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Partial Permutation and Alternating Sign Matrix Polytopes

Dylan Heuer*[1] and Jessica Striker†[1]

[1]*North Dakota State University*

Abstract. We define and study a new family of polytopes which are formed as convex hulls of partial alternating sign matrices. We determine the inequality descriptions and number of facets of these polytopes. We also study partial permutohedra that we show arise naturally as projections of these polytopes. We enumerate vertices and facets and also characterize the face lattices of partial permutohedra in terms of chains in the Boolean lattice. Finally, we have a result and a conjecture on the volume of partial permutohedra when one parameter is fixed to be two.

Keywords: polytope; alternating sign matrix; Birkhoff polytope; permutohedron

1 Introduction

Many examples of polytopes are either *simple* (every vertex contained in the minimal number of facets), such as the n-cube, or *simplicial* (every proper face is a simplex), such as the tetrahedron. A quintessential example of a non-simple and non-simplicial polytope is the nth Birkhoff polytope (for $n > 3$), defined as the convex hull of $n \times n$ permutation matrices [2, 13]. Another such example is the nth alternating sign matrix polytope (for $n \geq 3$), defined as the convex hull of $n \times n$ alternating sign matrices [1, 20]. These matrices are interesting in their own right. The poset of alternating sign matrices is the MacNeille completion of the Bruhat order on permutation matrices [9], and alternating sign matrices are in bijection with many interesting objects (see, for example [16]).

In this paper, we define and study a more general class of polytopes composed as convex hulls of $m \times n$ *partial alternating sign matrices*, denoted $\mathrm{PASM}(m, n)$. These matrices are also of independent interest, and there are analogous results about posets [6] and bijections [7]. This paper continues the study of analogous results in the realm of polytopes and reveals new connections to graph associahedra.

In Section 2, we introduce definitions and notation for the families of matrices used to create our polytopes. In Section 3, we define partial alternating sign matrix polytopes,

*dylan.k.heuer@ndsu.edu

†jessica.striker@ndsu.edu JS was supported by a grant from the Simons Foundation/SFARI (527204, JS)

and determine the inequality descriptions and facet enumerations of these polytopes using machinery developed in the study of sign matrix polytopes [18]. Finally, in Section 4, we investigate the *partial permutohedron*, $\mathcal{P}(m,n)$, and show it is a projection of the polytopes from the first part of the paper. This projection gives us a way to connect matrix polytopes to graph associahedra. We also explore connections to chains in the Boolean lattice. These connections are helpful conceptually, as they relate the face structure of these polytopes to familiar combinatorial objects. Finally, we have a result and a conjecture on the volume of $\mathcal{P}(m,n)$ when one parameter is fixed to be two. This paper is an extended abstract; for further details see the full paper on the arXiv [8].

2 Partial alternating sign matrices

In this section, we define matrices which generalize permutation matrices and alternating sign matrices, then in the next section, we study their corresponding polytopes.

Definition 2.1. An $m \times n$ *partial alternating sign matrix* is an $m \times n$ matrix $M = \left(M_{ij}\right)$ with entries in $\{-1, 0, 1\}$ such that:

$$\sum_{i'=1}^{i} M_{i'j} \in \{0,1\}, \qquad \text{for all } 1 \leq i \leq m, 1 \leq j \leq n. \tag{2.1}$$

$$\sum_{j'=1}^{j} M_{ij'} \in \{0,1\}, \qquad \text{for all } 1 \leq i \leq m, 1 \leq j \leq n. \tag{2.2}$$

We denote the set of all $m \times n$ partial alternating sign matrices as $\mathrm{PASM}_{m,n}$. The set of matrices in $\mathrm{PASM}_{m,n}$ with no -1 entries is the set of *partial permutation matrices* $P_{m,n}$.

Example 2.2. $M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \end{pmatrix}$ is one of the 924 matrices in $\mathrm{PASM}_{4,4}$.

The cardinality of $\mathrm{PASM}_{n,n}$ is given by OEIS sequence A202751 [14]. The existence of large primes indicates it is unlikely that there exists a product formula for $|\mathrm{PASM}_{m,n}| = |\mathrm{PASM}_{n,m}|$, which is given by OEIS sequence A202756. The bijection between partial alternating sign matrices and the objects described in these sequences is given by an analog of the corner-sum map in the usual alternating sign matrix setting.

