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Refined dual Grothendieck polynomials, integrability, and the Schur measure

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Abstract. We prove a Jacobi–Trudi formula, a Littlewood identity, a Cauchy identity, and symmetries for refined dual Grothendieck polynomials by using the Lindström–Gessel–Viennot lemma and an interpretation as integrable vertex models. We give an alternative definition of refined dual Grothendieck polynomials from the last passage percolation model. We then prove a Jacobi–Trudi formula for skew shapes and give a new proof of a relation with the Schur measure due to Baik and Rains.

Keywords: dual Grothendieck polynomial, vertex model, integrable system

1 Introduction

(Symmetric) Grothendieck polynomials were first introduced by Lascoux and Schützenberger [12] as polynomial representatives corresponding to Schubert varieties in the (connective) K-theory ring of the Grassmannian, the set of k-dimensional planes in $\mathbb{C}^n$. An integrable systems interpretation was given by Motegi and Sakai, where they were linked with the Totally Asymmetric Simple Exclusion Process (TASEP) in [13].

Dual Grothendieck polynomials were introduced as the dual basis to Grothendieck polynomials under the Hall inner product by Lam and Pylyavskyy in [9], where a combinatorial interpretation was given. Furthermore, their decomposition in terms of Schur functions was described using elegant tableaux via an RSK-like process. Recently, Yeliussizov connected dual Grothendieck polynomials to TASEP and the corresponding random matrix process called last passage percolation (LPP) [17, 18]. A refined version of dual Grothendieck polynomials $g_\lambda(\mathbf{x};\mathbf{t})$ were introduced by Galashin, Grinberg, and Liu [4], which can be seen as encoding the usual weight on the elegant tableaux.

In this extended abstract, we combine the combinatorial, integrable systems, and probabilistic approaches to obtain both new results and new proofs of recent results.

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In Section 3 we first translate the elegant tableau decomposition into nonintersecting lattice paths (NILPs), which allow us to describe $g_\lambda(\mathbf{x};\mathbf{t})$ as a multi-Schur function [10] by refining [11] and using [3]. We use this interpretation with the Lindström–Gessel–Viennot (LGV) lemma to obtain refined versions of formulas from [17, 18]. Using the integrable systems interpretation, we prove new symmetries not previously noticed.

In Section 4, we use Yeliussizov's bijection from [18] with the Robinson–Schensted–Knuth (RSK) bijection to relate $g_\lambda(\mathbf{x};\mathbf{t})$ with the LPP model, deconstructing the map from [9]. Next, we refine the proofs by Johansson [7] to prove a (dual) Jacobi–Trudi formula for $g_{\lambda/\mu}(\mathbf{x};\mathbf{t})$ conjectured by Grinberg. This was done independently by Kim [8] and previously the dual version by Amanov and Yeliussizov [1] and the $\mathbf{t} = \beta$ version by Iwao [5], all using different techniques. We then further refine Johansson's method to relate $g_\lambda(\mathbf{x};\mathbf{t})$ with the Schur measure [14], recovering his results [6] when $\mathbf{x} = \mathbf{t} = q^{1/2}$ and the general case by Baik and Rains [2].

2 Background and combinatorics

Let $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n)$, and $\mathbf{t} = (t_1, \dots, t_n)$ be (possibly infinite) sequences of commuting indeterminates. Let $\mathbf{x} \sqcup \mathbf{y} := (x_1, \dots, x_n, y_1, \dots, y_n)$ denote the concatenation. For $\lambda = (\lambda_1, \dots, \lambda_n)$, define $\mathbf{x}^\lambda := x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$. Let $e_\lambda(\mathbf{x})$ and $h_\lambda(\mathbf{x})$ denote the elementary and homogeneous symmetric functions.

We consider partitions λ inside an $n \times k$ box drawn in English convention. The *01-sequence* of λ is a 1 for north steps and 0 for east steps read from the lower-left corner. Let $\lambda^\dagger$ denote the complement of λ in the $n \times k$ box. The dual partition $\lambda^\vee$ interchanges $0 \leftrightarrow 1$ in the 01-sequence. The conjugate partition λ' is reflecting over the $y = -x$ line.

A *reverse plane partition (RPP)* is a filling of λ with positive integers that weakly increase across rows and down columns. If the columns are strictly increasing, then it is a *semistanard Young tableau (SSYT)*. Let $\mathrm{RPP}^n(\lambda)$ (resp. $\mathrm{SSYT}^n(\lambda)$) denote the set of reverse plane partitions (resp. SSYT) of shape λ with max entry n. A *flagging* on a RPP is an upper bound on the entries that appear in each row. An *elegant tableau* is a SSYT of skew shape λ/μ with the i-th row is strictly less than i; the set of which is denoted $\mathrm{ET}(\lambda/\mu)$.

