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Subdivisions of Shellable Complexes

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Abstract. This extended abstract is a summary of a recent paper which studies the enumeration of faces of subdivisions of cell complexes. Motivated by a conjecture of Brenti and Welker on the real-rootedness of the h-polynomial of the barycentric subdivision of the boundary complex of a convex polytope, we introduce a framework for proving real-rootedness of h-polynomials for subdivisions of polytopal complexes by relating interlacing polynomials to shellability via the existence of so-called *stable shellings*. We show that any shellable cubical, or simplicial, complex admitting a stable shelling has barycentric and edgewise subdivisions with real-rooted h-polynomials. Such shellings are shown to exist for well-studied families of cubical polytopes, giving a positive answer to the conjecture of Brenti and Welker in these cases. The framework of stable shellings is also applied to answer to a conjecture of Mohammadi and Welker on edgewise subdivisions in the case of shellable simplicial complexes.

Keywords: polytope, shellability, real-rooted, unimodality, barycentric subdivision, edgewise subdivision

1 Introduction

Currently, there is interest in the inequalities that can be shown to hold amongst the coefficients of a *generating polynomial* $p = p_0 + p_1x + \cdots + p_dx^d$ where the sequence $p_0, \ldots, p_d$ encodes algebraic, geometric, and/or topological data [8, 11, 10, 24]. For instance, p is called *unimodal* if $p_0 \leq \cdots \leq p_t \geq \cdots \geq p_d$ for some $t \in \{0\} \cup [d]$, and it is called *real-rooted* (or *(real) stable*) if $p \equiv 0$ or p has only real zeros. A classic result states that p is unimodal whenever it is real-rooted [10, Theorem 1.2.1]. Most proofs of real-rootedness rely on interlacing polynomials [8], which are inherently tied to recursions associated to the generating polynomials of interest.

In algebraic, geometric, and topological combinatorics, the generating polynomials studied are typically the f- or h-polynomial associated to a cell complex. A foundational

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result in the field, known as the g-theorem, implies that the h-polynomial associated to the boundary complex of a simplicial polytope is unimodal [24]. In [12], Brenti and Welker strengthened this unimodality result by showing that the h-polynomial of the barycentric subdivision of the boundary complex of a simplicial polytope is real-rooted. They then asked if their result generalizes to all polytopes [12, Question 1].

While proofs of real-rootedness via interlacing polynomials typically rely on polynomial recursions, proofs pertaining to the geometry of polytopal complexes often use the recursive structure of the complex (when it exists). This recursive property of polytopal complexes is called *shellability*. Here, shellability is related to interlacing polynomials so as to provide a framework for proving real-rootedness of h-polynomials of subdivisions of polytopal complexes. It is demonstrated that the existence of special shelling orders of a polytopal complex implies real-rootedness of the h-polynomial of a given subdivision of the complex. It is shown that a family of shellings, termed *stable shellings*, have this property for simplicial and cubical complexes. Moreover, these stable shellings can be applied to different subdivisions of the same complex, yielding multiple real-rootedness results simultaneously. As one application of this framework, we positively answer the question of Brenti and Welker for well-known families of cubical polytopes; namely, the *cuboids* [17], the *capped cubical polytopes* [20], and the *neighborly cubical polytopes* [4]. We also apply stable shellings to a problem proposed by Mohammadi and Welker [22] pertaining to edgewise subdivisions of simplicial complexes. Moreover, the stable shelling approach also yields a positive answer to their question in the case of cubical complexes. See the full paper [18] for proofs of these results.

