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Combinatorial Howe duality of symplectic type

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Abstract. We give a symplectic analog of the RSK algorithm for Howe dual pairs of the form $(\mathfrak{g}, \mathrm{Sp}_{2\ell})$, where $\mathfrak{g}$ is a Lie (super)algebra of classical type. We introduce an analog of jeu de taquin sliding for spinor model of irreducible characters of a Lie superalgebra $\mathfrak{g}$ to define P-tableau, and then define the associated Q-tableau in terms of a symplectic tableau due to King.

Keywords: crystal graphs, Howe duality, RSK correspondence, jeu de taquin

1 Introduction

Let $\mathscr{P}$ be the set of partitions or Young diagrams $\lambda = (\lambda_1, \lambda_2, \ldots)$ and, for $n \geq 1$, let $\mathscr{P}_n = \{\, \lambda \in \mathscr{P} \mid \ell(\lambda) \leq n \,\}$, where $\ell(\lambda)$ is the length of λ.

Let $\mathcal{A}$ be a $\mathbb{Z}_2$-graded linearly ordered set and let $\mathscr{E}_{\mathcal{A}}$ be the exterior algebra generated by the superspace with a linear basis indexed by $\mathcal{A}$. Then $\mathscr{F}_{\mathcal{A}} = \mathscr{E}_{\mathcal{A}}^* \otimes \mathscr{E}_{\mathcal{A}}$ is a semisimple module over a classical Lie (super)algebra $\mathfrak{g}_{\mathcal{A}}$, the type of which depends on $\mathcal{A}$, and the ℓ-fold tensor power $\mathscr{F}_{\mathcal{A}}^{\otimes \ell}$ ($\ell \geq 1$) is a $(\mathfrak{g}_{\mathcal{A}}, \mathrm{Sp}_{2\ell})$-module with the following multiplicity-free decomposition:

$$\mathscr{F}_{\mathcal{A}}^{\otimes \ell} \cong \bigoplus_{(\lambda,\ell) \in \mathscr{P}(\mathrm{Sp})_{\mathcal{A}}} V_{\mathfrak{g}_{\mathcal{A}}}(\lambda, \ell) \otimes V_{\mathrm{Sp}_{2\ell}}(\lambda), \tag{1.1}$$

where the direct sum is over a set $\mathscr{P}(\mathrm{Sp})_{\mathcal{A}}$ of pairs $(\lambda, \ell) \in \mathscr{P} \times \mathbb{Z}_+$ with $\ell(\lambda) \leq \ell$ (see [1, 4, 3, 5, 12, 18] for various choices of $\mathcal{A}$). Here $V_{\mathrm{Sp}_{2\ell}}(\lambda)$ is the irreducible $\mathrm{Sp}_{2\ell}$-module corresponding to λ, and $V_{\mathfrak{g}_{\mathcal{A}}}(\lambda, \ell)$ is the irreducible highest weight $\mathfrak{g}$-module corresponding to $V_{\mathrm{Sp}_{2\ell}}(\lambda)$ appearing in $\mathscr{F}_{\mathcal{A}}^{\otimes \ell}$. This decomposition is obtained from a more general principle called Howe duality [3].

There exists a combinatorial object called a spinor model of type C, which gives the character of $V_{\mathfrak{g}_{\mathcal{A}}}(\lambda, \ell)$ in (1.1) in a uniform way [9]. As a set, the spinor model $\mathbf{T}_{\mathcal{A}}(\lambda, \ell)$ consists of sequences of usual semistandard tableaux of two-columned shapes with letters in $\mathcal{A}$, where two adjacent tableaux satisfy certain configuration.

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In this extended abstract, we give an analog of RSK algorithm for (1.1) in terms of spinor model. More precisely, we construct an explicit bijection

$$\mathbf{F}^{\ell}_{\mathcal{A}} \longrightarrow \bigsqcup_{\lambda \in \mathscr{P}(\mathrm{Sp})_{\mathcal{A}}} \mathbf{T}_{\mathcal{A}}(\lambda, \ell) \times \mathbf{K}(\lambda, \ell)\,, \tag{1.2}$$

where $\mathbf{F}^{\ell}_{\mathcal{A}}$ is the set of 2ℓ-tuple of $\mathcal{A}$-semistandard tableaux of single-columned shapes with letters in $\mathcal{A}$, and $\mathbf{K}(\lambda, \ell)$ is the set of symplectic tableaux of shape λ due to King [8] giving the character of $V_{\mathrm{Sp}_{2\ell}}(\lambda)$. The bijection (1.2) yields the Cauchy type identity which follows from the decomposition (1.1) for arbitrary $\mathcal{A}$, and it recovers well-known identities [7, 16, 19] when $\mathcal{A}$ is a finite set of homogeneous degree. A full version of this paper including detailed proofs has appeared in [2].

2 Preliminaries

2.1 Notations

Let $\mathbb{Z}_+$ denote the set of non-negative integers. We denote by $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ the conjugate of λ. Let $\mathcal{A}$ be a linearly ordered set with a $\mathbb{Z}_2$-grading $\mathcal{A} = \mathcal{A}_0 \sqcup \mathcal{A}_1$. For example, for a positive integer n, let

$$[n] = \{1 < 2 < \cdots < n\}, \quad [\overline{n}] = \{\overline{n} < \overline{n-1} < \cdots < \overline{1}\},$$

where we assume that all the entries are assumed to be of degree 0. For positive integers m and n, let $\mathbb{I}_{m|n} = \{1 < 2 < \cdots < m < 1' < 2' < \cdots < n'\}$ with $(\mathbb{I}_{m|n})_0 = \{1 < \cdots < m\}$ and $(\mathbb{I}_{m|n})_1 = \{1' < \cdots < n'\}$.