Let $a(T) = (a_1, \dots, a_n)$ with a_i the number of columns containing an i. Let $b(T) = (b_1, \dots, b_n)$ with b_r the number of boxes b in the r-th row equal to the box directly below b. Grinberg, Galashin, and Liu [4] defined the *refined dual Grothendieck polynomial* as

$$g_\lambda(\mathbf{x};\mathbf{t}) := \sum_{T \in \mathrm{RPP}^n(\lambda)} \mathbf{t}^{b(T)} \mathbf{x}^{a(T)}.$$

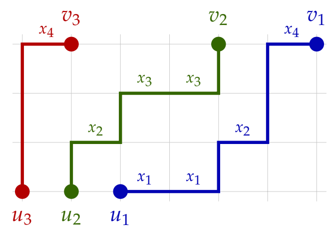
A *Schur function* $s_\lambda(\mathbf{x}) := g_\lambda(\mathbf{x};0)$ is the sum over semistandard tableaux. From the bijection $\phi\colon \mathrm{RPP}^n(\lambda) \mapsto \bigsqcup_{\mu \subseteq \lambda} \mathrm{SSYT}^n(\mu) \times \mathrm{ET}(\lambda/\mu)$ from [9, Thm. 9.8], which we call the *inflation map*, we have the following (see also [16, Eq. (72)]).

Theorem 2.1. *We have* $g_\lambda(\mathbf{x}, \mathbf{t}) = \sum_{\mu \subseteq \lambda} e^{\mu}_{\lambda}(\mathbf{t}) s_\mu(\mathbf{x})$, *where* $e^{\mu}_{\lambda}(\mathbf{t}) := \sum_{T \in \mathrm{ET}(\lambda/\mu)} \mathbf{t}^T$.

We can interpret a SSYT as a *family of non-intersecting lattice paths (NILP)* from the Jacobi–Trudi formula via the *Lindström–Gessel–Viennot (LGV) Lemma*. Additionally, a *Gelfand–Tselin (GT) pattern* is sequence of partitions $(\lambda^{(i)})_{i=1}^{n}$ representing a SSYT where $\lambda^{(i)}$ is the shape containing all entries at most i.

Example 2.2. We have a SSYT as a NILP and a GT pattern:

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array} \quad \longleftrightarrow$$



$$\longleftrightarrow \quad \begin{array}{ccccccc} 4 & & 3 & & 1 & & 0 \\ & 3 & & 3 & & 0 & \\ & & 3 & & 1 & & \\ & & & 2 & & & \end{array} .$$

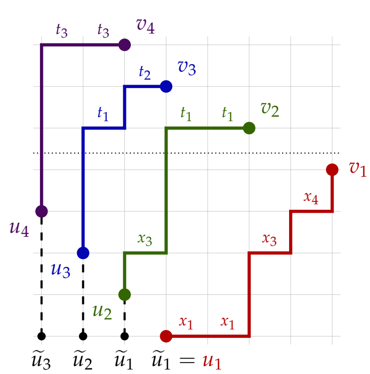
3 Lattice paths, identities, and integrability

In this section, we describe $g_\lambda(\mathbf{x}; \mathbf{t})$ using NILPs and interpret these as integrable lattice models. We fix a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with $\lambda_\ell > 0$. We build a lattice model by using Theorem 2.1, noting that an elegant tableau can be thought of as larger entries on a semistandard tableau. We obtain a refined version of Lascoux and Naruse [11, Eq. (3)] showing $g_\lambda(\mathbf{x}; \mathbf{t})$ is a multi-Schur function [10] by using the results of Chen, Li, and Louck [3, Thm. 3.2], although with the starting vertices begin at the same y-coordinate.

Define a *multi-Schur function* [10] by $s_\lambda(\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(\ell)}) := \det[S_{\lambda_{\ell+1-k}+h-k}(\mathbf{x}^{(k)})]_{h,k=1}^{\ell}$, where $\sum_{i \geq 0} S_i(\mathbf{x}^{(k)}) u^i = \prod_{i=1}^{n_k} (1 - x_i^{(k)} u)^{-1}$ for indeterminates $\mathbf{x}^{(k)} = (x_1^{(k)}, \ldots, x_{n_k}^{(k)})$.

Theorem 3.1. *We have* $g_\lambda(\mathbf{x}; \mathbf{t}) = s_\lambda(\mathbf{x}, \mathbf{x} + (t_1), \mathbf{x} + (t_1, t_2), \ldots, \mathbf{x} + \mathbf{t})$.

Example 3.2. Let $\lambda = 4322$ and $n = 5$. We give a NILP and the corresponding semistandard tableau of shape $\mu = 41$ and elegant tableau in the shaded portion:



$$\longleftrightarrow \quad \begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 4 \\ \hline 3 & 1 & 1 \\ \cline{1-3} 1 & 2 \\ \cline{1-2} 3 & 3 \\ \cline{1-2}\end{array}$$

The extension of the NILP from [3] is given by the dashed black lines, which are fixed.

Our first identity is a Jacobi–Trudi formula for refined dual Grothendieck polynomials, which is a dual version of [16, Eq. (73)], from the LGV lemma. This was shown for $\mathbf{t} = \beta$ in [16, Cor. 10.3] and [5], which is also implicit from [3, 11].

Corollary 3.3. *We have* $g_\lambda(\mathbf{x};\mathbf{t}) = \det\big[h_{\lambda_i+j-i}(\mathbf{x}, t_1, \dots, t_{i-1})\big]_{i,j=1}^{n}$.

