{n-1}.

Proposition 3.3 ([5]). *For $1 \le i < j \le n$, we have an edge $w \xrightarrow{(i,j)} w \cdot (i,j)$ in* QBG(W) *if and only if there is no k such that $i < k < j$ and $w(i) \prec w(k) \prec w(j)$.*

Example 3.4. Consider the dominant weight $\lambda = 3\varepsilon_1 + 2\varepsilon_2 = \omega_1 + 2\omega_2$ in the root system A_2, which corresponds to the Young diagram $\begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline \; & \; \\ \cline{1-2}\end{array}$. The corresponding λ-chain is

$$\Gamma = \Gamma_1 \Gamma_2 \Gamma_3 = \Gamma(2)\Gamma(2)\Gamma(1) = \{\underline{(2,3)}, \underline{(1,3)} | \underline{(2,3)}, (1,3) | \underline{(1,2)}, (1,3)\}.$$

Consider $J = \{1, 2, 3, 5\}$, cf. the underlined roots, with $T = \{(2,3), (1,3) | (2,3) | (1,2)\}$.

We write the permutations in Definition 2.1 as broken columns. Note that J is admissible since, based on Proposition 3.3 and the Bruhat cover notation from Section 2.1, we have

$$\begin{array}{ccccccccccccc}
\begin{array}{|c|}\hline 1 \\ \hline \mathbf{2} \\ \hline \end{array} & & \begin{array}{|c|}\hline \mathbf{1} \\ \hline 3 \\ \hline \end{array} & & \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array} & & \begin{array}{|c|}\hline 2 \\ \hline \mathbf{3} \\ \hline \end{array} & & \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline \end{array} & & \begin{array}{|c|}\hline \mathbf{2} \\ \hline \end{array} & & \begin{array}{|c|}\hline 1 \\ \hline \end{array} \\
 & \lessdot & & \lessdot & & | & & \lhd & & | & & \lhd & & | \, ; \\
\begin{array}{|c|}\hline \mathbf{3} \\ \hline \end{array} & & \begin{array}{|c|}\hline \mathbf{2} \\ \hline \end{array} & & \begin{array}{|c|}\hline 1 \\ \hline \end{array} & & \begin{array}{|c|}\hline \mathbf{1} \\ \hline \end{array} & & \begin{array}{|c|}\hline 3 \\ \hline \end{array} & & \begin{array}{|c|}\hline \mathbf{1} \\ \hline 3 \\ \hline \end{array} & & \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array}
\end{array}$$

the bold entries are those swapped by the underlined transpositions. By considering the top part of the last permutation in each segment, and by concatenating them left to right, we obtain *fill_A*$(J) = \begin{array}{|c|c|c|}\hline 2 & 2 & 1 \\ \hline 3 & 1 \\ \cline{1-2}\end{array}$ and *sfill_A*$(J) = \begin{array}{|c|c|c|}\hline 2 & 1 & 1 \\ \hline 3 & 2 \\ \cline{1-2}\end{array}$.

Theorem 3.5 ([5, 6]). *The map "sfill_A" is an affine crystal isomorphism between $\mathcal{A}(\lambda)$ (recall Definition 2.1) and $B^{\lambda'} := B^{\lambda'_1, 1} \otimes B^{\lambda'_2, 1} \otimes \dots$.*

The proof of bijectivity is given in [5] by exhibiting an inverse map. We will now present the algorithm for constructing this map, as the corresponding construction in the other classical types has this algorithm as a starting point.

3.2 The inverse map in type A_{n-1}

Consider $B^{\lambda'} := B^{\lambda'_1,1} \otimes B^{\lambda'_2,1} \otimes \ldots = B(\omega_{\lambda'_1}) \otimes B(\omega_{\lambda'_2}) \otimes \ldots$. This is simply the set of column-strict fillings of the Young diagram λ with integers in $[n]$. Fix a filling b in $B^{\lambda'}$ written as a concatenation of columns $b_1 \ldots b_{\lambda_1}$.