For a skew Young diagram λ/μ, let $SST_{\mathcal{A}}(\lambda/\mu)$ be the set of semistandard (or $\mathcal{A}$-semistandard) tableaux of shape λ/μ, that is, tableaux with entries in $\mathcal{A}$ such that (1) the entries in each row (resp. column) are weakly increasing from left to right (resp. from top to bottom), (2) the entries in $\mathcal{A}_0$ (resp. $\mathcal{A}_1$) are strictly increasing in each column (resp. row).

For $a, b, c \in \mathbb{Z}_+$, let $\lambda(a, b, c) = (2^{b+c}, 1^a)/(1^b)$ be a skew Young diagram with two columns. For an $\mathcal{A}$-semistandard tableau T with two columns, let $T^{\mathtt{L}}$ and $T^{\mathtt{R}}$ denote the left and right columns of T, respectively.

We place a tableau so that its top or bottom edges is parallel with or same as a given horizontal line L. More precisely, let $U_1, \ldots, U_r$ be column tableaux (that is, tableaux of single-columned shapes), which are $\mathcal{A}$-semistandard. For $(u_1, \ldots, u_r) \in \mathbb{Z}_+^r$, let

$$\lfloor U_1, \ldots, U_r \rfloor_{(u_1, \ldots, u_r)}, \quad \lceil U_1, \ldots, U_r \rceil^{(u_1, \ldots, u_r)}$$

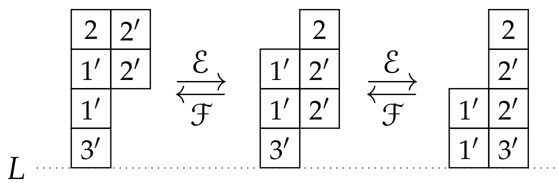
be the tableaux such that the i-th column from the left is U_i and the bottom (resp. top) edge of U_i is slid by u_i positions up (resp. down) from L (see Examples 2.1, 2.4, 3.3, 3.6). In particular, we do not record the tuple $(u_1, \ldots, u_r)$ when $(u_1, \ldots, u_r) = (0, \ldots, 0)$.

2.2 Crystal and Schützenberger's jeu de taquin

Note that Schützenberger's jeu de taquin is also available for $\mathcal{A}$-semistandard tableaux. We use this algorithm in terms of crystal operator $\mathcal{E}$ and $\mathcal{F}$ for $\mathfrak{sl}_2$ (cf. [13]).

For $T = \lfloor T^{\mathtt{L}}, T^{\mathtt{R}} \rfloor_{(0,a)} \in SST_{\lambda(a,b,c)}$ such that $\lfloor T^{\mathtt{L}}, T^{\mathtt{R}} \rfloor_{(0,a-1)}$ is not $\mathcal{A}$-semistandard, we define $\mathcal{E}T$ and $\mathcal{F}T$ to be the tableaux obtained by applying the usual jeu de taquin to the outer and inner corners of T, respectively. Here we define $\mathcal{E}T = \mathbf{0}$ and $\mathcal{F}T = \mathbf{0}$ when $a = 0$ and $b = 0$, respectively, where $\mathbf{0}$ is a formal symbol.

Example 2.1. *Suppose that* $\mathcal{A} = \mathbb{I}_{4|3}$.



Now, for $(U, V) \in SST_{\mathcal{A}}((1^u)) \times SST_{\mathcal{A}}((1^v))$ ($u, v \in \mathbb{Z}_+$), we define

$$\mathcal{X}(U,V) = \begin{cases} ((\mathcal{X}T)^{\mathtt{L}}, (\mathcal{X}T)^{\mathtt{R}}) & \text{if } \mathcal{X}T \neq \mathbf{0}, \\ \mathbf{0} & \text{if } \mathcal{X}T = \mathbf{0}, \end{cases} \qquad (\mathcal{X} = \mathcal{E}, \mathcal{F}), \tag{2.1}$$

where T is the unique tableau in $SST_{\mathcal{A}}(\lambda(u-k, v-k, k))$ for some $0 \leq k \leq \min\{u, v\}$ such that $(T^{\mathtt{L}}, T^{\mathtt{R}}) = (U, V)$ and $\lfloor T^{\mathtt{L}}, T^{\mathtt{R}} \rfloor_{(0,u-k-1)}$ is not $\mathcal{A}$-semistandard. Put $\varphi(U, V) = \max\{\, k \,|\, \mathcal{F}^k(U,V) \neq \mathbf{0} \,\}$ and $\mathcal{F}^{\max}(U,V) = \mathcal{F}^{\varphi(U,V)}(U,V)$.

For $r \geq 2$, let $\mathbf{E}^r_{\mathcal{A}}$ be the set of r-tuples of single-columned $\mathcal{A}$-semistandard tableaux. For $(U_r, \ldots, U_1) \in \mathbf{E}^r_{\mathcal{A}}$ and $1 \leq i \leq r-1$, we define

$$\mathcal{X}_i(U_r, \ldots, U_1) = \begin{cases} (U_r, \ldots, \mathcal{X}(U_{i+1}, U_i), \ldots, U_1) & \text{if } \mathcal{X}(U_{i+1}, U_i) \neq \mathbf{0}, \\ \mathbf{0} & \text{if } \mathcal{X}(U_{i+1}, U_i) = \mathbf{0}, \end{cases} \tag{2.2}$$

where $\mathcal{X}(\,\cdot\,,\,\cdot\,)$ for $\mathcal{X} = \mathcal{E}, \mathcal{F}$ is defined in (2.1). Then we have the following.

Lemma 2.2 ([2, Lemma 2.3])**.** $\mathbf{E}^r_{\mathcal{A}}$ *is a regular* $\mathfrak{sl}_r$*-crystal with respect to* $\mathcal{E}_i$ *and* $\mathcal{F}_i$ *for* $1 \leq i \leq r-1$.