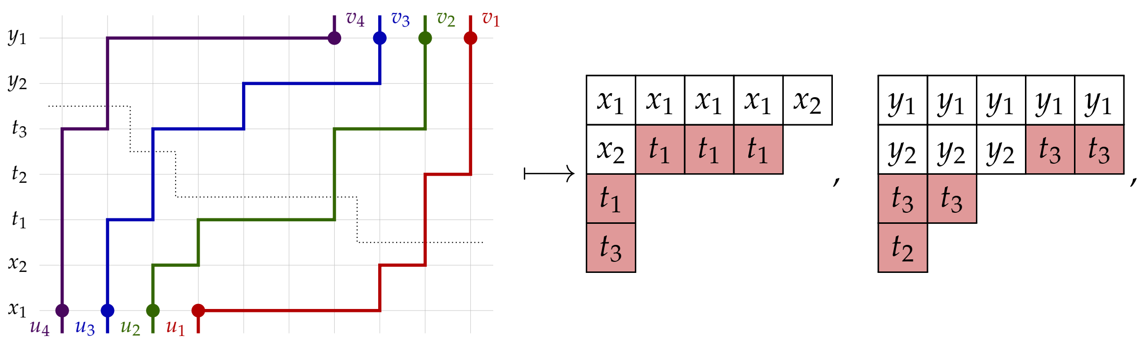
We obtain a dual Jacobi–Trudi formula by refining the computation for [11, Eq. (5)].

Corollary 3.4. *We have* $g_\lambda(\mathbf{x};\mathbf{t}) = \det\big[e_{\lambda'_i+j-i}(\mathbf{x}, t_1, \dots, t_{\lambda'_i-1})\big]_{i,j=1}^{n}$.

We give a new analog of the Cauchy identity for refined dual Grothendieck polynomials. We prove this by putting the two lattice paths together in the common $\mathbf{t}$ region.

Corollary 3.5. *We have* $s_{m^\ell}(\mathbf{x},\mathbf{t},\mathbf{y}) = \sum_{\lambda\subseteq m^\ell} g_\lambda(\mathbf{x};\mathbf{t}) g_{\lambda^\dagger}(\mathbf{y};\mathbf{t}^\dagger)$, *where* $\mathbf{t}^\dagger = (t_{\ell-1}, \dots, t_1)$.

Example 3.6. Let $n = 2$. An NILP for $s_{6^4}(\mathbf{x},\mathbf{t},\mathbf{y})$ with the dotted line indicating the cut for the decomposition from Corollary 3.5 and the corresponding semistandard tableaux is



where the second tableau is semistandard with respect to the alphabet $y_1 < y_2 < t_3 < t_2 < t_1$. When we rotate the second tableau by π and join it to the first tableau, we obtain a semistandard tableau of (rectangular) shape 6^4 for $x_1 < x_2 < t_1 < t_2 < t_3 < y_2 < y_1$.

We also have a generalized Littlewood identity; the $\mathbf{t} = 1$ was proven in [17, Cor. 3.5].

Corollary 3.7. *Let* $m \geq \ell$. *We have* $s_{m^\ell}(\mathbf{x},\mathbf{t},t_\ell) = \sum_{\lambda\subseteq m^\ell} \prod_{i=1}^{\ell} t_i^{m-\lambda_i} g_\lambda(\mathbf{x};\mathbf{t})$.

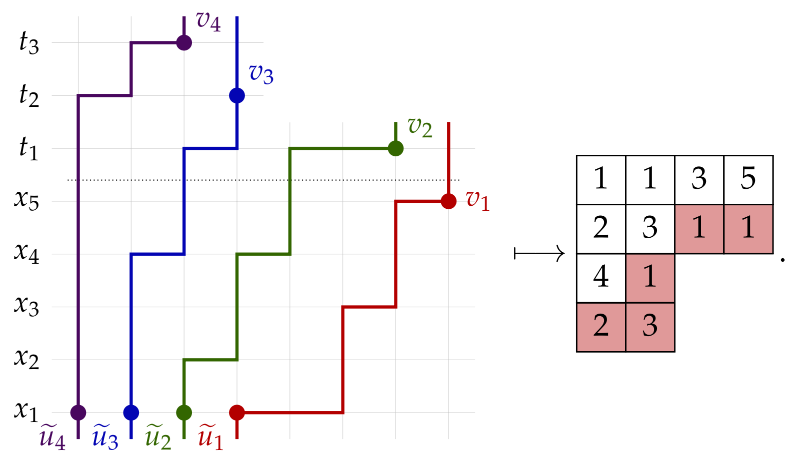
If we remove the topmost row in the previous lattice paths, then the upper-right region become fixed with only one allowable partition of λ being the rectangle. This is exactly analogous to how we are able to fix the lower-left portion. Therefore, we obtain the following, where the $\mathbf{t} = 1$ specialized version was first given in [17, Lemma 3.4].

Corollary 3.8. *We have* $s_{m^\ell}(\mathbf{x},\mathbf{t}) = g_{m^\ell}(\mathbf{x};\mathbf{t})$.

Corollary 3.8 implies that $g_\lambda(\mathbf{x};\mathbf{t})$ is symmetric in $\mathbf{x} \sqcup \mathbf{t}$ when λ is a rectangle. We can interpret the NILP as a state of an integrable 5-vertex model (see, *e.g.*, [13]), where the symmetry comes from the fact it satisfies the *Yang–Baxter equation*. Generalizing these fixed regions for more general shapes λ, we obtain the following.