2 Preliminaries

This section contains the basic definitions and results on polytopal complexes and interlacing polynomials needed to follow the remainder of the paper. Experts can likely skip this section, using it mainly as a reference for notation. A finite collection $\mathcal{C}$ of polytopes in $\mathbb{R}^n$ is called a *polytopal complex* if the empty polytope $\emptyset$ is in $\mathcal{C}$, if when $P \in \mathcal{C}$ then all faces of P are also in $\mathcal{C}$, and if when $P, Q \in \mathcal{C}$, then their intersection $P \cap Q$ is a face of both P and Q. The elements of $\mathcal{C}$ are called its *faces*, and its maximal faces (with respect to inclusion) are called its *facets*. When all facets of $\mathcal{C}$ are simplices, $\mathcal{C}$ is a *simplicial complex*, and when all facets are cubes, it is a *cubical complex*. Given a polytopal complex we can then work with its associated abstract cell complex (forgetting the embedding), whose faces are abstract polytopes. Given an abstract polytope P, or convex polytope $P \subset \mathbb{R}^n$, $\mathcal{C}(P)$ denotes the complex consisting of all faces in P and $\mathcal{C}(\partial P)$ is the complex of all faces in ∂P, the boundary of P. The *facets* of the polytope P are the facets of the complex $\mathcal{C}(\partial P)$. Given a collection of polytopes $P_1, \ldots, P_m$, let $\mathcal{C}(P_1 \cup \cdots \cup P_m) = \cup_{i \in [m]} \mathcal{C}(P_i)$. We refer to the difference $\mathcal{C} \setminus \mathcal{D} = \{P \in \mathcal{C} : P \notin \mathcal{D}\}$ as a *relative (polytopal) complex*, and we define the *dimension* of $\mathcal{C} \setminus \mathcal{D}$ to be the largest dimension of a polytope in $\mathcal{C} \setminus \mathcal{D}$. When $\mathcal{D} = \emptyset$, note that $\mathcal{C} \setminus \mathcal{D} = \mathcal{C}$.

The *f-polynomial* of a $(d-1)$-dimensional polytopal complex $\mathcal{C}$ is the polynomial

$$f(\mathcal{C};x) := f_{-1}(\mathcal{C}) + f_0(\mathcal{C})x + f_1(\mathcal{C})x^2 + \cdots + f_{d-1}(\mathcal{C})x^d,$$

where $f_{-1}(\mathcal{C}) := 1$ when $\mathcal{C} \neq \emptyset$ and $f_k(\mathcal{C})$ denotes the number of k-dimensional faces of $\mathcal{C}$ for $0 \leq k \leq d-1$. Given a subcomplex $\mathcal{D}$ of $\mathcal{C}$, the *f-polynomial* of the relative complex $\mathcal{C} \setminus \mathcal{D}$ is then

$$f(\mathcal{C} \setminus \mathcal{D};x) := f(\mathcal{C};x) - f(\mathcal{D};x).$$

The *h-polynomial* of the $(m-1)$-dimensional relative complex $\mathcal{C} \setminus \mathcal{D}$ is the polynomial

$$h(\mathcal{C} \setminus \mathcal{D};x) := (1-x)^m f\left(\mathcal{C} \setminus \mathcal{D}; \frac{x}{1-x}\right).$$

We write $h(\mathcal{C} \setminus \mathcal{D};x) = h_0(\mathcal{C} \setminus \mathcal{D}) + h_1(\mathcal{C} \setminus \mathcal{D})x + \cdots + h_m(\mathcal{C} \setminus \mathcal{D})x^m$ when expressing $h(\mathcal{C} \setminus \mathcal{D};x)$ in the standard basis, and we similarly write $f(\mathcal{C} \setminus \mathcal{D};x) = f_0(\mathcal{C} \setminus \mathcal{D}) + f_1(\mathcal{C} \setminus \mathcal{D})x + \cdots + f_m(\mathcal{C} \setminus \mathcal{D})x^m$.

Let $\mathcal{C}$ be a pure d-dimensional polytopal complex. A *shelling* of $\mathcal{C}$ is a linear ordering $(F_1, F_2, \ldots, F_s)$ of the facets of $\mathcal{C}$ such that either $\mathcal{C}$ is zero-dimensional (and thus the facets are points), or it satisfies the following two conditions:

1. The boundary complex $\mathcal{C}(\partial F_1)$ of the first facet in the linear ordering has a shelling, and

2. for $j \in [s]$, the intersection of the facet F_j with the union of the previous facets is nonempty and it is the beginning segment of a shelling of the $(d-1)$-dimensional boundary complex of F_j; that is,

$$F_j \cap \bigcup_{i=1}^{j-1} F_i = G_1 \cup G_2 \cup \cdots G_r$$

   for some shelling $(G_1, \ldots, G_r, \ldots, G_t)$ of the complex $\mathcal{C}(\partial F_j)$ and $r \in [t]$.

A polytopal complex is *shellable* if it is pure and admits a shelling.