Remark 2.3. Partial permutations matrices are sometimes called *subpermutation matrices*, see, for example, [3]. Partial alternating sign matrices are a subset of *sign matrices*, which allow for any non-negative integer row sums rather than restricting them to $\{0,1\}$; see [18].

3 Partial alternating sign matrix polytopes

In this section, we define partial alternating sign matrix polytopes, enumerate their vertices and facets, and prove an inequality description.

Definition 3.1. Let $\mathrm{PASM}(m,n)$ be the polytope defined as the convex hull, as vectors in $\mathbb{R}^{mn}$, of all the matrices in $\mathrm{PASM}_{m,n}$. Call this the (m,n)-*partial alternating sign matrix polytope*.

The proof of the following proposition is in the arXiv verson [8].

Proposition 3.2. *The vertices of* $\mathrm{PASM}(m,n)$ *are exactly the matrices in* $\mathrm{PASM}_{m,n}$.

We now give the following definitions from [18], which we will use in the proof of Theorem 3.5.

Definition 3.3 (Definition 3.3 [18])**.** We define the $m \times n$ *grid graph* $\Gamma_{(m,n)}$ as follows. The vertex set is $V(m,n) := \{(i,j) : 1 \le i \le m+1, 1 \le j \le n+1\} - \{(m+1,n+1)\}$. We separate the vertices into two categories. We say the *internal vertices* are $\{(i,j) : 1 \le i \le m, 1 \le j \le n\}$ and the *boundary vertices* are $\{(m+1,j)$ and $(i,n+1) : 1 \le i \le m, 1 \le j \le n\}$. The edge set is

$$E(m,n) := \begin{cases} (i,j) \text{ to } (i+1,j) & 1 \le i \le m, 1 \le j \le n \\ (i,j) \text{ to } (i,j+1) & 1 \le i \le m, 1 \le j \le n. \end{cases}$$

Edges between internal vertices are called *internal edges* and any edge between an internal and boundary vertex is called a *boundary edge*. We draw the graph with i increasing to the right and j increasing down, to correspond with matrix indexing.

Definition 3.4 (Definition 3.4, [18])**.** Given an $m \times n$ matrix X, we define a labeled graph, $\hat{X}$, which is a labeling of the vertices and edges of $\Gamma_{(m,n)}$ from Definition 3.3. The internal vertices (i,j), $1 \le i \le m$, $1 \le j \le n$ are each labeled with the corresponding entry of X: $\hat{X}_{ij} = X_{ij}$. The edges from (i,j) to $(i,j+1)$ are each labeled by the corresponding row partial sum $r_{ij} = \sum_{j'=1}^{j} X_{ij'}$ $(1 \le i \le m, 1 \le j \le n)$. Likewise, the edges from (i,j) to $(i+1,j)$ are each labeled by the corresponding column partial sum $c_{ij} = \sum_{i'=1}^{i} X_{i'j}$ $(1 \le i \le m, 1 \le j \le n)$.

Theorem 3.5. $\mathrm{PASM}(m,n)$ *consists of all* $m \times n$ *real matrices* $X = (X_{ij})$ *such that:*

$$0 \le \sum_{i'=1}^{i} X_{i'j} \le 1, \qquad \textit{for all } 1 \le i \le m, 1 \le j \le n. \tag{3.1}$$

$$0 \le \sum_{j'=1}^{j} X_{ij'} \le 1, \qquad \textit{for all } 1 \le i \le m, 1 \le j \le n. \tag{3.2}$$

Proof. First we need to show that any $X \in \mathrm{PASM}(m,n)$ satisfies (3.1) - (3.2). Then $X = \sum_\gamma \mu_\gamma M_\gamma$ where $\sum_\gamma \mu_\gamma = 1$ and the $M_\gamma \in \mathrm{PASM}_{m,n}$. Since we have a convex combination of partial alternating sign matrices, by Definition 2.1, we obtain (3.1) - (3.2) immediately. Thus $\mathrm{PASM}(m,n)$ fits the inequality description.

Let X be a real-valued $m \times n$ matrix satisfying (3.1) - (3.2). We wish to show that X can be written as a convex combination of partial alternating sign matrices in $\mathrm{PASM}_{m,n}$, so that X is in $\mathrm{PASM}(m,n)$.