Corollary 3.9. *Suppose $\lambda_i = \lambda_{i+1}$, then $g_\lambda(\mathbf{x};\mathbf{t})$ is symmetric in t_{i-1} and t_i, where $t_0 = x_n$.*

Example 3.10. Let $\lambda = 4422$ and $n = 5$. Then in the extended case, we have



We can see that $g_\lambda(\mathbf{x};\mathbf{t})$ is symmetric in $x_n \leftrightarrow t_1$ and $t_2 \leftrightarrow t_3$ as

$$\begin{aligned} g_\lambda(\mathbf{x};\mathbf{t}) &= s_{4422}(\mathbf{x},t_1) + (t_2+t_3)s_{4421}(\mathbf{x},t_1) + (t_2^2+t_2t_3+t_3^2)s_{442}(\mathbf{x},t_1) \\ &\quad + t_2t_3s_{4411}(\mathbf{x},t_1) + (t_2^2t_3+t_2t_3^2)s_{441}(\mathbf{x},t_1) + t_2^2t_3^2s_{44}(\mathbf{x},t_1). \end{aligned}$$

We also have a refined version of the branching rule of [16, Thm. 8.6] at $\alpha = 0$.

Corollary 3.11. *We have $g_\lambda(\mathbf{x},\gamma;\mathbf{t}) = \sum_{\mu\subseteq\lambda} \gamma^{\lambda_1-\mu_1} t_1^{\lambda_2-\mu_2} t_2^{\lambda_3-\mu_3} \cdots t_{\ell-1}^{\lambda_\ell-\mu_\ell} g_\mu(\mathbf{x};\gamma,\mathbf{t})$.*

Additionally, Corollary 3.8 is generalized to arbitrary λ using the LGV lemma by using the same cut approach as for Corollary 3.5 (see also Example 3.6).

Corollary 3.12. *Let ν be a partition, $\ell = \ell(\nu)$, and $\widetilde{\mathbf{t}} = (t_1,\dots,t_m)$ for some $m \geq \ell - 1$. Then*

$$s_\nu(\mathbf{x},\widetilde{\mathbf{t}}) = \sum_{\lambda\subseteq\nu} p_\nu^\lambda(\widetilde{\mathbf{t}}) g_\lambda(\mathbf{x};\mathbf{t}), \qquad \textit{where } p_\nu^\lambda(\widetilde{\mathbf{t}}) = \det\left[h_{\nu_i-\lambda_j-i+j}(t_m,\dots,t_j)\right]_{i,j=1}^{\ell} = \sum_T \widetilde{\mathbf{t}}^T$$

with the sum is over all semistandard skew tableaux T of shape ν/λ with max entry m and lower *flagging $f = (0,1,\dots,\ell-1)$, and for $m = \ell-1$, we consider $h_k(t_m,\dots,t_\ell) = \delta_{k0}$.*

Taking the combined SSYT for the RPP to a GT pattern, we use the $\beta = 0$ 5-vertex model from [13] and the corresponding Cauchy identity (see, *e.g.*, [13, Cor. 3.6, Cor. 5.4]) to obtain a refined Cauchy-type identity of [17, Thm 5.2(iv)] with λ in an $\ell \times m$ box.

Corollary 3.13. *We have*

$$\sum_{\lambda\subseteq m^\ell} \prod_{i=1}^{\ell} t_i^{m-\lambda_i} g_\lambda(\mathbf{x};\mathbf{t}) = \prod_{i=1}^{\ell} t_i^m \prod_{1\leq i<j\leq n} \frac{1}{(x_i-x_j)(t_i^{-1}-t_j^{-1})} \det\left[\frac{(x_it_j^{-1})^{m+n}-1}{x_it_j^{-1}-1}\right]_{i,j=1}^{n}.$$

By using the general β version of the Boltzmann weights from [13], we have a new basis of symmetric functions that expands positively in the Grothendieck polynomials, where the coefficients are the number of set-valued elegant tableaux. Furthermore, this expansion is given by counting set-valued elegant tableaux of a fixed shape λ/μ.

4 Probability theory

We begin this section by showing $g_\lambda(\mathbf{x};\mathbf{t})$ is the probability of a stochastic model, refining Yeliussizov relation [18] of $g_\lambda(\mathbf{x};1)$ with random matrix theory. This also essentially deconstructs the inflation map into the bijection from [18, Thm. 1] and RSK. Consider a random matrix $W = (w_{ij})_{i,j,\geq 1}$ with entries w_{ij} with geometric distribution $P(w_{ij} = k) = (1 - t_i x_j)(t_i x_j)^k$, where $t_i, x_j \in (0,1)$ and $k \in \mathbb{Z}_{\geq 0}$. The *last passage time* for W is

$$G(m,n) = \max_{\Pi} \sum_{(i,j)\in\Pi} w_{ij},$$

where the maximum is taken over all paths from $(1,1)$ to (m,n) with steps to the right or up, *i.e.*, for (i_a, j_a), we either have $(i_{a+1}, j_{a+1}) = (i_a, j_a+1), (i_a+1, j_a)$. We index our matrices using the natural indexing on $\mathbb{Z}^2$, so the bottom-left corner is $(1,1)$.