The algorithm for mapping b to a sequence of roots $S \subset \Gamma$ consists of two sub-algorithms, which we call the *Reorder algorithm* (this reverses the ordering in the column b_i back to that of the corresponding column in *fill_A*(S)), and the *Path algorithm* (this provides the corresponding path in the quantum Bruhat graph). The Reorder algorithm (Algorithm 3.6) takes b as input and outputs a filling *ord_A*$(b) = C$, a reordering of the column entries, based on the circle order given in Definition 3.2.

Algorithm 3.6. *("ord_A")*
let $C_1 := b_1$*;*
for i *from* 2 *to* λ_1 *do*
for j *from* 1 *to* λ'_i *do*
let $C_i(j) := min_{\prec_{C_{i-1}(j)}}(b_i \setminus \{C_i(1), \ldots, C_i(j-1)\})$
end do;
end do;
return $C := C_1 \ldots C_{\lambda_1}$.

Example 3.7. Algorithm 3.6 gives the filling C from b below.

$$b = \begin{array}{|c|c|c|c|}\hline 3&2&1&2\\\hline 5&3&2\\\cline{1-3} 6&4&4\\\cline{1-3}\end{array} \xrightarrow{ord\_A} \begin{array}{|c|c|c|c|}\hline 3&3&4&2\\\hline 5&2&2\\\cline{1-3} 6&4&1\\\cline{1-3}\end{array} = C$$

The Path algorithm (Algorithm 3.8) inputs the reordered filling C and outputs a sequence of roots *path_A*$(C) = S \subset \Gamma$. Let C_0 be the increasing column $(1, 2, \ldots, n)$.

Algorithm 3.8. *("path_A")*
for i *from* 1 *to* λ_1 *do*
let $S_i := \emptyset$, $A := C_{i-1}$*;*
for (l, m) *in* Γ_i *do*
if $A(l) \neq C_i(l)$ *and* $A(l) \prec A(m) \prec C_i(l)$ *then let* $S_i := S_i, (l, m)$ *and* $A := A(l, m)$*;*
end if;
end do;
end do;
return $S := S_1 \ldots S_{\lambda_1}$.

Example 3.9. In Example 3.4, we recover J from *sfill_A*(J) via "*path_A* $\circ$ *ord_A*."

Theorem 3.10 ([5]). *If fill_A*$(T) = C$*, then the output of the Path algorithm* $C \mapsto S$ *is such that* $S = T$*. Moreover, the map "path_A* $\circ$ *ord_A" is the inverse of "sfill_A".*

3.3 The quantum alcove model and filling map in type C_n

The filling map naturally extends to all classical types, however the corresponding inverse maps become more and more complex as we progress beyond type A_{n-1}. This fact is a direct consequence of the differences between the structures of the corresponding KN columns and quantum Bruhat graphs.

Recall from the construction of the filling map in type A_{n-1} that we treated the columns of a filling as initial segments of permutations. However, the KN columns of type C_n allow for both i and $\bar{\imath}$ to appear as entries in such a column. In order to pursue the analogy with type A_{n-1}, we need to replace a KN column with its *split* version, i.e., two columns of the same height as the initial column. The splitting procedure, described in [3], gives an equivalent definition of KN columns, cf. Section 2.2.

Given our fixed dominant weight λ, an element b of $B^{\lambda'}$ can be viewed as a concatenation of KN columns $b_1 \dots b_{\lambda_1}$, with b_i of height λ'_i. Let $b' := b^l_1 b^r_1 \dots b^l_{\lambda_1} b^r_{\lambda_1}$ be the associated filling of the shape 2λ, where $(b^l_i, b^r_i) := (lb_i, rb_i) = split\_C(b_i)$ is the splitting of the KN column b_i.