Consider the corresponding labeled graph $\hat{X}$ of Definition 3.4. We will construct a trail in $\hat{X}$ all of whose edges are labeled by inner numbers and show it is a simple path or cycle. (A number α is *inner* if $0 < \alpha < 1$.)

If there exists i or j such that $\hat{X}_{i,n+1}$ or $\hat{X}_{m+1,j}$ is inner, begin constructing the trail at the adjacent boundary edge. If no such i or j exist, start the trail on any edge with inner label. If there are no inner edge labels, then X is already a partial alternating sign matrix. From the starting point, construct the circuit as follows. Go along a row or column from the starting point along edges with inner labels. When possible, switch from going along a row to a column, or vice versa (*i.e.* make a "turn"). Continue in this manner until either (1) you reach an edge that was previously in the trail, or (2) you reach a new boundary edge. If (1), then the part of the trail constructed between the first and second time you reached that edge will be a simple cycle. That is, we cut off any part that was constructed before the first time that edge was reached. If (2), then the starting point for the trail must have been a boundary edge, since there is at least one boundary vertex with inner label. Thus our trail is actually a path.

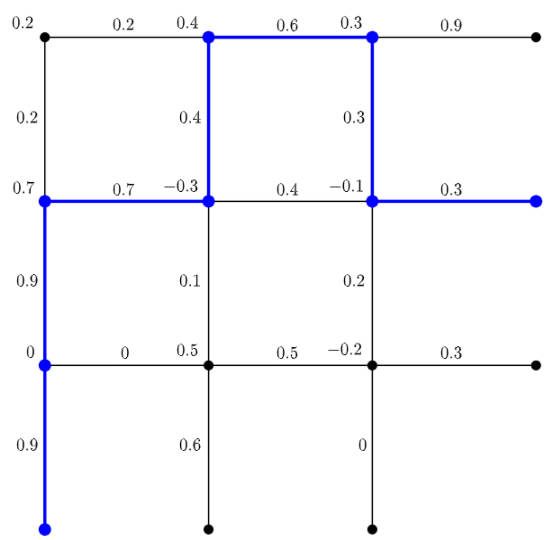

Figure 1: An example of the path construction described in the proof of Theorem 3.5. The bold blue edges and vertices are those included in the path.

Label the corners of the path or cycle (not the boundary vertices) alternately $(+)$ and

$(-)$. Set ℓ^+ equal to the largest number that we could subtract from the $(-)$ entries and add to the $(+)$ entries while still satisfying (3.1) - (3.2). Construct a matrix X^+ by subtracting and adding in this way. X^+ is a matrix which stills satisfies (3.1) - (3.2) and which has least one more non-inner edge label than X.

Now give opposite labels to the corners and set ℓ^- equal to the largest number we could subtract from $(-)$ entries and add to $(+)$ entries while still satisfying (3.1) - (3.2). Add and subtract in a similar way to create X^-, another matrix satisfying (3.1) - (3.2) and which has at least one more non-inner edge label than X.

Both X^+ and X^- satisfy (3.1) - (3.2) by construction. Also by construction,

$$X = \frac{\ell^-}{\ell^+ + \ell^-}X^+ + \frac{\ell^+}{\ell^+ + \ell^-}X^-$$

and $\frac{\ell^-}{\ell^+ + \ell^-} + \frac{\ell^+}{\ell^+ + \ell^-} = 1$. So X is a convex combination of the two matrices X^+ and X^- that still satisfy the inequalities and are each at least one step closer to being partial alternating sign matrices, since that have at least one more partial sum attaining its maximum or minimum bound. By repeatedly applying this procedure, X can be written as a convex combination of partial alternating sign matrices.

See Figure 1 and Example 3.6 for an example of this construction. □

Example 3.6. Let $X = \begin{pmatrix} 0.2 & 0.4 & 0.3 \\ 0.7 & -0.3 & -0.1 \\ 0 & 0.5 & -0.2 \end{pmatrix}$. Then by the construction described in the proof of Theorem 3.5 and shown in Figure 1, X can be decomposed as $X = \frac{0.3}{0.1+0.3}X^+ + \frac{0.1}{0.1+0.3}X^-$, where $X^+ = \begin{pmatrix} 0.2 & 0.5 & 0.2 \\ 0.8 & -0.4 & 0 \\ 9 & 0.5 & -0.2 \end{pmatrix}$ and $X^- = \begin{pmatrix} 0.2 & 0.1 & 0.6 \\ 0.4 & 0 & -0.4 \\ 0 & 0.5 & -0.2 \end{pmatrix}$. In this step of decomposing, $\ell^+ = 0.1$ and $\ell^- = 0.3$. Continuing the process of decomposition, one could write X as a convex combination of partial alternating sign matrices.