Define $\mathbf{G}(i) := (G(\ell,i),\ldots,G(1,i))$. For partitions λ,μ, denote the transition probability by $P(\mathbf{G}(i) = \lambda | \mathbf{G}(j) = \mu)$ of the *last passage percolation (LPP)* model. Our goal is to prove the following ($\mathbf{t} = 1$ case was proven in [17, Thm. 8.1]).

Theorem 4.1. *We have* $P(\mathbf{G}(n) = \lambda) = \prod_{i=1}^{\ell}\prod_{j=1}^{n}(1 - t_i x_j)\mathbf{t}^\lambda g_\lambda(\mathbf{x};\mathbf{t}^{-1})$.

Yeliussizov [18] gave a direct combinatorial proof of Theorem 4.1 at $\mathbf{t} = 1$ by constructing a bijection Φ between RPPs and the matrices that records the number of times a box is not equal to the box below it (after a trivial modification of [18, Thm. 1]). The factor $\prod_{i=1}^{\ell}\prod_{j=1}^{n}(1 - t_i x_j)$ trivially factors out from the definition.

We note that these random matrices W are in bijection with two-line arrays with w_{ij} denoting the number of bi-letters (i,j), which are sorted by lex order. Let $W \mapsto (P,Q)$ under RSK with λ being the shape of P and Q. Recall that under RSK, the length of a longest increasing subsequence in the lower word corresponds to λ_1.

Example 4.2. We have

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 1 & 3 & 4 \\ \hline 3 & 3 \\ \cline{1-2} \end{array} \xmapsto{\Phi} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 1 & 1 & 2 & 2 & 3 & 3 \\ 3 & 3 & 1 & 4 & 1 & 3 \end{bmatrix} \xmapsto{\text{RSK}} \left(\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 3 & 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 & 3 \\ \hline \end{array} \right).$$

Using the definition of Φ and jeu-de-taquin is equivalent to RSK insertion, we have:

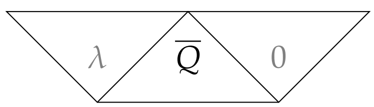
Proposition 4.3. *The insertion tableau of* RSK $\circ\, \Phi$ *is given by removing any i such that there is an i directly below it and using jeu-de-taquin to write the remaining boxes as a straight shape μ.*

The longest increasing subsequence of the bottom row when we restrict to $\ell \times n$ matrices corresponds to $G(\ell, n)$ (see, *e.g.*, [6]). Hence, we compute $G(\ell - k, n)$ by setting the top k rows of the matrix W to 0, which we denote this matrix by $W^{(k)}$. Let $W^{(k)} \mapsto (P^{(k)}, Q^{(k)})$ under RSK, which have shape $\lambda^{(k)}$. Hence, we have $G(\ell - k, n) = \lambda_1^{(k)}$. If we look at the recording tableau, we have that $Q^{(k)}$ is equal to removing all entries at least $n - k$ from $Q = Q^{(0)}$. Thus any matrix W such that $\big(G(\ell - k, n)\big)_{k=0}^{\ell-1} = \lambda$, we require that the left side of the GT pattern representation of Q must be λ. Therefore, we obtain

$$g_\lambda(\mathbf{x}; \mathbf{t}) = \mathbf{t}^\lambda \sum_{\mu \subseteq \lambda} \sum_{\substack{Q \in \mathrm{SSYT}^\ell(\mu) \\ \mathrm{left}(Q) = \lambda}} \mathbf{t}^{-Q} s_\mu(\mathbf{x}),$$

where $\mathrm{left}(Q)$ is the leftmost entries of the GT pattern corresponding to Q. Thus we have proven Theorem 4.1. Note that $\mathbf{t}^\lambda \mathbf{t}^{-Q} = \mathbf{t}^{b(T)}$ for all corresponding RPPs T to Q.

To see that these GT patterns correspond to the elegant tableaux, we simply reflect the GT pattern vertically and consider it inside of a larger skew GT pattern. Pictorially:



Example 4.4. The elegant tableaux from Example 3.2 as a GT pattern and SSYT are

$$\begin{ytableau} \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 1 \\ 1 & 2 \\ 3 & 3 \end{ytableau} \quad \longmapsto \quad \begin{matrix} 4 & & 1 & & 0 & & 0 \\ & 3 & & 1 & & 0 & \\ & & 2 & & 0 & & \\ & & & 2 & & & \end{matrix} \quad \longleftrightarrow \quad \begin{ytableau} 1 & 1 & 3 & 4 \\ 3 \end{ytableau}.$$

Subsequently, we also have a new bijective proof of Theorem 2.1. Moreover, this is a new probabilistic proof of Corollary 3.13 with $m \to \infty$ by summing all λ with $\ell(\lambda) \leq \ell$.

Next, we take Theorem 4.1 as the *definition* of $g_\lambda(\mathbf{x}; \mathbf{t})$. Then using the transition probability, a natural definition of skew refined dual Grothendieck polynomials is

$$g_{\lambda/\mu}(\mathbf{x}; \mathbf{t}) := \prod_{i=1}^{\ell} \prod_{j=1}^{n} (1 - t_i x_j)^{-1} \mathbf{t}^{\mu - \lambda} P(\mathbf{G}(n) = \lambda | \mathbf{G}(0) = \mu) \tag{4.1}$$

which gives us a natural branching formula of $g_{\lambda/\nu}(\mathbf{x} \sqcup \mathbf{y}; \mathbf{t}) = \sum_\mu g_{\lambda/\mu}(\mathbf{x}; \mathbf{t}) g_{\mu/\nu}(\mathbf{y}; \mathbf{t})$.