Given a polytopal complex $\mathcal{C}$, a *(topological) subdivision* of $\mathcal{C}$ is a polytopal complex $\mathcal{C}'$ such that each face of $F \in \mathcal{C}$ is subdivided into a ball by faces of $\mathcal{C}'$ such that the boundary of this ball is a subdivision of the boundary of F. The subdivision is further called *geometric* if both $\mathcal{C}$ and $\mathcal{C}'$ admit *geometric realizations*, G and G', respectively; that is to say, each face of $\mathcal{C}$ and $\mathcal{C}'$ is realized by a convex polytope in some real-Euclidean space such that G and G' both have the same underlying set of vertices and each face of G' is contained in a face of G. When referring to a subdivision $\mathcal{C}'$ of $\mathcal{C}$, we may instead refer to its associated inclusion map $\varphi : \mathcal{C}' \to \mathcal{C}$. While the main result of this paper applies to general topological subdivisions, the applications of these results will pertain to some special families of subdivisions that are well-studied in the literature. These include the barycentric subdivision and the edgewise subdivision of a complex.

2.1 Interlacing polynomials

Two real-rooted polynomials $p, q \in \mathbb{R}[x]$ are said to *interlace* if there is a zero of p between each pair of zeros of q (counted with multiplicity) and vice versa. If p and q are interlacing, it follows that the *Wronskian* $W[p,q] = p'q - pq'$ is either nonpositive or nonnegative on all of $\mathbb{R}$. We will write $p \prec q$ if p and q are real-rooted, interlacing, and the Wronskian $W[p,q]$ is nonpositive on all of $\mathbb{R}$. We also assume that the zero polynomial 0 is real-rooted and that $0 \prec p$ and $p \prec 0$ for any real-rooted polynomial p. In this work, we will use the fact that $p \prec q$ is equivalent to p and q being real-rooted and interlacing when the leading coefficients of p and q are both positive.

Let $(p_i)_{i=0}^s = (p_0, \ldots, p_s)$ be a sequence of real-rooted polynomials. We say that the sequence of polynomials $(p_i)_{i=0}^s$ is an *interlacing sequence* if $p_i \prec p_j$ for all $1 \leq i \leq j \leq s$. Any convex combination of polynomials in an interlacing sequence is real-rooted [7, Lemma 2.6]. For a polynomial $p \in \mathbb{R}[x]$ of degree at most d, we let $\mathcal{I}_d(p) := x^d p(1/x)$. In [9], it is characterized when a degree d real-rooted polynomial satisfies $\mathcal{I}_d(p) \prec p$.

3 Stable Shellings of Polytopal Complexes

A shelling of a polytopal complex presents a natural way to decompose the complex into disjoint, relative polytopal complexes. Given a shelling order $(F_1, \ldots, F_s)$ of a polytopal complex $\mathcal{C}$, and a subdivision $\varphi : \mathcal{C}' \to \mathcal{C}$, we can let

$$\mathcal{R}_i := \mathcal{C}'\big|_{F_i} \setminus \left(\bigcup_{k=1}^{i-1} \mathcal{C}'\big|_{F_k} \right),$$

to produce the decomposition of $\mathcal{C}'$ into disjoint relative complexes $\mathcal{C}' = \bigsqcup_{i=1}^s \mathcal{R}_i$, with respect to the shelling $(F_1, \ldots, F_s)$ of $\mathcal{C}$. For a fixed shelling $(F_1, \ldots, F_s)$ of a polytopal complex $\mathcal{C}$ and subdivision $\varphi : \mathcal{C}' \to \mathcal{C}$, for $i \in [s]$, we call the relative complex $\mathcal{R}_i$ the *relative complex associated to F_i by $(F_1, \ldots, F_s)$ and φ*. If $\mathcal{C}$ is $(d-1)$-dimensional, then so is each $\mathcal{R}_i$. Hence, $h(\mathcal{C}'; x) = \sum_{i=1}^s h(\mathcal{R}_i; x)$. The additive nature of the h-polynomials of their associated relative simplicial complexes pairs nicely with the properties of interlacing polynomials discussed in Subsection 2.1, leading to the following observation.

Theorem 3.1. *Let $\mathcal{C}$ be a shellable polytopal complex with shelling $(F_1, \ldots, F_s)$ and subdivision $\varphi : \mathcal{C}' \to \mathcal{C}$. If $(h(\mathcal{R}_{\sigma(i)}; x))_{i=1}^s$ is an interlacing sequence for some permutation $\sigma \in \mathfrak{S}_s$, then $h(\mathcal{C}'; x)$ is real-rooted.*

A proof of Theorem 3.1 can be derived directly from the observations in Subsection 2.1. To apply Theorem 3.1, our goal becomes to find the right shelling of our complex for a given subdivision. Ideally, we would identify a single shelling that applies to numerous subdivisions, thereby yielding many real-rootedness results simultaneously. To this end, we introduce the family of *stable shellings*, which are defined via a generalization of *reciprocal domains* for convex embeddings of polytopes, as studied by Ehrhart [15, 16] and Stanley [23]. We then apply Theorem 3.1 to show that the existence of a stable shelling implies real-rootedness of, not just one but, multiple subdivisions of cubical and simplicial complexes.