Theorem 3.7. *The number of facets of* $\mathrm{PASM}(m, n)$ *is* $4mn - 3m - 3n + 5$.

We have used SageMath to compute the Ehrhart polynomials for $\mathrm{PASM}(m, n)$ for $m, n \leq 4$ and note that in all of these cases their coefficients are positive.

Remark 3.8. There are analogous results for the case where we consider the convex hull of the matrices in $P_{m,n}$, which we state in the arXiv version [8]. Some of these results are simply rectangular analogues of Mirsky's work [12], and many follow as special cases of the matching polytope (see, for example, [17]).

4 Partial permutohedron

In this section, we study partial permutohedra that arise naturally as projections of $\mathrm{PASM}(m, n)$. After giving the definition, we count vertices and facets and find an inequality description. By relating the partial permutohedron to the stellohedron, we describe its face lattice combinatorially. Finally we show that the partial alternating sign matrix polytope projects to it, and discuss its volume.

Definition 4.1. Given a partial permutation matrix $M \in P_{m,n}$, its *one-line notation* $w(M)$ is a word $w_1 w_2 \dots w_m$ where $w_i = j$ if there exists j such that $M_{ij} = 1$ and 0 otherwise.

Example 4.2. Let $M = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$. Then $w(M) = 3502$.

Proposition 4.3. *$w(P_{m,n})$ can be characterized as the set of all words of length m whose entries are in $\{0, 1, \dots, n\}$ and whose non-zero entries are distinct.*

Definition 4.4. Let $\mathcal{P}(m, n)$ be the polytope defined as the convex hull, as vectors in $\mathbb{R}^m$, of the words in $w(P_{m,n})$. Call this the *(m, n)-partial permutohedron*.

Definition 4.5. Let $z \in \mathbb{R}^n$ be a vector with distinct nonzero entries. Define $\phi_z : \mathbb{R}^{m\times n} \to \mathbb{R}^m$ as $\phi_z(X) = Xz$. Also define $w_z(P_{m,n})$ as the set of all words of length m whose entries are in $\{0, z_1, z_2, \dots, z_n\}$ and whose nonzero entries are distinct. Then $\mathcal{P}_z(m, n)$ is the polytope defined as the convex hull, as vectors in $\mathbb{R}^m$ of the words in $w_z(P_{m,n})$.

We will not use Definition 4.5 until later in this section, but the upcoming results about the structure of partial permutohedra can also be extended to the $\mathcal{P}_z$ polytopes.

Proposition 4.6. *The number of vertices of $\mathcal{P}(m, n)$ is equal to* $\displaystyle\sum_{k=\max(m-n,0)}^{m} \frac{m!}{k!}$.

The proof of this proposition follows from a straightforward counting argument; see the arXiv version for details [8].

For the proof of the next theorem, and for that of Theorem 4.23, we need the concept of (weak) majorization [11].

Definition 4.7. Let u and v be vectors of length N. Then $u \prec_w v$ (that is, u is *weakly majorized* by v) if $\displaystyle\sum_{i=1}^{k} u_{[i]} \le \sum_{i=1}^{k} v_{[i]}$, for all $1 \le k \le N$, where the vector $\left(u_{[1]}, u_{[2]}, \dots, u_{[N]}\right)$ is obtained from u by rearranging its components so that they are in decreasing order.

Proposition 4.8 (4.C.2 in [11])**.** *For vectors u and v of length n, $u \prec_w v$ if and only if u lies in the convex hull of the set of all vectors z which have the form $z = \left(\varepsilon_1 v_{\pi(1)}, \dots, \varepsilon_n v_{\pi(n)}\right)$, where π is a permutation and each ε_i is either 0 or 1.*

Theorem 4.9. *$\mathcal{P}(m,n)$ consists of all nonnegative vectors $u \in \mathbb{R}^m$ such that:*

$$\sum_{i \in S} u_i \leq \binom{n+1}{2} - \binom{n-k+1}{2}, \qquad \text{where } S \subseteq [m], |S| = k \neq 0. \tag{4.1}$$

Proof. First, note that if $P \in P_{m,n}$, then $w(P)$ is nonnegative and satisfies (4.1). This is because the largest values that may appear are the m largest non-negative integers less than or equal to n, and the non-zero integers must be distinct. Since $w(P)$ satisfies the inequalities for any P, so must any convex combination.