The algorithm for mapping b' to a sequence of roots $S \subset \Gamma$ is similar to the type A_{n-1} one. The Reorder algorithm "*ord_C*" for type C_n is the obvious extension from type A_{n-1}. The Path algorithm "*path_C*" is also similar to its type A_{n-1} counterpart.

Theorem 3.11 ([5, 6])**.** *The map "path_C ∘ ord_C ∘ split_C" is the inverse of the sorted filling map "sfill_C" in type C_n, which is an affine crystal isomorphism between $\mathcal{A}(\lambda)$ and $B^{\lambda'}$.*

4 The bijection in types B_n and D_n

We now move to the main content of this paper: extending the work done in types A_{n-1} and C_n to both types B_n and D_n, by addressing the complexities in these types.

4.1 The quantum alcove model and filling map in type B_n

We start with the basic facts about the root system of type B_n. We can identify the space $\mathfrak{h}^*_{\mathbb{R}}$ with $V := \mathbb{R}^n$, with coordinate vectors $\varepsilon_1, \dots, \varepsilon_n \in V$. The root system is $\Phi = \{\pm\varepsilon_i \pm \varepsilon_j : i \neq j, 1 \leq i < j \leq n\} \cup \{\pm\varepsilon_i : 1 \leq i \leq n\}$. The Weyl group W is the group of signed permutations, which acts on V by permuting the coordinates and changing their signs. A signed permutation is a bijection w from $[\overline{n}] := \{1 < 2 < \dots < n < \overline{n} < \overline{n-1} < \dots < \overline{1}\}$ to $[\overline{n}]$ which satisfies $w(\bar{\imath}) = \overline{w(i)}$. Here, $\bar{\imath}$ is viewed as $-i$, so that $\bar{\bar{\imath}} = i$, and we can define $|i|$ and $sign(i) \in \{\pm 1\}$, for $i \in [\overline{n}]$. We will use the so-called *window notation* $w = w(1)w(2)\dots w(n)$. For simplicity, given $1 \leq i < j \leq n$, we denote by (i, j) and $(i, \bar{\jmath})$ the positive roots $\varepsilon_i - \varepsilon_j$ and $\varepsilon_i + \varepsilon_j$, respectively; the corresponding reflections, denoted in the same way, are identified with the composition of transpositions $t_{ij}t_{\bar{\jmath}\bar{\imath}}$ and $t_{i\bar{\jmath}}t_{j\bar{\imath}}$, respectively. Finally, we denote by $(i, \bar{\imath})$ the root ε_i and the corresponding reflection, identified with the transposition $t_{i\bar{\imath}}$.

We now consider the specialization of the quantum alcove model to type B_n. For any $k = 1, \ldots, n$, we have the following (split) ω_k-chain, denoted by $\Gamma^l(k)\Gamma^r(k)$ [5], where:

$$\Gamma^l(k) := \Gamma^{kk} \ldots \Gamma^{k1}, \quad \Gamma^r(k) := \Gamma^k \ldots \Gamma^2 , \tag{4.1}$$

$$\begin{aligned}\Gamma^{ki} := (\, & (i, k+1), \quad (i, k+2), \quad \ldots, \quad (i, n) \,, \\ & (i, \bar{\imath}) \,, \\ & (i, \bar{n}), \qquad (i, \overline{n-1}), \quad \ldots, \quad (i, \overline{k+1}) \,, \\ & (i, \overline{i-1}), \quad (i, \overline{i-2}), \quad \ldots, \quad (i, \bar{1}) \,) \,,\end{aligned}$$

$$\Gamma^i := ((i, \overline{i-1}), (i, \overline{i-2}), \ldots, (i, \bar{1})) \,.$$