2.3 Spinor model

Let

$$\mathscr{P}(\mathrm{Sp}) = \{\, (\lambda, \ell) \,|\, \ell \geq 1,\ \lambda \in \mathscr{P}_\ell \,\}.$$

For $a \in \mathbb{Z}_+$, let

$$\mathbf{T}_{\mathcal{A}}(a) = \bigsqcup_{c \in \mathbb{Z}_+} SST_{\mathcal{A}}(\lambda(a, 0, c)).$$

For $T \in \mathbf{T}_{\mathcal{A}}(a)$, we define
$$ {}^{\mathtt{L}}T = (\mathcal{E}^a T)^{\mathtt{L}}, \quad {}^{\mathtt{R}}T = (\mathcal{E}^a T)^{\mathtt{R}}. $$

Definition 2.3 ([9, Definition 6.7, Definition 6.10]). *(1) For $a_1, a_2 \in \mathbb{Z}_+$ with $a_2 \le a_1$ and $(T_2, T_1) \in \mathbf{T}_{\mathcal{A}}(a_2) \times \mathbf{T}_{\mathcal{A}}(a_1)$, we define*
$$ T_2 \prec T_1 \quad \text{if } \lfloor {}^{\mathtt{R}}T_2, T_1^{\mathtt{L}} \rfloor \text{ and } \lfloor T_2^{\mathtt{R}}, {}^{\mathtt{L}}T_1 \rfloor_{(a_2,a_1)} \text{ are } \mathcal{A}\text{-semistandard.} $$

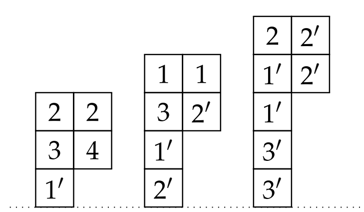
(2) For $(\lambda, \ell) \in \mathscr{P}(\mathrm{Sp})$, we define
$$ \mathbf{T}_{\mathcal{A}}(\lambda, \ell) = \{ \mathbf{T} = (T_\ell, \ldots, T_1) \mid T_i \in \mathbf{T}_{\mathcal{A}}(\lambda_i) \text{ for all } 1 \le i \le \ell \text{ and } T_\ell \prec \cdots \prec T_1 \}. $$

Put
$$ \mathscr{P}(\mathrm{Sp})_{\mathcal{A}} = \{ (\lambda, \ell) \in \mathscr{P}(\mathrm{Sp}) \mid \mathbf{T}_{\mathcal{A}}(\lambda, \ell) \neq \emptyset \}. $$

Let $\mathbf{x}_{\mathcal{A}} = \{ x_a \mid a \in \mathcal{A} \}$ be the set of formal commuting variables indexed by $\mathcal{A}$. For an $\mathcal{A}$-semistandard tableau T, let $\mathbf{x}_{\mathcal{A}}^T = \prod_{a \in \mathcal{A}} x_a^{m_a}$, where m_a is the number of occurrences of a in T. Let t be a variable commuting with all x_a ($a \in \mathcal{A}$). For $(\lambda, \ell) \in \mathscr{P}(\mathrm{Sp})_{\mathcal{A}}$, we define the character of $\mathbf{T}_{\mathcal{A}}(\lambda, \ell)$ to be
$$ S_{(\lambda,\ell)}(\mathbf{x}_{\mathcal{A}}) = t^\ell \sum_{(T_\ell,\ldots,T_1) \in \mathbf{T}_{\mathcal{A}}(\lambda,\ell)} \mathbf{x}_{\mathcal{A}}^{T_\ell} \cdots \mathbf{x}_{\mathcal{A}}^{T_1}. \tag{2.3} $$

The character $S_{(\lambda,\ell)}(\mathbf{x}_{\mathcal{A}})$ gives various irreducible characters of Lie (super)algebras under suitable choices of $\mathcal{A}$ (cf. [9]). In particular, it is the character of a finite-dimensional irreducible $\mathfrak{sp}_{2n}$-module when $\mathcal{A} = [\overline{n}]$.

Example 2.4. *Suppose $\mathcal{A} = \mathbb{I}_{4|3}$ and $(\lambda, \ell) = ((3,2,1), 3)$. Let $\mathbf{T} = (T_3, T_2, T_1)$ be given by*



T_3 T_2 T_1

where the dotted line is the common bottom line. In this case, $T_3 \prec T_2 \prec T_1$ and so $\mathbf{T} \in \mathbf{T}_{\mathcal{A}}(\lambda, \ell)$.

Definition 2.5 ([2, Definition 5.1]). *Let $\mathbf{T} = (T_\ell, \ldots, T_1) \in \mathbf{T}_{\mathcal{A}}(a_\ell) \times \cdots \times \mathbf{T}_{\mathcal{A}}(a_1)$ be given for some $a_1, \ldots, a_\ell \in \mathbb{Z}_+$. Let λ/μ be a skew diagram with $\lambda, \mu \in \mathscr{P}_\ell$. We say that*

(1) $\mathbf{T}$ is of shape λ/μ if
$$ a_i = \lambda_i - \mu_i, \quad \mathrm{ht}(T_{i+1}^{\mathtt{L}}) + \mu_{i+1} \le \mathrm{ht}(T_i^{\mathtt{L}}) + \mu_i \quad (1 \le i \le \ell), $$
where $\mathrm{ht}(U)$ denotes the height of a single-columned tableau U,

(2) $\mathbf{T}$ *is* $\mathcal{A}$*-admissible of shape* λ/μ *if* $\mathbf{T}$ *is of shape* λ/μ *and*

$$\lfloor {}^{\mathtt{R}}T_{i+1}, T_i^{\mathtt{L}} \rfloor_{(\mu_{i+1},\mu_i)} \text{ and } \lfloor T_{i+1}^{\mathtt{R}}, {}^{\mathtt{L}}T_i \rfloor_{(\lambda_{i+1},\lambda_i)} \text{ are } \mathcal{A}\text{-semistandard} \quad (1 \le i \le \ell - 1).$$