Now, suppose $x \in \mathbb{R}^m$ is nonnegative and satisfies (4.1). We will proceed by using Proposition 4.8. Fix n and let $v = (n, n-1, n-2, \ldots, 1, 0, 0, \ldots, 0)$ be the decreasing vector whose biggest entry is n, and whose subsequent non-zero entries decrease by 1 and for which all other entries are 0. Note that if $n \geq m$, then v will have no 0 entries: it will be $(n, n-1, \ldots, n-m+1)$. Since x is nonnegative and satisfies (4.1), it is by definition weakly majorized by v; note in particular that (4.1) is requiring that the sum of the k largest entries is never more than the k biggest integers less than or equal to n. But now the convex hull described in Proposition 4.8 is actually $\mathcal{P}(m,n)$, thus $x \in \mathcal{P}(m,n)$. □

Theorem 4.10. *The number of facets of $\mathcal{P}(m,n)$ is equal to $m + 2^m - 1 - \sum_{r=1}^{m-n} \binom{m}{m-r}$.*

Proof. There are $2^m - 1$ total inequalities given in (4.1), and m inequalities given by the fact that the vectors have nonnegative entries. When $m \leq n$, none of these are redundant, as there are no k values such that the right hand sides are the same. In other words, there is at most one k-value such that $\binom{n-k+1}{2} = 0$. When $m > n$, there are $m - n$ values of k such that $\binom{n-k+1}{2} = 0$, creating redundancies. For each r between 1 and $m-n$, we have redundant inequalities for the subsets of $[m]$ of size $m - r$, which are counted by $\binom{m}{m-r}$. □

We now relate $\mathcal{P}(m,m)$ to a specific graph associahedron, the stellohedron. We will need the following definitions.

Definition 4.11 (Definition 2.2 in [4])**.** Let G be a connected graph. A *tube* is a proper nonempty set of vertices of G whose induced graph is a proper, connected subgraph of G. There are three ways that two tubes t_1 and t_2 may interact on the graph:

1. Tubes are *nested* if $t_1 \subset t_2$.

2. Tubes *intersect* if $t_1 \cap t_2 \neq \emptyset$ and $t_1 \not\subset t_2$ and $t_2 \not\subset t_1$.

3. Tubes are *adjacent* if $t_1 \cap t_2 = \emptyset$ and $t_1 \cup t_2$ is a tube in G.

Tubes are *compatible* if they do not intersect and they are not adjacent. A *tubing* T is a set of tubes of G such that every pair of tubes is compatible. A *k-tubing* has k tubes.

Definition 4.12 (Definition 2 in [5])**.** For a graph G, the *graph associahedron* $\mathrm{Assoc}(G)$ is a simple, convex polytope whose face poset is isomorphic to the set of tubings of G, ordered such that $T < T'$ if T obtained from T' by adding tubes.

Of particular interest to us is the graph associahedron of the star graph, $\mathrm{Assoc}(K_{1,m})$, sometimes called the stellohedron.

Definition 4.13. The *star graph* (with $m+1$ vertices) is the complete bipartite graph $K_{1,m}$. We label the lone vertex $*$, and call it the *inner vertex*. We label the other m vertices $x_1, x_2, \ldots, x_m$, and call them *outer vertices*.

Remark 4.14. Note that if G has n nodes, vertices of $\mathrm{Assoc}(G)$ correspond to maximal tubings of G (*i.e.* $(n-1)$-tubings), and in general, faces of dimension k correspond to $(n-k-1)$-tubings of G. Thus for the star graph $K_{1,m}$, which has $m+1$ nodes, vertices of $\mathrm{Assoc}(K_{1,m})$ correspond to m-tubings, and in general, faces of dimension k correspond to $(m-k)$-tubings.