We refer to the four rows above in Γ^{ki} as stages *I*, *II*, *III*, and *IV* respectively. We can construct a λ-chain as a concatenation $\Gamma := \Gamma_1^l \Gamma_1^r \ldots \Gamma_{\lambda_1}^l \Gamma_{\lambda_1}^r$, where $\Gamma_i^l := \Gamma^l(\lambda_i')$ and $\Gamma_i^r := \Gamma^r(\lambda_i')$. We will use interchangeably the set of positions J in the λ-chain Γ and the sequence of roots T in Γ in those positions, which we call a *folding sequence*. The factorization of Γ with factors Γ_i^l, Γ_i^r induces a factorization of T with factors T_i^l, T_i^r. We define the circle order $\prec_a$ in a similar way to Definition 3.2, but on the set $[\bar{n}]$. Below is a criterion for $\mathrm{QBG}(W)$ in type B_n, analogous to Proposition 3.3.

Proposition 4.1 ([1]). *The quantum Bruhat graph of type B_n has the following edges.*

1. *Given $1 \le i < j \le n$, we have an edge $w \xrightarrow{(i,j)} w \cdot (i, j)$ if and only if there is no k such that $i < k < j$ and $w(i) \prec w(k) \prec w(j)$.*

2. *Given $1 \le i < j \le n$, we have an edge $w \xrightarrow{(i,\bar{\jmath})} w \cdot (i, \bar{\jmath})$ if and only if one of the following conditions holds:*

   (a) $w(i) < w(\bar{\jmath})$, $sign(w(i)) = sign(w(\bar{\jmath}))$, and there is no k such that $i < k < \bar{\jmath}$ and $w(i) < w(k) < w(\bar{\jmath})$;

   (b) $sign(w(i)) = -1$, $sign(w(\bar{\jmath})) = 1$, and there is no k such that $i < k \neq j < \bar{\jmath}$ and $w(i) \prec w(k) \prec w(\bar{\jmath})$.

3. *Given $1 \le i \le n$, we have an edge $w \xrightarrow{(i,\bar{\imath})} w \cdot (i, \bar{\imath})$ if and only if:*

   (a) $w(i) < w(\bar{\imath})$ and there is no k such that $i < k < \bar{\imath}$ and $w(i) \prec w(k) \prec w(\bar{\imath})$;

   (b) or $w(\bar{\imath}) < w(i)$ and $i = n$.

Definition 4.2. *Given a folding sequence T, we consider the signed permutations $\pi_i^l := T_1^l T_1^r \ldots T_{i-1}^l T_{i-1}^r T_i^l$, $\pi_i^r := \pi_i^l T_i^r$ (cf. the notation in Section 3.1). Then the filling map is* *the map "fill_B" from folding sequences T in $\mathcal{A}(\lambda)$ to fillings $fill\_B(T) = C_1^l C_1^r \dots C_{\lambda_1}^l C_{\lambda_1}^r$ of the shape 2λ, which are viewed as concatenations of columns; here $C_i^l := \pi_i^l[1, \lambda_i']$ and $C_i^r := \pi_i^r[1, \lambda_i']$, for $i = 1, \dots, \lambda_1$. We then define sfill_B : $\mathcal{A}(\lambda) \to B^{\lambda'}$ to be the composition "sort $\circ$ fill_B", where "sort" reorders the entries of each column increasingly; here we represent a KR crystal $B^{r,1}$ as a* split *(also known as* doubled*) KN column of height r, see Section 4.2.*

4.2 The type B_n inverse map

Recall from Section 2.2 that $B^{k,1}$, as a classical type crystal, is isomorphic to the crystal $B(\omega_k) \sqcup B(\omega_{k-2}) \sqcup B(\omega_{k-4}) \sqcup \dots$ where, as before, the elements of the set $B(\omega_r)$ are given by type B KN columns of height r. This presents the following two issues.

1. As in type C_n, the KN columns in type B_n are allowed to contain both i and $\bar{\imath}$ values; in addition, they may contain the value 0. There is a type B_n splitting algorithm "*split_B;*" see [4] and Definition 4.3.