We show that $g_{\lambda/\nu}(\mathbf{x}; \mathbf{t})$ is equivalent to the combinatorial definition from [4]. Define the Heaviside step function $H\colon \mathbb{Z} \to \mathbb{C}$ and the convolution product of $f, g\colon \mathbb{Z} \to \mathbb{C}$ as

$$H(\nu) = \begin{cases} 0 & \text{if } \nu < 0, \\ 1 & \text{if } \nu \geq 0, \end{cases} \qquad \text{and} \qquad f * g(\nu) = \sum_{\xi=-\infty}^{\infty} f(\nu - \xi) g(\xi).$$

Next, one can compute the transition probability explicitly as

$$P(\mathbf{G}(n) = \lambda | \mathbf{G}(n-1) = \mu) = \prod_{j=1}^{\ell} (1 - t_j x_n)(t_j x_n)^{\lambda_j - \max(\mu_j, \lambda_{j+1})} H(\lambda_j - \max(\mu_j, \lambda_{j+1})),$$

from which we find the factorized form, where $v(\nu) := x_n^{\nu} H(\nu)$,

$$g_{\lambda/\mu}(x_n; \mathbf{t}) = \prod_{j=1}^{\ell-1} t_j^{\max(\mu_j, \lambda_{j+1}) - \mu_j} \prod_{j=1}^{\ell} v(\lambda_j - \max(\mu_j, \lambda_{j+1})). \tag{4.2}$$

We can see that (4.2) is precisely the generating function for the number of RPPs of shape λ/μ with a single letter n by reading the RPP row-by-row. This is equivalent to the definition in [4] by (recursively on n) removing the boxes in a RPP containing an n.

Corollary 4.5. *We have* $g_{\lambda/\mu}(\mathbf{x}; \mathbf{t}) = \sum_{T \in \mathrm{RPP}^n(\lambda/\mu)} \mathbf{t}^{b(T)} \mathbf{x}^{a(T)}$.

We introduce the weighted difference operators Δ_t and Δ_t^{-1} acting on $f\colon \mathbb{Z} \to \mathbb{C}$ as

$$\Delta_t f(\nu) = f(\nu+1) - t f(\nu),\ \Delta_t^{-1} f(\nu) = \sum_{\mu=-\infty}^{\nu-1} t^{\nu-1-\mu} f(\mu),\ \Delta_{\mathbf{t}}^{j-i} := \begin{cases} \Delta_{t_i} \cdots \Delta_{t_{j-1}} & \text{if } j \geq i, \\ \Delta_{t_j}^{-1} \cdots \Delta_{t_{i-1}}^{-1} & \text{if } j < i. \end{cases}$$

Lemma 4.6. *The following identity holds:*

$$g_{\lambda/\mu}(x_n; \mathbf{t}) = \det\left[\Delta_{\mathbf{t}}^{j-i} v(\lambda_i - \mu_j)\right]_{i,j=1}^{\ell} = \prod_{j=1}^{\ell} t_j^{-\mu_j + \max(\mu_j, \lambda_{j+1})} v(\lambda_j - \max(\mu_j, \lambda_{j+1})).$$

Lemma 4.6 is essentially a refined version of [7, Lemma 3.1] with a similar proof. We also have a refined version of [7, Lemma 3.2].

Proposition 4.7. *For $f, g\colon \mathbb{Z} \to \mathbb{C}$ such that $f(\nu) = g(\nu) = 0$ for some M and all $\nu < M$,*

$$\sum_{\nu_\ell \leq \cdots \leq \nu_2 \leq \nu_1} \det\left[\Delta_{\mathbf{t}}^{j-i} f(\nu_i - \mu_j)\right]_{i,j=1}^{\ell} \det\left[\Delta_{\mathbf{t}}^{j-i} g(\lambda_i - \nu_j)\right]_{i,j=1}^{\ell} = \det\left[(\Delta_{\mathbf{t}}^{j-i} f * g)(\lambda_i - \mu_j)\right]_{i,j=1}^{\ell}.$$

Combining the above, we find that $g_{\lambda/\mu}(\mathbf{x}; \mathbf{t})$ can be expressed as a single determinant

$$g_{\lambda/\mu}(\mathbf{x}; \mathbf{t}) = \det\left[(\Delta_{\mathbf{t}}^{j-i} f_1 * \cdots * f_n)(\lambda_i - \mu_j)\right]_{i,j=1}^{\ell}, \text{ where } f_j(\nu) = h_\nu(x_j) = x_j^{\nu} H(\nu). \tag{4.3}$$