We examine the polytope $\mathrm{Assoc}(K_{1,m})$ through the lens of partial permutations, which allows to understand it in a different way. Lemmas 4.18 and 4.19, and Corollary 4.20, which culminate in Theorem 4.21, shed light on a way to view these tubings, and thus the faces of the stellohedron, as certain chains in the Boolean lattice. Furthermore, in Conjecture 4.22 we attempt to describe what happens for $\mathcal{P}(m,n)$, where $n \neq m$. But first, we review the following result that relates $\mathcal{P}(m,m)$ to the stellohedron; this can be found, in other language, in [10]; we describe the explicit map in the remark below.

Theorem 4.15 (Proposition 56 [10])**.** *$\mathcal{P}(m,m)$ is a realization of $\mathrm{Assoc}(K_{1,m})$.*

Remark 4.16. The map which sends maximal tubings of the star graph to coordinates (which can be thought of as the one-line notation of partial permutations) is as follows. Let T be a maximal tubing of $K_{1,m}$, and for each outer vertex x_i, let t_i be the smallest tube containing x_i. Then the coordinate in $\mathbb{R}^m$ corresponding to T is simply $(|t_1|-1, |t_2|-1, \ldots, |t_m|-1)$.

One can view a tubing instead as its corresponding spine, defined below.

Definition 4.17. Let T be a tubing of the star graph. The *spine* of T is the poset of tubes of T ordered by inclusion, and whose elements are labeled not by the tubes themselves but by the set of new vertices in each tube. For simplicity, we will use the label i in place of x_i.

Note that two tubes are of the star graph are compatible only if they each contain only a single outer vertex, or one is contained in the other. Spines are defined (in more generality) in Remark 10 of [10], and are called B-trees in [15]. See Figure 2 for examples of tubings with their corresponding spines, as well as their corresponding chains from the bijection in the following lemma. See the arXiv version for the proof of Lemma 4.19.

Lemma 4.18. *Tubings of $K_{1,m}$ are in bijection with chains in the boolean lattice $\mathcal{B}_m$.*

Proof. Given a spine S of a tubing T of $K_{1,m}$, we can recover the corresponding chain in the boolean lattice as follows. The bottom element of the chain is the subset including anything that is with $*$ in S. Each subsequent subset is made by adding in the elements in the next level of S, until we reach the top level. Any elements not used in the subsets of the chain will be those that appear below the $*$ in S. Starting with a chain $C \in \mathcal{B}_m$, we can recover the corresponding spine S (and thus the tubing) by reversing this process. Any elements not in the maximal chain of C will be in the bottom level of S as singletons. Any elements in the minimal chain of C will appear with $*$ in S. The new elements that appear in each subsequent subset in C appear together as a new level in S. Once we have S, we can of course recover T. □

Lemma 4.19. *Let T be a k-tubing and T' be a $(k+j)$-tubing of $K_{1,m}$, and let C and C' be their corresponding chains in $\mathcal{B}_m$ via the bijection in Lemma 4.18. $T \subset T'$ if and only if C' can be obtained from C by j iterations of the following:*

1. *adding a non-maximal subset, or*
2. *removing the same element from every subset.*

Corollary 4.20. *A face of $\mathcal{P}(m,m)$ is of dimension k if and only if the corresponding chain has k missing ranks.*

Proof. We know that adding a tube reduces the dimension of the corresponding face by one. Also, by Lemma 4.19, we know that adding a tube corresponds to either adding a non-maximal subset or removing an element from every subset in the corresponding chain. In either case, this reduces the number of missing ranks in the chain by one. So, having k missing ranks in the chain corresponds to having $m-k$ tubes, which by definition of the graph associahedron corresponds to a face being of dimension k. □

The theorem below follows directly from the above lemmas and corollary.

Theorem 4.21. *The face lattice of $\mathcal{P}(m,m)$ is isomorphic to the lattice of chains in $\mathcal{B}_m$, where $C < C'$ if C' can be obtained from C by iterations of (1) and/or (2) from Lemma 4.19. A face of $\mathcal{P}(m,m)$ is of dimension k if and only if the corresponding chain has k missing ranks.*

As chains in the Boolean lattice are generally more familiar objects than tubings of graphs, presenting results in terms of these chains is helpful conceptually. In fact, because of the description of the faces of $\mathcal{P}(m,m)$ in terms of chains, we are able to form the following conjecture for $\mathcal{P}(m,n)$, which we have verified for $m,n \leq 4$.