Let $q \in \mathbb{R}^n$, and let $P \subset \mathbb{R}^n$ a d-dimensional convex polytope. A point $p \in P$ is called *visible* from q if the open line segment (q, p) in $\mathbb{R}^n$ does not meet the interior of P. Let $B \subset \partial P$ denote the collection of all points visible from q, and set $D := \overline{\partial P \setminus B}$, the closure of $\partial P \setminus B$. Given a facet F of P, the point q is said to be *beyond* F if $q \notin T_F(P)$, the tangent cone of F in P. It follows that q is beyond F if and only if the closed line segment $[q, p]$ satisfies $[q, p] \cap P = \{p\}$ for all $p \in F$ [6, Section 3.7]. Otherwise, the point q is said to be *beneath* F. Hence, B consists of all points in ∂P that lie in a facet which q is beyond; that is, $P \setminus B = P \setminus \mathcal{H}_B$, where $\mathcal{H}_B$ denotes the collection of facets which q is beyond. Similarly, D consists of all points in ∂P that lie in a facet which q is beneath; that is, $P \setminus D = P \setminus \mathcal{H}_D$, where $\mathcal{H}_D$ denotes the collection of facets which q is beneath. In [15, 16], Ehrhart referred to the half-open polytopes $P \setminus B$ and $P \setminus D$ as *reciprocal domains*.

Let $\mathcal{P} = ([n], <_{\mathcal{P}})$ be a partially ordered set on elements $[n]$ with partial order $<_{\mathcal{P}}$. If $\mathcal{P}$ has a unique minimal element, which we will denote by $\hat{0}$, we can define its set of *atoms* to be all elements $i \in [n]$ such that $\hat{0} <_{\mathcal{P}} i$ and there is no $j \in [n]$ for which $\hat{0} <_{\mathcal{P}} j <_{\mathcal{P}} i$. Given a poset $\mathcal{P}$ with a unique minimal element, we will denote its set of atoms by $A(\mathcal{P})$. The *dual poset* of $\mathcal{P}$ is the poset $\mathcal{P}^*$ on elements $[n]$ with partial order $<_{\mathcal{P}^*}$ in which $i <_{\mathcal{P}^*} j$ if and only if $j <_{\mathcal{P}} i$. Given $i, j \in [n]$, the *closed interval* between i and j in $\mathcal{P}$ is the set $[i, j] := \{k \in [n] : i \leq_{\mathcal{P}} k \leq_{\mathcal{P}} j\}$. Note that we can view the closed interval $[i, j]$ as a subposet of $\mathcal{P}$ by allowing it to inherit the partial order $<_{\mathcal{P}}$ from $\mathcal{P}$.

Let P be a d-dimensional polytope with face lattice $L(P)$; that is, $L(P)$ is the partially ordered set whose elements are the faces of P and for which the partial order is given by inclusion. Since $L(P)$ is a *lattice* (see [25, Chapter 3.3]), it follows that $L(P)$ has a unique minimal and maximal element, corresponding to the faces $\emptyset$ and P of P. Let $\mathcal{C}(P)$ denote the polytopal complex consisting of all faces of P. Given a face F of P we call the pair of relative complexes

$$\mathcal{C}(P) \setminus \mathcal{C}(A([F, P]^*)) \qquad \text{and} \qquad \mathcal{C}(P) \setminus \mathcal{C}(A(L(P)^*) \setminus A([F, P]^*))$$

the *reciprocal domains* associated to F in P. We call a relative complex $\mathcal{R}$ *stable* if it is isomorphic to one of the reciprocal domains associated to a face F in a polytope P, for some polytope P. Using this terminology, we can now define the family of shellings, to which we will apply Theorem 3.1.