2. $B^{k,1}$ contains columns of height less than k, so we need to extend them to full height k, such that the reflections corresponding to $\Gamma^l(k)\Gamma^r(k)$ in (4.1) may be correctly applied. The respective algorithm "*extend_B*" is given in Definition 4.4.

Definition 4.3 ([4])**.** *Let C be a column and $I = \{z_1 > \dots > z_r\}$ be the set letters 0 in C and the unbarred letters z such that the pair $(z, \overline{z})$ occurs in C. The column C can be split when there exists a set of r unbarred letters $J = \{t_1 > \dots > t_r\} \subset [n]$ such that t_1 is the greatest letter in $[n]$ satisfying: $t_1 < z_1, t_1 \notin C$, and $\overline{t_1} \notin C$, and for $i = 2, \dots, r$, the letter t_i is the greatest value in $[n]$ satisfying $t_i < min(t_{i-1}, z_i), t_i \notin C$, and $\overline{t_i} \notin C$. In this case we write:*

1. *rC for the column obtained by changing $\overline{z_i}$ into $\overline{t_i}$ in C for each letter $z_i \in I$, and by reordering if necessary,*

2. *lC for the column obtained by changing z_i into t_i in C for each letter $z_i \in I$, and by reordering if necessary.*

The pair (lC, rC) is then called a split *(or* doubled*) column.*

Definition 4.4 ([1])**.** *Given a split column (lC, rC) of length $1 \le r < n$ and $r \le k < n$, append $\{\bar{\imath}_1 < \dots < \bar{\imath}_{r-k}\}$ to lC and $\{i_1 < \dots < i_{r-k}\}$ to rC, where i_1 is the minimal value in $[\overline{n}]$ such that $i_1, \bar{\imath}_1 \notin lC, rC$, and i_t for $2 \le t \le r-k$ is minimum value in $[\overline{n}]$ such that $i_t, \bar{\imath}_t \notin lC, rC$ and $i_t > i_{t-1}$. Sort the extended columns increasingly. Let $(\widehat{lC}, \widehat{rC})$ be the extended split column.*

Example 4.5. *A type B_8 KN column with its split and extended height 6 columns.*

$$C = \begin{array}{|c|}\hline 5 \\ \hline 0 \\ \hline \overline{8} \\ \hline \overline{5} \\ \hline \end{array} \quad (lC, rC) = \begin{array}{|c|c|}\hline 4 & 5 \\ \hline 7 & \overline{8} \\ \hline \overline{8} & \overline{7} \\ \hline \overline{5} & \overline{4} \\ \hline \end{array} \quad (\widehat{lC}, \widehat{rC}) = \begin{array}{|c|c|}\hline 4 & 1 \\ \hline 7 & 2 \\ \hline \overline{8} & 5 \\ \hline \overline{5} & \overline{8} \\ \hline \overline{2} & \overline{7} \\ \hline \overline{1} & \overline{4} \\ \hline \end{array}$$

There are additional complexity issues due to the structure of $\mathrm{QBG}(W)$ in type B_n. The subtle differences in the quantum Bruhat criteria for types C_n and B_n make the natural extensions of the reorder rule and Path algorithm fail in type B_n. We enhance these algorithms, based on the following pattern avoidance in two adjacent columns.

Definition 4.6. *We say that columns $C = (l_1, l_2, ..., l_k)$ and $C' = (r_1, r_2, ..., r_k)$ are* blocked off at i by $b = r_i$ *if and only if the following hold:*

1. $|l_i| \leq b < n$, *where* $|l_i| = b$, *if and only if* $l_i = \overline{b}$;
2. $\{1, 2, ..., b\} \subset \{|l_1|, |l_2|, ..., |l_i|\}$ *and* $\{1, 2, ..., b\} \subset \{|r_1|, |r_2|, ..., |r_i|\}$;
3. $|\{j : 1 \leq j \leq i, l_j < 0, r_j > 0\}|$ *is odd.*

Proposition 4.7. *If columns C and C' are blocked off at i by b, then there is no subsequence of the respective part of Γ producing a path between C and C' in* $\mathrm{QBG}(W)$.