We denote by $\mathbf{T}_{\mathcal{A}}(\lambda/\mu, \ell)$ the set of $\mathcal{A}$-admissible tableaux of shape λ/μ. When $\mathbf{T}$ is of shape λ/μ, let us often identify $\mathbf{T}$ with a tableau

$$\lfloor T_\ell, \dots, T_1 \rfloor_{(\mu_\ell,\dots,\mu_1)} := \lfloor T_\ell^{\mathtt{L}}, T_\ell^{\mathtt{R}}, \dots, T_1^{\mathtt{L}}, T_1^{\mathtt{R}} \rfloor_{(\mu_\ell,\lambda_\ell,\dots,\mu_1,\lambda_1)}.$$

3 Symplectic jeu de taquin

3.1 Symplectic jeu de taquin for KN tableaux

Recall that there exists a well-known combinatorial model for the irreducible highest weight $\mathfrak{sp}_{2n}$-module, called Kashiwara–Nakashima tableaux of type C (KN tableaux for short) [6]. For $\lambda \in \mathscr{P}_n$, denote by $\mathbf{KN}_\lambda$ the set of KN tableaux of shape λ with letters in $\{1 < \cdots < n < \overline{n} < \cdots < \overline{1}\}$.

Note that a tableau with letters in $\{1 < \cdots < n < \overline{n} < \cdots < \overline{1}\}$ is a KN tableau if and only if it satisfies the *admissibility* condition (see [17, Section 4]). With this characterization, one can define the set $\mathbf{KN}_{\lambda/\mu}$ of admissible tableaux of a skew shape λ/μ with $\lambda, \mu \in \mathscr{P}_n$ [14, Definition 6.1.1].

For $(\lambda, \ell) \in \mathscr{P}(\mathrm{Sp})_{[\overline{n}]}$, put $\rho_n(\lambda, \ell) = (n - \lambda_\ell, n - \lambda_{\ell-1}, \dots, n - \lambda_1)'$, which is the conjugate of the rectangular complement of λ in (n^ℓ).

For $U \in SST_{[\overline{n}]}((1^m))$, let U^c be the tableau in $SST_{[n]}((1^{n-m}))$ such that k appears in U^c if and only if $\overline{k}$ does not appear in U for each $k \in [n]$. For $T \in \mathbf{T}_n(a)$, define T^{ad} to be the tableau obtained by putting ${}^{\mathtt{L}}T$ below $({}^{\mathtt{R}}T)^c$. Then the map $T \longmapsto T^{\mathrm{ad}}$ is a bijection from $\mathbf{T}_n(a)$ to $\mathbf{KN}_{(1^{n-a})}$ [11, Lemma 3.11]. Moreover, we have a bijection [2, Corollary 5.6]

$$\begin{array}{ccc} \mathbf{T}_{[\overline{n}]}(\lambda/\mu, \ell) & \longrightarrow & \mathbf{KN}_{\rho_{n+\mu_1}(\lambda,\ell)/\rho_{\mu_1}(\mu,\ell)} \\ \mathbf{T} = \lfloor T_\ell, \dots, T_1 \rfloor_{(\mu_\ell,\dots,\mu_1)} & \longmapsto & \mathbf{T}^{\mathrm{ad}} := \lceil T_\ell^{\mathrm{ad}}, \dots, T_1^{\mathrm{ad}} \rceil^{\rho_{\mu_1}(\mu,\ell)} \end{array} . \tag{3.1}$$

For $\mathbf{T} \in \mathbf{T}_{[\overline{n}]}(\lambda/\mu, \ell)$ and an inner corner c of T (if exists), one may apply the symplectic analog of jeu de taquin in [17] (see also [14, Section 6]) to obtain another admissible tableau, say $\mathtt{jdt}_{KN}(T, c)$. It is proved in [14] that there exists a unique KN tableau in $\mathbf{KN}_\nu$ for some $\nu \in \mathscr{P}_n$, which is obtained from T by applying $\mathtt{jdt}_{KN}$ successively.

3.2 Symplectic jeu de taquin for spinor model

Note that $\mathbf{T} \in \mathbf{T}_{\mathcal{A}}(\lambda/\mu, \ell)$ does not necessarily correspond to a KN tableau of skew shape for arbitrary $\mathcal{A}$ as in (3.1). In order to define an analog of jeu de taquin, we first introduce the notion of n-conjugate of $\mathbf{T}$.

Let us assume that $\mathbf{T} = (T_\ell^{\mathtt{L}}, T_\ell^{\mathtt{R}}, \dots, T_1^{\mathtt{L}}, T_1^{\mathtt{R}}) \in \mathbf{E}_{\mathcal{A}}^{2\ell}$. Then we have a map

$$\begin{aligned} \mathbf{E}_{\mathcal{A}}^{2\ell} &\longrightarrow \bigsqcup_{\mu\in\mathscr{P}} SST_{\mathcal{A}}(\mu) \times SST_{[2\ell]}(\mu')\,, \\ \mathbf{U} &\longmapsto (P_{\mathcal{A}}(\mathbf{U}), Q_{\mathcal{A}}(\mathbf{U})) \end{aligned}$$

where $P_{\mathcal{A}}(\mathbf{U})$ is obtained by the usual Schensted's column insertion and $Q_{\mathcal{A}}(\mathbf{U})$ is the associated recording tableau.

Definition 3.1. *Let $\mathbf{T} = (T_\ell, \dots, T_1) \in \mathbf{T}_{\mathcal{A}}(\lambda/\mu, \ell)$ be given and set $\nu = \mathrm{sh}(P_{\mathcal{A}}(\mathbf{T}))$. For $n \geq \ell(\nu)$, we define the n-conjugate of $\mathbf{T}$ to be the unique tableau $\overline{\mathbf{T}} \in \mathbf{T}_{[\overline{n}]}(\lambda/\mu, \ell)$ such that*

$$P_{[\overline{n}]}(\overline{\mathbf{T}}) = H_\nu \quad \textit{and} \quad Q_{[\overline{n}]}(\overline{\mathbf{T}}) = Q_{\mathcal{A}}(\mathbf{T}),$$

where $H_\nu \in SST_{[\overline{n}]}(\nu)$ is such that the i-th row from the top is filled with $\overline{n-i+1}$ for $1 \leq i \leq n$.