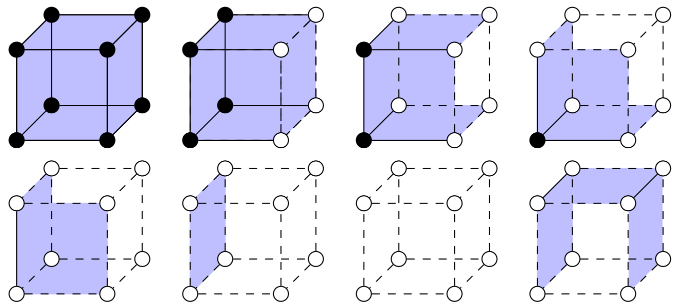

Figure 1: The eight possible relative complexes $\mathcal{R}_i$ for a facet F_i in a shelling order $(F_1, \ldots, F_s)$ if F_i is a 3-dimensional cube. All of the complexes are stable, excluding the bottom-right complex.

Definition 3.1. Let $\mathcal{C}$ be a polytopal complex. A shelling $(F_1, \ldots, F_s)$ of $\mathcal{C}$ is *stable* if for all $i \in [s]$ the relative complex $\mathcal{R}_i$ associated to F_i by the shelling $(F_1, \ldots, F_s)$ and the trivial subdivision $\varphi : \mathcal{C} \to \mathcal{C}$ is stable.

In Section 4, we demonstrate that the existence of a stable shelling can imply real-rootedness of the h-polynomial for numerous subdivisions of a given complex. Any shelling of a simplicial complex is stable. This observation coincides with our desired goal given the recent results on $\mathcal{F}$-uniform triangulations of simplicial complexes [3].

3.1 Stable shellings of cubical complexes

Unlike simplicial complexes, it is not the case that all shellings of a general polytopal complex are stable. Already in the case of cubical complexes, stable shellings become a proper subclass of the class of all shellings. Let $\square_d$ denote the (abstract) d-dimensional cube. When we consider a standard geometric realization of $\square_d$, such as the cube $[-1,1]^d \in \mathbb{R}^d$, we assign each facet F of $\square_d$ to a facet-defining hyperplane $x_i = \pm 1$ of $[-1,1]^d$. Given a facet F of $\square_d$ we will say that F is *opposite (or opposing)* the facet G of $\square_d$ whenever their vertex sets are disjoint. In this case, we call the pair of facets F, G an *opposing pair*. The following lemma gives a characterization of stable relative subcomplexes of the cube in terms of opposing pairs of facets.

Lemma 3.2. *Let $\square_d$ be the d-dimensional (combinatorial) cube, and let $\mathcal{D}$ be a subcomplex of $\mathcal{C}(\square_d)$. Then the following are equivalent:*

1. *The relative complex $\mathcal{C}(\square_d) \setminus \mathcal{D}$ is stable,*
2. *The set of codimension one faces of $\mathcal{C}(\square_d) \setminus \mathcal{D}$ or the set of facets of $\mathcal{D}$ does not contain an opposing pair.*

Already in three dimensions there exist relative complexes that are not stable, and hence are forbidden from being associated to a facet of a cubical complex in any stable shelling. For example, the eighth relative complex in Figure 1, with its table-top shape, is such that both the codimension one faces of $\mathcal{C}(\square_3) \setminus \mathcal{D}$ (depicted in blue) and the facets of $\mathcal{D}$ (depicted by their absence) contain an opposing pair. Given that not all relative subcomplexes of the cube are stable, it is natural to ask which cubical complexes admit stable shellings.

Question 3.1. Does the boundary complex of every polytope (cubical or otherwise) admit a stable shelling?

In the following section, we will observe that, for any cubical (or simplicial) complex, real-rootedness of h-polynomials for multiple uniform subdivisions is implied by the existence of a stable shelling. This suggests that the family of stable shellings is a good one for applying Theorem 3.1. As one of the subdivisions to which our results will apply is the barycentric subdivision, proving the existence of a stable shelling may well suffice to prove the conjecture of Brenti and Welker for general polytopes.

4 Applications to Cubical and Simplicial Complexes

In this section, we show that if a cubical or simplicial complex admits a stable shelling then the h-polynomial of both its barycentric subdivision and any of its edgewise subdivisions is real-rooted. The real-rootedness results will follow from an application of Theorem 3.1, suggesting that the existence of a stable shelling is enough to imply real-rootedness of the h-polynomial of the subdivided complex. As applications of these results, we positively answer a question of Brenti and Welker [12, Question 1] for well-known constructions of *cubical polytopes* (i.e., polytopes whose facets are all cubes) by showing that these polytopes admit a stable shelling. In its most general form, the question is as follows:

Problem 4.1 ([12, Question 1])**.** *Let $\mathcal{C}$ be the boundary complex of an arbitrary polytope. Is the h-polynomial of the barycentric subdivision of $\mathcal{C}$ real-rooted?*

We also apply these techniques to the edgewise subdivision of simplicial and cubical complexes so as to solve a second problem of Mohammadi and Welker [22, Problem 27] for shellable complexes.