So (4.3) becomes $g_{\lambda/\mu}(\mathbf{x};\mathbf{t}) = \det\big[(\Delta_{\mathbf{t}}^{j-i} h_\nu(\mathbf{x}))|_{\nu=\lambda_i-\mu_j}\big]_{i,j=1}^{\ell}$, as it is easy to see $f_1 * \cdots * f_n(\nu) = h_\nu(\mathbf{x})$, with $\Delta_{\mathbf{t}}^{j-i}$ acting on the subscript ν of $h_\nu(\mathbf{x})$. By induction on k, we have

$$\Delta_{t_k} \cdots \Delta_{t_2}\Delta_{t_1} f(\nu) = \sum_{m=0}^{\infty} e_m(-t_1, -t_2, \ldots, -t_k) f(\nu + k - m), \tag{4.4a}$$

$$\Delta_{t_k}^{-1} \cdots \Delta_{t_2}^{-1}\Delta_{t_1}^{-1} f(\nu) = \sum_{m=0}^{\infty} h_m(t_1, t_2, \ldots, t_k) f(\nu - k - m). \tag{4.4b}$$

Using (4.4) and the above determinant formula, we obtain the following.

Theorem 4.8. *The refined dual Grothendieck polynomial $g_{\lambda/\mu}(\mathbf{x};\mathbf{t})$ equals*

$$\det\left[\sum_{m=0}^{\infty} \alpha_m^{ij}(\mathbf{t}) h_{\lambda_i-\mu_j-i+j-m}(\mathbf{x})\right]_{i,j=1}^{\ell}, \text{ where } \alpha_m^{ij}(\mathbf{t}) = \begin{cases} h_m(t_j, \ldots, t_{i-1}) & \text{if } i \geq j, \\ e_m(-t_i, \ldots, -t_{j-1}) & \text{if } i < j. \end{cases}$$

This is the refined version of the Jacobi–Trudi type formula derived by Iwao [5, Prop. 5.2] at $\mathbf{t} = \beta$ using the boson-fermion correspondence and by Amanov and Yeliussizov [1, Thm. 14]. Moreover, Theorem 4.8 refines [7, Thm. 2.1] for the transition probability as a corollary and was proven independently by Kim [8] using plethystic techniques.

The approach in Section 3 is not amenable to the skew setting by comparing Equation (4.2) and Corollary 3.11 (by disregarding the $\mathbf{x}$ variables) in the single variable case.

We prove a dual Jacobi–Trudi type identity similar to Theorem 4.8, which then describes $P(\mathbf{G}(n) = \lambda' | \mathbf{G}(0) = \mu')$. Instead of $H(\nu)$, we use an indicator function of $[0,1]$.

Theorem 4.9. *The refined dual Grothendieck polynomial $g_{\lambda/\mu}(\mathbf{x};\mathbf{t})$ equals*

$$\det\left[\sum_{m=0}^{\infty} \widetilde{\alpha}_m^{ij}(\mathbf{t}^{-1}) e_{\lambda_i-\mu_j-i+j-m}(\mathbf{x})\right]_{i,j=1}^{\ell}, \text{ where } \widetilde{\alpha}_m^{ij}(\mathbf{t}) = \begin{cases} e_m(t_{\mu_j+1}, \ldots, t_{\lambda_i-1}) & \text{if } \mu_j \geq \lambda_i - 1, \\ h_m(-t_{\lambda_i}, \ldots, -t_{\mu_j}) & \text{otherwise.} \end{cases}$$

Theorem 4.9 was also proven in [1, 8] and for $\mathbf{t} = \beta$ in [5] using different techniques.

Proving refined versions of [7], we recover Baik and Rains [2] result on the *Schur measure* [14] on partitions defined as, for $t_i, x_j \in (0,1)$ such that $0 < \prod_{i,j}(1 - t_i x_j) < \infty$,

$$P_{\text{Schur}}(\lambda) = \prod_{i,j}(1 - t_i x_j) s_\lambda(\mathbf{t}) s_\lambda(\mathbf{x}).$$

Theorem 4.10 ([6, 2])**.** *We have*

$$P(G(\ell, n) \leq m) = \sum_{\lambda:\lambda_1 \leq m} P_{\text{Schur}}(\lambda) = \prod_{i=1}^{\ell}\prod_{j=1}^{n}(1 - t_i x_j) \sum_{\lambda:\lambda_1 \leq m} s_\lambda(\mathbf{x}) s_\lambda(\mathbf{t}).$$

Using Theorem 4.1, our determinant formulas, and elementary transformations, we obtain the refinement of [7, Thm. 2.2], which we then transform to a refined [7, Prop. 2.4].

Lemma 4.11. *Let $\mathbf{t}_i = (t_1, \ldots, t_i)$. We have $P(G(\ell, n) \leq m)$ equal to*

$$\prod_{i=1}^{\ell} t_i^m \prod_{i=1}^{\ell}\prod_{j=1}^{n}(1 - t_i x_j) \det\big[\Delta_{\mathbf{t}^{-1}}^{j-i-1} h_\nu(\mathbf{x})|_{\nu=m+1}\big]_{i,j=1}^{\ell}$$
$$= \frac{1}{\ell!}\prod_{i=1}^{\ell} t_i^m \prod_{i=1}^{\ell}\prod_{j=1}^{n}(1 - t_i x_j) \sum_{\nu_1,\ldots,\nu_\ell=0}^{m+\ell-1} \det\big[h_{m+\ell-\nu_j-i}(\mathbf{t}_i^{-1})\big]_{i,j=1}^{\ell} \det\big[\Delta_{\mathbf{t}^{-1}}^{j-1} h_{\nu_i-\ell+1}(\mathbf{x})\big]_{i,j=1}^{\ell}.$$

Under additional analysis and manipulation of the determinants, we rewrite

$$P(G(\ell, n) \leq m) = \prod_{i=1}^{\ell} t_i^m \prod_{i=1}^{\ell}\prod_{j=1}^{n}(1 - t_i x_j) s_{m^\ell}(\mathbf{x}, \mathbf{t}^{-1}). \tag{4.5}$$

Then using the combinatorics of Schur functions, we finish our proof of Theorem 4.10.