Now, let us introduce an analog of jeu de taquin for spinor model. We first consider the case when $\ell = 2$. Suppose that $\mathbf{T} = (T_2, T_1) \in \mathbf{T}_{\mathcal{A}}(a_2) \times \mathbf{T}_{\mathcal{A}}(a_1)$ is given for some $a_1, a_2 \in \mathbb{Z}_+$. Let

$$d(T_1, T_2) = \min \left\{ d \in \mathbb{Z}_+ \,\middle|\, \lfloor T_2, T_1 \rfloor_{(0,d)} \text{ is } \mathcal{A}\text{-admissible (of a skew shape)} \right\}.$$

We assume that $\mathbf{T} = (T_2^{\mathtt{L}}, T_2^{\mathtt{R}}, T_1^{\mathtt{L}}, T_1^{\mathtt{R}}) \in \mathbf{E}_{\mathcal{A}}^4$ and consider the $\mathfrak{sl}_4$-crystal structure on $\mathbf{E}_{\mathcal{A}}^4$ given in Lemma 2.2.

Definition 3.2 ([2, Section 5.2])**.** *Under the above hypothesis, suppose that $d = d(T_1, T_2) > 0$. Define*

$$\mathtt{jdt}_{spin}(\mathbf{T}) = \mathbf{T}' = (T_2', T_1')$$

to be the pair (T_2', T_1') obtained by applying a sequence of crystal operators as follows:

Case 1. *Suppose that $\lfloor {}^{\mathtt{R}}T_2, T_1^{\mathtt{L}} \rfloor_{(0,d-1)}$ is not $\mathcal{A}$-semistandard. Then*

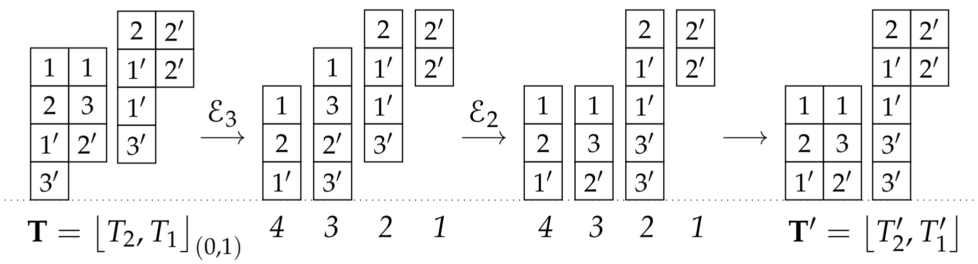
$$\mathbf{T}' := (U_4, U_3, U_2, U_1) = \begin{cases} \mathcal{F}_3^{a_2-1}\mathcal{E}_2\mathcal{E}_3^{a_2}\mathbf{T} & \textit{if } \varepsilon_3(\mathcal{E}_2\mathcal{E}_3^{a_2}\mathbf{T}) = 0, \\ \mathcal{F}_3^{a_2}\mathcal{E}_2\mathcal{E}_3^{a_2}\mathbf{T} & \textit{if } \varepsilon_3(\mathcal{E}_2\mathcal{E}_3^{a_2}\mathbf{T}) = 1. \end{cases} \tag{3.2}$$

Case 2. *Suppose that $\lfloor {}^{\mathtt{R}}T_2, T_1^{\mathtt{L}} \rfloor_{(0,d-1)}$ is $\mathcal{A}$-semistandard, but $\lfloor T_2^{\mathtt{R}}, {}^{\mathtt{L}}T_1 \rfloor_{(0,d-1)}$ is not. Then*

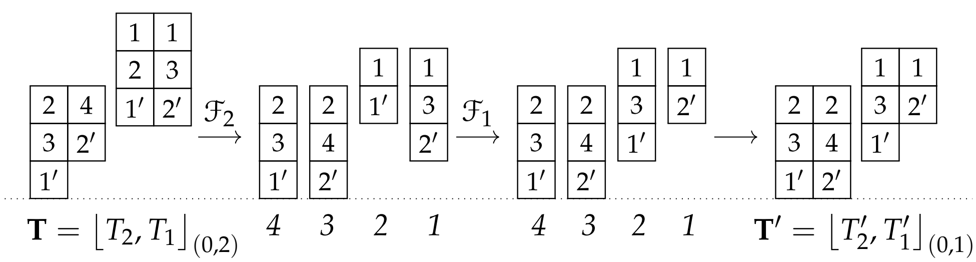
$$\mathbf{T}' := (U_4, U_3, U_2, U_1) = \mathcal{F}_1^{a_1+1}\mathcal{F}_2\mathcal{E}_1^{a_1}\mathbf{T}. \tag{3.3}$$

Example 3.3. *Suppose that* $\mathcal{A} = \mathbb{I}_{4|3}$.

(1) The following is an example of Case 1 *with* $\varepsilon_3(\mathcal{E}_2\mathcal{E}_3^{a_2}\mathbf{T}) = 0$.



(2) The following is an example of Case 2.



One can check that $\mathtt{jdt}_{spin}$ is compatible with the n-conjugate for a sufficiently large n, that is, $\overline{\mathtt{jdt}_{spin}(\mathbf{T})} = \mathtt{jdt}_{spin}(\overline{\mathbf{T}})$. Furthermore, $\mathtt{jdt}_{spin}$ can be viewed as a generalization of $\mathtt{jdt}_{KN}$ in the following sense.

$$\left(\mathtt{jdt}_{spin}(\overline{\mathbf{T}})\right)^{\mathrm{ad}} = \mathtt{jdt}_{KN}\left(\overline{\mathbf{T}}^{\mathrm{ad}}, c\right), \tag{3.4}$$

where $(\,\cdot\,)^{\mathrm{ad}}$ is given in (3.1), and c is the inner corner of $\overline{\mathbf{T}}^{\mathrm{ad}}$.