We now define the enhanced versions of the *Reorder* and *Path* algorithms. Let $b := b_1^l b_1^r \dots b_{\lambda_1}^l b_{\lambda_1}^r = b_1 \dots b_{2\lambda_1}$ be extended split columns indexing a vertex of the crystal $B^{\lambda'}$ of type B_n. Similarly, let $\Gamma := \Gamma_1^l \Gamma_1^r \dots \Gamma_{\lambda_1}^l \Gamma_{\lambda_1}^r = \Gamma_1 \dots \Gamma_{2\lambda_1}$.

Algorithm 4.8. *("ord_B")*
let $C_1 := b_1$;
for i *from* 2 *to* $2\lambda_1$ *do*
for j *from* 1 *to* $\lambda_i' - 1$ *do*
let $C_i(j) := min_{\prec_{C_{i-1}(j)}}(b_i \setminus \{C_i(1), \dots, C_i(j-1)\}$ *so that* C_{i-1}, C_i *not blocked off at* j*)*
end do;
let $C_i(\lambda_i') := min_{\prec_{C_{i-1}(j)}}(b_i \setminus \{C_i(1), \dots, C_i(\lambda_i' - 1)\}$
end do;
return $C := C_1 \dots C_{2\lambda_1} = C_1^l C_1^r \dots C_{\lambda_1}^l C_{\lambda_1}^r$.

Example 4.9. Algorithm 4.8 gives the filling C from b below. Note that Algorithm 3.6 would have paired the 3 with the $\overline{3}$ in the 4^{th} row. However, this would cause the two columns to be blocked off at 4 by 3, so the type B algorithm skips to the next value and pairs the 8 with the $\overline{3}$ instead.

$$b = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 4 & 3 \\ \hline \overline{2} & 5 \\ \hline \overline{3} & 8 \\ \hline 5 & \overline{2} \\ \hline \end{array} \xrightarrow{ord\_B} \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 4 & 5 \\ \hline \overline{2} & \overline{2} \\ \hline \overline{3} & 8 \\ \hline 5 & 3 \\ \hline \end{array} = C$$

The algorithm "*path_B*" (Algorithm 4.10) takes the type B reordered, extended, split filling $C = C_1 \dots C_{2\lambda_1}$ given by Algorithm 4.8, and outputs a sequence of roots $path\_B(C) = S \subset \Gamma$. We let C_0 be the increasing column $(1, 2, \dots, n)$.

Algorithm 4.10. *("path_B")*
for i from 1 *to* $2\lambda_1$ *do*
let $S_i := \emptyset$, $A := C_{i-1}$;
for (l, m) *in* Γ_i *do*
if $(l, m) = (i, i+1)$ *and* A, C_i *are blocked off at* i *by* $C_i(i)$, *then let* $S_i := S_i, (i, i+1)$, $A := A(i, i+1)$;
elsif $A(l) \neq C_i(l)$ *and* $A(l) \prec A(m) \prec C_i(l)$ *and* $A(l, m), C_i$ *not blocked off at* l *by* $C_i(l)$, *then let* $S_i := S_i, (l, m)$, $A := A(l, m)$;
end if;
end do;
end do;
return $S := S_1 \dots S_{2\lambda_1} = S_1^l S_1^r \dots S_{\lambda_1}^l S_{\lambda_1}^r$.