We can give additional probabilistic proofs of some of our identities. The first is the finite weighted Littlewood formula in Corollary 3.7 by combining (4.5) and

$$P(G(\ell, n) \leq m) = \sum_{\lambda \subseteq m^\ell} P(\mathbf{G}(n) = \lambda) = \sum_{\lambda \subseteq m^\ell} \prod_{i=1}^{\ell}\prod_{j=1}^{n}(1 - t_i x_j)\mathbf{t}^\lambda g_\lambda(\mathbf{x}; \mathbf{t}^{-1}).$$

Next we prove the Cauchy identity of Corollary 3.5 by using two expressions for $P(G(\ell, 2n) \leq m)$. Build $\mathbf{y}$ by $y_j := x_{n+j}$. On one hand, we have (4.5). On the other hand,

$$P(G(\ell, 2n) \leq m) = \sum_{\mu \subseteq m^\ell} P(\mathbf{G}(n) = \mu) P(G(\ell, 2n) \leq m | \mathbf{G}(n) = \mu).$$

By algebraic manipulations, we show

$$P(G(\ell, 2n) \leq m | \mathbf{G}(n) = \mu) = \mathbf{t}^\mu \prod_{i=1}^{\ell} t_i^m \prod_{i=1}^{\ell}\prod_{j=1}^{n}(1 - t_i y_j) \det\big[\Delta_{\mathbf{t}^{-1}}^{j-i-1} h_\nu(\mathbf{y})|_{\nu=m+1-\mu_j}\big]_{i,j=1}^{\ell}.$$

We obtain the Cauchy identity (Corollary 3.5) by taking the $t_\ell \to \infty$ limit and using

$$\lim_{t_\ell \to \infty} \det\big[\Delta_{\mathbf{t}^{-1}}^{j-i-1} h_\nu(\mathbf{y})|_{\nu=m+1-\mu_j}\big]_{i,j=1}^{\ell} = g_{\mu^\dagger}(\mathbf{y}; (\mathbf{t}^\dagger)^{-1}).$$

Finally, we give an integral formula for $g_\lambda(\mathbf{x};\mathbf{t})$ by again refining [7]. We show

$$\Delta_{t_k}\cdots\Delta_{t_2}\Delta_{t_1}(f_1 * f_2 * \cdots * f_n)(\nu) = \frac{1}{2\pi i}\int_{\gamma_r}\frac{\prod_{m=1}^{k}(1-t_m z)}{\prod_{m=1}^{n}(1-x_m z)z^{\nu+k+1}}dz,$$
$$\Delta_{t_k}^{-1}\cdots\Delta_{t_2}^{-1}\Delta_{t_1}^{-1}(f_1 * f_2 * \cdots * f_n)(\nu) = \frac{1}{2\pi i}\int_{\gamma_r}\frac{dz}{\prod_{m=1}^{n}(1-x_m z)\prod_{m=1}^{k}(1-t_m z)z^{\nu-k+1}},$$

where the integration circle γ_r satisfies $0 < r < t_m^{-1}, x_m^{-1}$ for all m. Applying this to (4.3):

Proposition 4.12. *We have $g_{\lambda/\mu}(\mathbf{x};\mathbf{t}) = \det[F_{ij}]_{i,j=1}^{\ell}$, where*

$$F_{ij} = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\gamma_r}\frac{\prod_{m=i}^{j-1}(1-t_m z)}{\prod_{m=1}^{n}(1-x_m z)z^{\lambda_i-\mu_j+j-i+1}}\,dz & \text{if } j \geq i, \\ \dfrac{1}{2\pi i}\displaystyle\int_{\gamma_r}\frac{1}{\prod_{m=1}^{n}(1-x_m z)\prod_{m=j}^{i-1}(1-t_m z)z^{\lambda_i-\mu_j+j-i+1}}\,dz & \text{if } j < i. \end{cases}$$

For $\mu = \varnothing$ and $\mathbf{t} = \beta$, then we have

$$g_\lambda(\mathbf{x};\beta) = \frac{1}{(2\pi i)^\ell}\oint\cdots\oint\frac{\prod_{i=1}^{\ell} z_i^{\lambda_i+\ell-i}\prod_{1\leq i<j\leq \ell}(z_j-z_i)(1-\beta z_j-\beta z_i)}{\prod_{i=1}^{\ell}\prod_{m=1}^{\ell}(z_i-x_m)\prod_{1\leq i<j\leq\ell}(1-\beta z_j)}\,dz_1\cdots dz_\ell.$$

Moreover, for $\mu = \varnothing$ and $\mathbf{t} = 0$, this is the famous integral representation of $s_\lambda(\mathbf{x})$.

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