Now consider a general case. Let $\mathbf{T} = (T_\ell, \ldots, T_1) \in \mathbf{T}_{\mathcal{A}}(\lambda/\mu, \ell)$ be given. Let c be the inner corner of λ/μ in the i-th column from the right.

Let b be the inner corner of $\overline{\mathbf{T}}^{\mathrm{ad}}$ in the $(i+1)$-th column from the right, where $\overline{\mathbf{T}}$ is the n-conjugate of $\mathbf{T}$ for a sufficiently large n, and consider $\mathtt{jdt}_{KN}\left(\overline{\mathbf{T}}^{\mathrm{ad}}, b\right)$. By Definition 3.2 and (3.4), there exists a composite of operators $\mathcal{E}_i$ and $\mathcal{F}_i$, say $\mathcal{X}$, such that

$$\left(\mathcal{X}\,\overline{\mathbf{T}}\right)^{\mathrm{ad}} = \mathtt{jdt}_{KN}\left(\overline{\mathbf{T}}^{\mathrm{ad}}, b\right). \tag{3.5}$$

We remark that the composite $\mathcal{X}$ is independent of the choice of n.

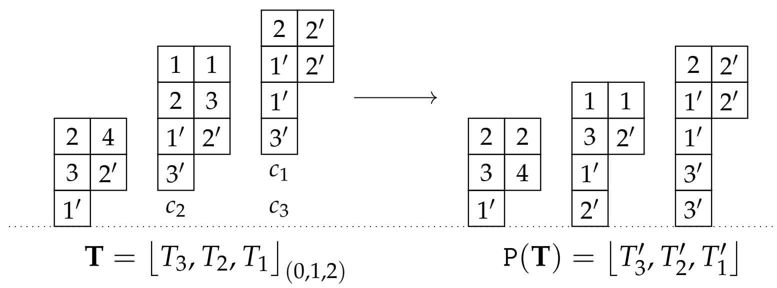
Definition 3.4 ([2, Definition 5.14]). *Under the above hypothesis, we define*

$$\mathtt{jdt}_{spin}(\mathbf{T}, c) = \mathcal{X}\,\mathbf{T}. \tag{3.6}$$

Theorem 3.5 ([2, Theorem 5.15]). *Let $\mathbf{T} = (T_\ell, \ldots, T_1) \in \mathbf{T}_{\mathcal{A}}(\lambda/\mu, \ell)$ be given. There exists a unique $\mathtt{P}(\mathbf{T}) \in \mathbf{T}_{\mathcal{A}}(\nu, \ell)$ for some $(\nu, \ell) \in \mathscr{P}(\mathrm{Sp})_{\mathcal{A}}$, which can be obtained from $\mathbf{T}$ by applying $\mathtt{jdt}_{spin}(\,\cdot\,, c)$ finitely many times with respect to inner corners c.*

Example 3.6. *Let $\mathbf{T} = \lfloor T_3, T_2, T_1 \rfloor_{(0,1,2)}$ be given as below. Then $\mathtt{P}(\mathbf{T})$ can be obtained by*

$$\mathtt{P}(\mathbf{T}) = \mathtt{jdt}_{spin}(\mathtt{jdt}_{spin}(\mathtt{jdt}_{spin}(\mathbf{T}, c_1), c_2), c_3) = \mathcal{E}_4\mathcal{F}_3\mathcal{F}_4\mathcal{E}_2\mathcal{E}_3\mathbf{T}.$$



4 Symplectic RSK correspondence

4.1 Oscillating tableaux and King tableaux

An oscillating tableau is a sequence of partitions $Q = (Q_1, \ldots, Q_s)$ for some $s \geq 1$ such that each pair (Q_i, Q_{i+1}) differs by one box for $1 \leq i \leq s-1$, i.e., $Q_i/Q_{i+1} = \square$ or $Q_{i+1}/Q_i = \square$. We say that $Q = (Q_1, \ldots, Q_s)$ is vertical if $Q_1 \subsetneq \cdots \subsetneq Q_r \supsetneq \cdots \supsetneq Q_s$ for some $1 \leq r \leq s$ and Q_r/Q_1 and Q_r/Q_s is a skew diagram of vertical strip. Here $A \subsetneq B \iff A \subseteq B$ and $A \neq B$. We denote by $|Q| = s$ the length of $Q = (Q_1, \ldots, Q_s)$.

Definition 4.1 ([2, Section 6.1]). *Let $(\lambda, \ell) \in \mathscr{P}(\mathrm{Sp})$ be given. For $n \geq \lambda_1$, define $\mathbf{O}(\lambda, \ell; n)$ to be a set of sequences of oscillating tableaux $Q = (Q^{(1)} : \cdots : Q^{(\ell)})$ such that*

- *Q is itself an oscillating tableau,*
- *$Q^{(i)} = (Q_{i,1}, \ldots, Q_{i,s_i})$ is a vertical oscillating tableau for $1 \leq i \leq \ell$,*
- *$\ell(Q_{i,j}) \leq n$ for $1 \leq i \leq \ell$ and $1 \leq j \leq s_i$,*
- *$Q_{1,1} = \square$ and $Q_{\ell,s_\ell} = \rho_n(\lambda, \ell)$.*

For $Q = (Q^{(1)} : \cdots : Q^{(\ell)}) \in \mathbf{O}(\lambda, \ell; n)$ with $|Q^{(i)}| = s_i$, let $\sigma(Q) = (\widehat{Q}^{(1)} : \cdots : \widehat{Q}^{(\ell)})$ be a sequence of oscillating tableaux $\widehat{Q}^{(i)}$ given as follows;

$$\widehat{Q}^{(i)} = ((i) \cup Q_{i-1,s_{i-1}}, (i) \cup Q_{i,1}, \ldots, (i) \cup Q_{i,s_i}) \quad (1 \leq i \leq \ell).$$