Problem 4.2 ([22, Problem 27])**.** *If $\mathcal{C}$ is a d-dimensional simplicial complex with $h_k(\mathcal{C}) \geq 0$ for all $0 \leq k \leq d+1$, is $h(\mathcal{C}^{\langle r \rangle}; x)$ real-rooted whenever $r > d$?*

4.1 The Barycentric and Edgewise Subdivisions

We now recall the definitions of the subdivisions to be considered. Given a polytopal complex $\mathcal{C}$, let $L(\mathcal{C})$ denote its face lattice with partial order $<_{\mathcal{C}}$ given by inclusion. The *barycentric subdivision* of $\mathcal{C}$ is the simplicial complex $\mathrm{sd}(\mathcal{C})$ whose k-dimensional faces are the subsets $\{F_0, F_1, \ldots, F_k\}$ of faces of $\mathcal{C}$ for which

$$\emptyset <_{\mathcal{C}} F_0 <_{\mathcal{C}} F_1 <_{\mathcal{C}} \cdots <_{\mathcal{C}} F_k$$

is a strictly increasing chain in $L(\mathcal{C})$.

The edgewise subdivision of a simplicial complex is another well-studied subdivision that arises frequently in algebraic and topological contexts (see for instance [13]). Within algebra, it is intimately tied to the Veronese construction, and it is considered to be the algebraic analogue of barycentric subdivision [13, Acknowledgements]. For $r \geq 1$, the r^{th} edgewise subdivision of a simplicial complex $\mathcal{C}$ is defined as follows: Identify the n 0-dimensional faces of $\mathcal{C}$ with the standard basis vectors in $\mathbb{R}^n$, and consider the lattice points in the r^{th} dilation of the standard simplex $\Delta := \mathrm{conv}(e^{(1)}, \ldots, e^{(n)}) \subset \mathbb{R}^n$. For $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$, we let $\mathrm{supp}(x) := \{i \in [n] : x_i \neq 0\}$, and we define the linear transformation $\iota : \mathbb{R}^n \to \mathbb{R}^n$ by $\iota : x \longmapsto (x_1, x_1 + x_2, \ldots, x_1 + \cdots + x_n)$. The r^{th} *edgewise subdivision* of $\mathcal{C}$ is the simplicial complex $\mathcal{C}^{\langle r \rangle}$ whose set of 0-dimensional faces are the lattice points in $r\Delta \cap \mathbb{Z}^n$ and for which $F \subset r\Delta \cap \mathbb{Z}^n$ is a face of $\mathcal{C}^{\langle r \rangle}$ if and only if $\bigcup_{x \in F} \mathrm{supp}(x) \in \mathcal{C}$, and for all $x, y \in F$ either $\iota(x) - \iota(y) \in \{0,1\}^n$ or $\iota(y) - \iota(x) \in \{0,1\}^n$.

Given a simplicial complex $\mathcal{C}$, the r^{th} *edgewise subdivision* of $\mathcal{C}$ can also be seen as gluing together the r^{th} edgewise subdivisions of each of its facets. This can be extended in a natural way to cubical complexes. Let $\Box_d$ denote the (abstract) d-dimensional cube, and consider its geometric realization $[0,1]^d$ and $r[0,1]^d = [0,r]^d$, the r^{th} dilation of $[0,1]^d$. The map ι then sends $(x_1, \ldots, x_d) \in [0,r]^d \cap \mathbb{Z}^d$ to $(x_1, x_1 + x_2, \ldots, x_1 + \cdots + x_d)$. We define the r^{th} *edgewise subdivision* $\Box_d^{\langle r \rangle}$ of the d-dimensional cube in terms of a subdivision of its geometric realization $[0,1]^d$ as follows: Let $A \subset [0,r]^d \cap \mathbb{Z}^d$. Then $\mathrm{conv}(A)$ is a face of the subdivision if and only if $\iota(v - v')$ or $-\iota(v - v')$ is in $\{0,1\}^d$ for all $v, v' \in A$. We first note that this a unimodular triangulation of $[0,1]^d$, as it splits the dilated cube $[0,r]^d$ into unit cubes which are each triangulated according to (a rotated version of) the standard unimodular triangulation of $[0,1]^d$; that is, the triangulation induced by the hyperplanes $x_i = x_j$ for all $1 \leq i < j \leq d$. Given a cubical complex $\mathcal{C}$, its r^{th} *edgewise subdivision*, denoted $\mathcal{C}^{\langle r \rangle}$ is given by gluing together the r^{th} edgewise subdivisions of each of its facets. A priori, such a subdivision may not exist for an arbitrary cubical complex. While we leave the general existence question open, we note that such a gluing can be seen to exist for cubical subcomplexes of an n-cube by extending the definition of the r^{th} edgewise subdivision of a simplicial complex accordingly. Additionally, such a gluing also exists for many examples of cubical complexes that are not subcomplexes of a cube, including the complexes listed in Theorem 4.9.