Example 4.11. Consider the crystal $B^{(2,2)}$ of type B_3. Then $\lambda' = \lambda = (2,2)$ and $\Gamma = \Gamma(2)\Gamma(2)$. Suppose that we have $\widehat{rC_1} = \begin{array}{|c|}\hline \overline{3} \\ \hline \overline{2} \\ \hline 1 \\ \hline \end{array}$ and $\widehat{lC_2} = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 2 \\ \hline \end{array}$. Algorithm 4.10 produces the following subset of $\Gamma^l(2) = \{(2,3), (2,\overline{2}), (2,\overline{3}), (2,\overline{1}), (1,3), (1,\overline{1}), (1,\overline{3})\}$:

$$\begin{array}{c}\begin{array}{|c|}\hline \overline{3} \\ \hline \overline{2} \\ \hline \end{array} \\ \\ \begin{array}{|c|}\hline 1 \\ \hline \end{array}\end{array} \xrightarrow{(2,3)} \begin{array}{c}\begin{array}{|c|}\hline \overline{3} \\ \hline 1 \\ \hline \end{array} \\ \\ \begin{array}{|c|}\hline \overline{2} \\ \hline \end{array}\end{array} \xrightarrow{(2,\overline{3})} \begin{array}{c}\begin{array}{|c|}\hline \overline{3} \\ \hline 2 \\ \hline \end{array} \\ \\ \begin{array}{|c|}\hline \overline{1} \\ \hline \end{array}\end{array} \xrightarrow{(2,\overline{1})} \begin{array}{c}\begin{array}{|c|}\hline \overline{2} \\ \hline 3 \\ \hline \end{array} \\ \\ \begin{array}{|c|}\hline \overline{1} \\ \hline \end{array}\end{array} \xrightarrow{(1,\overline{3})} \begin{array}{c}\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array} \\ \\ \begin{array}{|c|}\hline 2 \\ \hline \end{array}\end{array} .$$

Notice that Algorithm 3.8 would have called for the use of $(1,3)$ instead of $(1,\overline{3})$. This would have caused the resulting word to be blocked off with $\widehat{lC_2}$ at 1 by 1, and we can see that the original Path algorithm would not terminate correctly.

Theorem 4.12. *The map "path_B ∘ ord_B ∘ extend_B ∘ split_B" is the inverse of the type* B_n *map "sfill_B."*

4.3 The type D_n bijection

We briefly outline the major differences in the type D_n constructions. First, since KN columns of type D_n have no relation in the ordering of n and $\overline{n}$, the type D_n splitting algorithm "*split_D*" begins by converting all $(n, \overline{n})$ pairs in a given column to 0 values, and then it continues as in type B_n [4]. There is still need for the extending algorithm, and we use the same one as in type B_n, but we call it "*extend_D*". The quantum Bruhat graph criterion in type D_n differs from type B_n in that we no longer have any arrows of the form $(i, \bar{\imath})$, but in return we have less restriction concerning arrows of the form $(i, \bar{\jmath})$. This change requires further modifications to the Path and Reordering algorithms, based on the following "*type D_n blocked off*" condition.

Definition 4.13. *The columns $C = (l_1, l_2, ..., l_k)$ and $C' = (r_1, r_2, ..., r_k)$ are type D_n blocked off at i by $b = r_i$ if and only if C and C' are blocked off at i by $b = r_i$, or the following hold:*

1. *$-|l_i| \leq b < 0$, where $-|l_i| = b$ if and only if $l_i = \overline{b}$;*
2. *$\{b, b+1, \ldots, n\} \subset \{|l_1|, |l_2|, ..., |l_i|\}$ and $\{b, b+1, \ldots, n\} \subset \{|r_1|, |r_2|, ..., |r_i|\}$;*
3. *and $|\{j : 1 \geq j \geq i, l_j > 0, r_j < 0\}|$ is odd.*

We then define "path_D" and "ord_D" to be as in type B_n, but by replacing "blocked off" with "type D_n blocked off".

Theorem 4.14. *The map "path_D $\circ$ ord_D $\circ$ extend_D $\circ$ split_D" is the inverse of the type D_n map "sfill_D."*

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