Here we denote by $(i) \cup Q_{i,k}$ the partition obtained by adding i to $Q_{i,k}$ as its first part. Then we show $\sigma(Q) \in \mathbf{O}(\lambda, \ell; n+1)$. Indeed, $\sigma : \mathbf{O}(\lambda, \ell; n) \longrightarrow \mathbf{O}(\lambda, \ell; n+1)$ is a bijection for $n \geq \lambda_1$, and induces an equivalence relation on $\bigsqcup_{n \geq \lambda_1} \mathbf{O}(\lambda, \ell; n) \times \{n\}$, where $(Q', m) \sim (Q, n)$ if and only if $\sigma^{m-n}(Q) = Q'$ for $Q' \in \mathbf{O}(\lambda, \ell; m)$ and $Q \in \mathbf{O}(\lambda, \ell; n)$ with $m \geq n$. We define

$$\mathbf{O}(\lambda, \ell) = \{\, [Q, n] \mid Q \in \mathbf{O}(\lambda, \ell; n) \ (n \geq \lambda_1) \,\},$$

where $[Q, n]$ is the equivalence class of $Q \in \mathbf{O}(\lambda, \ell; n)$ with respect to $\sim$. We call $[Q, n]$ an oscillating tableau of shape (λ, ℓ).

For $(\lambda, \ell) \in \mathscr{P}(\mathrm{Sp})$, let $\mathbf{K}(\lambda, \ell)$ be the set of tableaux of shape λ with letters in $\{1 < \overline{1} < \cdots < \ell < \overline{\ell}\}$, whose letters are of degree 0, such that all entries in the ith row are larger than or equal to i. It is known as the set of King tableaux of shape λ [8]. Based on [15], we construct a bijection between King tableaux and oscillating tableaux.

Theorem 4.2 ([15, Theorem 2.7], [2, Corollary 6.8])**.** *For $\lambda \subseteq (n^\ell)$, we have an explicit bijection*

$$\mathbf{K}(\lambda, \ell) \ \longrightarrow \ \mathbf{O}(\lambda, \ell). \tag{4.1}$$

4.2 Symplectic RSK correspondence

For $\mathbf{a} = (a_1, \ldots, a_\ell) \in \mathbb{Z}_+^\ell$, let $\mathbf{T}_{\mathcal{A}}(\mathbf{a}) = \mathbf{T}_{\mathcal{A}}(a_\ell) \times \cdots \times \mathbf{T}_{\mathcal{A}}(a_1)$, and $\mathbb{Z}/(\mathbf{a}+\mathbf{1})\mathbb{Z} = \mathbb{Z}/(a_\ell+1)\mathbb{Z} \times \cdots \times \mathbb{Z}/(a_1+1)\mathbb{Z}$, where $\mathbb{Z}/(a+1)\mathbb{Z}$ is understood as the set $\{0, 1, \ldots, a\}$ for $a \in \mathbb{Z}_+$, and let $\mathbf{F}^\ell_{\mathcal{A}} = \mathbf{E}^{2\ell}_{\mathcal{A}}$.

Lemma 4.3 ([2, Corollary 7.5])**.** *We have a bijection*

$$\begin{array}{ccc} \mathbf{F}^\ell_{\mathcal{A}} & \longrightarrow & \displaystyle\bigsqcup_{\mathbf{a}} \mathbf{T}_{\mathcal{A}}(\mathbf{a}) \times \mathbb{Z}/(\mathbf{a}+\mathbf{1})\mathbb{Z}\,, \\ \mathbf{T} & \longmapsto & (\mathcal{F}^{\max}\mathbf{T}, \varphi(\mathbf{T})) \end{array} \tag{4.2}$$

where

$$\begin{aligned} \mathcal{F}^{\max}\mathbf{T} &= (\mathcal{F}^{\max}(U_{2\ell}, U_{2\ell-1}), \ldots, \mathcal{F}^{\max}(U_2, U_1)), \\ \varphi(\mathbf{T}) &= (\varphi(U_{2\ell}, U_{2\ell-1}), \ldots, \varphi(U_2, U_1)), \end{aligned}$$

for $\mathbf{T} = (U_{2\ell}, \ldots, U_1) \in \mathbf{F}^\ell_{\mathcal{A}}$ and the union is over $\mathbf{a} \in \mathbb{Z}^\ell_+$ such that $\mathbf{T}_{\mathcal{A}}(\mathbf{a}) \neq \emptyset$.

Let $\mathbf{T} = (T_\ell, \ldots, T_1) \in \mathbf{T}_{\mathcal{A}}(\mathbf{a})$ be given. Let us define a recording tableau for $\mathtt{P}(\mathbf{T})$ (see Theorem 3.5). We assume that $\mathbf{T} \in \mathbf{T}_{\mathcal{A}}(\lambda/\mu, \ell)$ for some skew shape λ/μ. Let $\overline{\mathbf{T}}$ be the n-conjugate of $\mathbf{T}$ for a sufficiently large n. Then $\overline{\mathbf{T}}^{\mathrm{ad}}$ is a KN tableau of shape $\rho_{n+\mu_1}(\lambda, \ell)/\rho_{\mu_1}(\mu, \ell)$.

By using an analog of Robinson-Schensted correspondence for KN tableaux of type C [14, Theorem 5.2.2] and given $(c_1,\ldots,c_\ell) \in \mathbb{Z}/(\mathbf{a}+\mathbf{1})\mathbb{Z}$, we obtain an oscillating tableaux $Q(\mathbf{T};n)$ from $\overline{\mathbf{T}}^{\mathrm{ad}}$ such that $Q(\mathbf{T};n) \in \mathbf{O}(\lambda,\ell;n)$ and $\sigma(Q(\mathbf{T};n)) = Q(\mathbf{T};n+1)$ [2, Lemma 6.2]. Then we define

$$Q(\mathbf{T}) = [Q(\mathbf{T};n),n] \in \mathbf{O}(\lambda,\ell).$$

Theorem 4.4 ([2, Proposition 6.4, Theorem 7.2]). *We have a bijection*

$$\begin{array}{ccc} \displaystyle\bigsqcup_{\mathbf{a}\in\mathbb{Z}_+^\ell} \mathbf{T}_{\mathcal{A}}(\mathbf{a}) \times \mathbb{Z}/(\mathbf{a}+\mathbf{1})\mathbb{Z} & \longrightarrow & \displaystyle\bigsqcup_{(\lambda,\ell)\in\mathscr{P}(\mathrm{Sp})_{\mathcal{A}}} \mathbf{T}_{\mathcal{A}}(\lambda,\ell) \times \mathbf{O}(\lambda,\ell)\ . \\ (\mathbf{T},(c_1,\ldots,c_\ell)) & \longmapsto & (\mathtt{P}(\mathbf{T}),Q(\mathbf{T})) \end{array} \tag{4.3}$$