4.2 Stable shellings and real-rootedness

We now show that any simplicial or cubical complex that admits a stable shelling will have real-rooted barycentric and edgewise subdivisions. To do so, we will make use of some well-studied real-rooted polynomials, which can be defined as follows: For $d, r \geq 1$ and $0 \leq \ell \leq d$, let $A_{d,\ell}^{(r)}$ be the polynomial defined by the relation

$$\sum_{t \geq 0} (rt)^{\ell}(rt+1)^{d-\ell} x^t = \frac{A_{d,\ell}^{(r)}}{(1-x)^{d+1}}. \tag{4.1}$$

We call $A_{d,\ell}^{(r)}$ the d^{th} *r-colored ℓ-Eulerian polynomial*. The following lemma will play a key role.

Lemma 4.3. *For $d, r \geq 1$ and $0 \leq \ell \leq d$, the polynomial $A_{d,\ell}^{(r)}$ has only simple, real zeros. Moreover, for a fixed $d, r \geq 1$, $\left(A_{d,\ell}^{(r)}\right)_{\ell=0}^{d}$ is an interlacing sequence.*

To establish the desired results it suffices, by Theorem 3.1, to show that the relative complexes used in a stable shelling of a cubical or simplicial complex form an interlacing sequence. Using techniques in Ehrhart theory and results from [5] and [9], the following can be deduced:

Lemma 4.4. *Let $\square_d$ be the d-dimensional cube, and let $\mathcal{D}$ be a subcomplex of $\mathcal{C}(\square_d)$ such that $\mathcal{C}(\square_d)\backslash\mathcal{D}$ is stable relative complex. Then $h(\mathrm{sd}(\mathcal{C}(\square_d)) \setminus \mathrm{sd}(\mathcal{D}), x)$ is either $A_{d,\ell}^{(2)}$ or $x\mathcal{I}_d A_{d,\ell}^{(2)}$ for some $0 \leq \ell \leq d$. Moreover, these polynomials form an interlacing sequence.*

In the case of the r^{th} edgewise subdivision of the cube, we can deduce the following.

Lemma 4.5. *Let $\square_d$ be the d-dimensional cube, and let $\mathcal{D}$ be a subcomplex of $\mathcal{C}(\square_d)$ such that $\mathcal{C}(\square_d)\backslash\mathcal{D}$ is stable relative complex. Then $h(\mathcal{C}(\square_d)^{\langle r\rangle}\backslash\mathcal{D}^{\langle r\rangle}; x)$ is either $A_{d,\ell}^{(r)}$ or $x\mathcal{I}_d A_{d,\ell}^{(r)}$ for some $0 \leq \ell \leq d$. Moreover, these polynomials form an interlacing sequence.*

Similarly, in the case of simplicial complexes, one can prove the following.

Lemma 4.6. *Let Δ_{d-1} be a $(d-1)$-dimensional simplex, and let $\mathcal{D}$ be a $(d-2)$-dimensional subcomplex of $\mathcal{C}(\Delta_{d-1})$ with ℓ facets. Then, $h(\mathrm{sd}(\Delta_{d-1}) \setminus \mathrm{sd}(\mathcal{D}); x) = A_{d,\ell}^{(1)}$.*

As for the edgewise subdivision of a simplex, we need a different family of polynomials than the colored Eulerian polynomials for which we can deduce an analogous result. The necessary family of polynomials is described in Section 4.3 of the full paper summarized by this extended abstract, [18].