Now we consider the composition of the following sequence of bijections (4.2), (4.3), and (4.1)

$$\begin{aligned} \mathbf{F}_{\mathcal{A}}^{\ell} &\xrightarrow{(4.2)} \bigsqcup_{\mathbf{a}} \mathbf{T}_{\mathcal{A}}(\mathbf{a}) \times \mathbb{Z}/(\mathbf{a}+\mathbf{1})\mathbb{Z} \\ &\xrightarrow{(4.3)} \bigsqcup_{(\lambda,\ell)} \mathbf{T}_{\mathcal{A}}(\lambda,\ell) \times \mathbf{O}(\lambda,\ell) \xrightarrow{(4.1)} \bigsqcup_{(\lambda,\ell)} \mathbf{T}_{\mathcal{A}}(\lambda,\ell) \times \mathbf{K}(\lambda,\ell) \end{aligned}$$

and denote by $(\mathtt{P}(\mathbf{T}),\mathtt{Q}(\mathbf{T}))$ the image of $\mathbf{T} \in \mathbf{F}_{\mathcal{A}}^{\ell}$ under the above composition.

Theorem 4.5 ([2, Theorem 7.7]). *The map* $\mathbf{T} \mapsto (\mathtt{P}(\mathbf{T}),\mathtt{Q}(\mathbf{T}))$ *gives a bijection*

$$\mathbf{F}_{\mathcal{A}}^{\ell} \longrightarrow \bigsqcup_{(\lambda,\ell)\in\mathscr{P}(\mathrm{Sp})_{\mathcal{A}}} \mathbf{T}_{\mathcal{A}}(\lambda,\ell) \times \mathbf{K}(\lambda,\ell).$$

4.3 Cauchy identity

Let $\mathbf{z} = \{z_1,\ldots,z_\ell\}$ be formal commuting variables, which commutes with $\mathbf{x} = \mathbf{x}_{\mathcal{A}}$. Let $(\lambda,\ell) \in \mathscr{P}(\mathrm{Sp})$ be given. For $K \in \mathbf{K}(\lambda,\ell)$, let $\mathbf{z}^K = \prod_{i=1}^{\ell} z_i^{m_i - m_{\bar{i}}}$, where m_i (resp. $m_{\bar{i}}$) is the number of occurrences of i (resp. $\bar{i}$) in K. Then put

$$sp_\lambda(\mathbf{z}) = \sum_{K\in\mathbf{K}(\lambda,\ell)} \mathbf{z}^K.$$

It is well-known that $sp_\lambda(\mathbf{z})$ is the character of the irreducible highest weight module of $\mathrm{Sp}_{2\ell}$ with highest weight corresponding to λ [8].

Let $\mathbf{U} = (U_{2\ell},\ldots,U_1) \in \mathbf{F}_{\mathcal{A}}^{\ell}$ be given with the $\mathrm{ht}(U_i) = u_i$. Let $\mathbf{x}^{\mathbf{U}} = \prod_{i=1}^{2\ell} \mathbf{x}^{U_i}$ and $\mathbf{z}^{\mathbf{U}} = \prod_{i=1}^{\ell} z_i^{u_{2i}-u_{2i-1}}$. Then we have

$$\mathrm{ch}\mathbf{F}_{\mathcal{A}}^{\ell} := \sum_{\mathbf{U}} \mathbf{x}^{\mathbf{U}}\mathbf{z}^{\mathbf{U}} = \prod_{j=1}^{\ell} \frac{\prod_{a\in\mathcal{A}_0}(1+x_a z_j)(1+x_a z_j^{-1})}{\prod_{a\in\mathcal{A}_1}(1-x_a z_j)(1-x_a z_j^{-1})}.$$

Theorem 4.6 ([2, Theorem 7.9]). *We have the following identity*

$$t^{\ell}\prod_{j=1}^{\ell}\frac{\prod_{a\in\mathcal{A}_0}(1+x_az_j)(1+x_az_j^{-1})}{\prod_{a\in\mathcal{A}_1}(1-x_az_j)(1-x_az_j^{-1})}=\sum_{(\lambda,\ell)\in\mathscr{P}(\mathsf{Sp})_{\mathcal{A}}}S_{(\lambda,\ell)}(\mathbf{x}_{\mathcal{A}})sp_{\lambda}(\mathbf{z}). \tag{4.4}$$

We recover well-known identities when $\mathcal{A}$ is homogeneous. If $\mathcal{A}=\mathcal{A}_0$, then (4.4) is the identity [7, (6.19)]. If $\mathcal{A}=\mathcal{A}_1$ and $\ell\geq n$, then (4.4) and the stability of $\mathbf{T}_{\mathcal{A}}(\lambda,\ell)$ [10, Theorem 6.5] imply the following identity [16, 19]

$$\frac{1}{\prod_{i=1}^{n}\prod_{j=1}^{\ell}(1-x_iz_j)(1-x_iz_j^{-1})}=\sum_{\ell(\lambda)\leq n}sp_{\lambda}(\mathbf{z})s_{\lambda}(\mathbf{x})\prod_{1\leq i<j\leq n}(1-x_ix_j)^{-1}, \tag{4.5}$$

where $s_\lambda(\mathbf{x})$ is the Schur polynomial in $x_1,\ldots,x_n$, and $\ell(\lambda)$ is the length of λ.

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