Lemma 4.7. *Let Δ_{d-1} be a $(d-1)$-dimensional simplex and $r > d$. The polynomials $h(\Delta_{d-1}^{\langle r\rangle} \setminus \mathcal{D}^{\langle r\rangle}, x)$, for all $(d-2)$-dimensional subcomplexes $\mathcal{D}$ of $\mathcal{C}(\Delta_{d-1})$, form an interlacing sequence.*

By combining Lemmas 4.3, 4.4, 4.5, 4.6, and 4.7 together with Theorem 3.1, we recover following theorem.

Theorem 4.8. *Let $\mathcal{C}$ be a cubical complex and $\mathcal{S}$ a simplicial complex of dimension d, and suppose $\mathcal{C}$ and $\mathcal{S}$ each admit a stable shelling. Then the following polynomials are real-rooted:*

1. $h(\mathrm{sd}(\mathcal{C});x)$,
2. $h(\mathrm{sd}(\mathcal{S});x)$,
3. $h(\mathcal{C}^{\langle r\rangle};x)$ *for* $r \geq 2$ *(if* $\mathcal{C}^{\langle r\rangle}$ *exists), and*
4. $h(\mathcal{S}^{\langle r\rangle};x)$ *for* $r > d$.

Since all shellings of a simplicial complex are stable, Theorem 4.8 (4) gives a positive answer to Problem 4.2 for shellable simplicial complexes. Moreover, Theorem 4.8 (3) shows that the claim of Problem 4.2 can shown to hold for more general complexes than simplicial complexes.

By a theorem of Bruggesser and Mani [14], the boundary complex of a polytope is always shellable and, in the case that the polytope is simplicial, it therefore always has a stable shelling. Hence, Theorem 4.8 (2) offers an alternative, geometric, proof of Brenti and Welker's original result that motivated Problem 4.1. Whereas Theorem 4.8 (1) gives a positive answer to Problem 4.1 for a new family of complexes, namely the boundary complexes of cubical polytopes admitting stable shellings. Indeed, it can be show that such complexes exist. The most well-known constructions of cubical polytopes are the *cuboids*, which were first introduced by Grünbaum in [17], the *capped cubical polytopes* [20], which are a cubical generalization of stacked simplicial polytopes [21], and the *neighborly cubical polytopes* [4]. Each of these constructions can be shown to admit a stable shelling.

Theorem 4.9. *The following cubical complexes admit stable shellings:*

1. *the boundary complex of a cuboid,*
2. *the boundary complex of a capped cubical polytope,*
3. *the boundary complex of a neighborly cubical polytope, and*
4. *any pile of cubes.*

It follows from Theorem 4.8 that the polynomials $h(\mathrm{sd}(\mathcal{C});x)$ and $h(\mathcal{C}^{\langle r\rangle};x)$ for $r \geq 2$ are real-rooted for any of the complexes listed in Theorem 4.9. Hence, the existence of a stable shelling positively answers Problem 4.1 in each of these cases, while also yielding additional real-rootedness results for other subdivisions.

5 Discussion and Future Directions

Since [18], the paper summarized by this extended abstract was released, Athanasiadis [2] gave a positive answer to Problem 4.1 for all cubical polytopes, the proof of which made use of cubical h-polynomials and the nonnegativity of their coefficients. While this result generalizes the results in Theorem 4.8 (1), the machinery used in the proof of [2] does not exist for arbitrary polytopes. On the other hand, a result of Bruggesser and Mani [14] states that the boundary complex of every polytope admits a so-called *line shelling*. Hence, a natural question (given the results mentioned above) is whether or not all boundary complexes of polytopes admit a stable line shelling. In general, it would be interesting to see other families of polytopes (i.e. non-cubical and non-simplicial) that admit stable shellings, and whether the existence of such shellings also implies the same real-rootedness results for these polytopes. It could also be interesting to continue to explore whether other classes of cubical polytopes admit a stable shelling, for e.g. the constructions in [1]. In a related direction, it would also be interesting to know if the existence of stable shellings of cubical and simplicial polytopes implies real-rootedness for other types of subdivisions (i.e., other than the barycentric and edgewise subdivisions). Finally, we note that some recent results of [19] can be applied to give a full answer to Problem 4.2 for simplicial complexes. As we saw above however, Theorem 4.8 (3) demonstrates that the claim of Problem 4.2 also holds for shellable cubical complexes. It would interesting to know if the same result holds for cubical complexes in general.

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