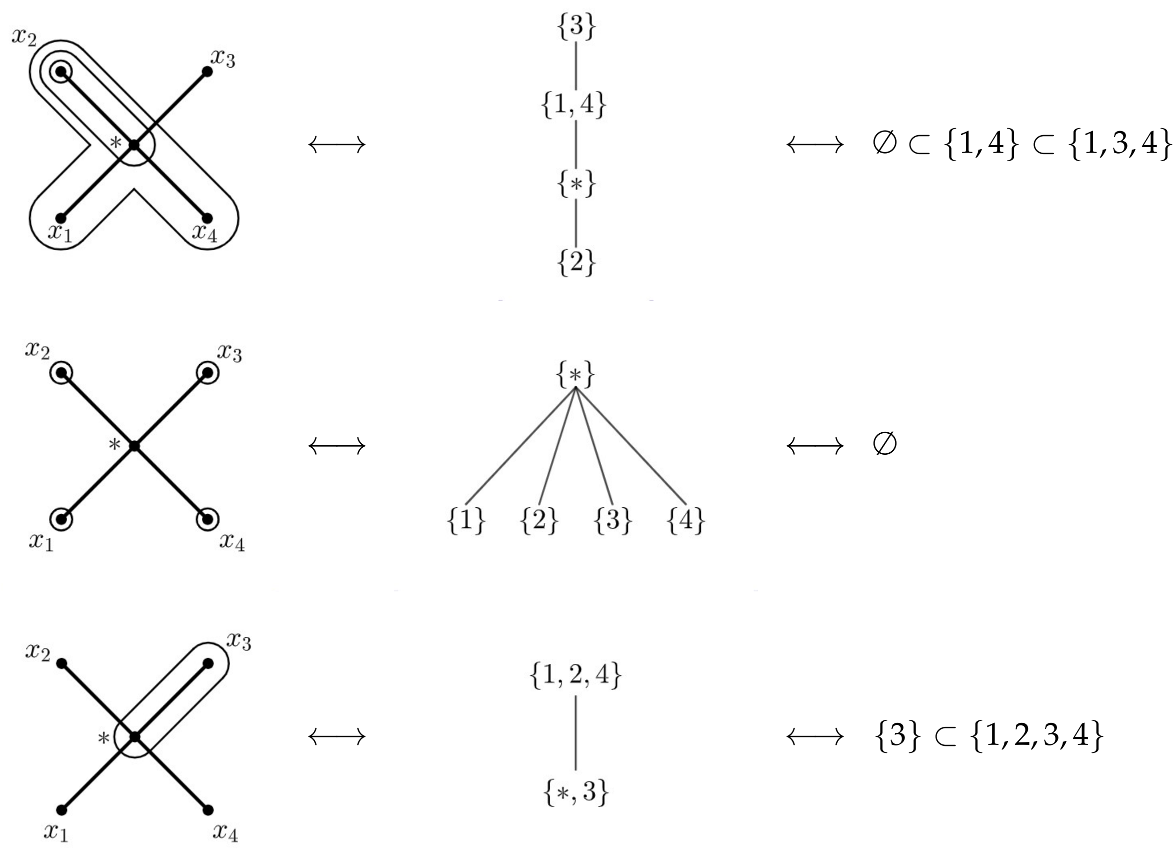

Figure 2: Examples of tubings of $K_{1,4}$ along with their corresponding spines (see Definition 4.17) and chains in $\mathcal{B}_4$ (via the bijection in Lemma 4.18)

Conjecture 4.22. *Faces of $\mathcal{P}(m,n)$ are in bijection with chains in $\mathcal{B}_m$ whose difference between largest and smallest nonempty subsets is at most $n-1$. A face of $\mathcal{P}(m,n)$ is of dimension k if and only if the corresponding chain has k missing ranks.*

We now show that the partial permutohedron is a projection of PASM(m,n). See the arXiv version [8] for the proof.

Theorem 4.23. *Let z be a strictly decreasing vector in $\mathbb{R}^m$. Then $\phi_z(\mathrm{PASM}(m,n)) = \mathcal{P}_z(m,n)$.*

We have the following volume theorem for $m = 2$ and conjecture for $n = 2$.

Theorem 4.24. *$\mathcal{P}(2,n)$ has normalized volume equal to $2n^2 - 1$.*

Proof. $\mathcal{P}(2,n)$ is a 2-dimensional polytope whose extreme points consist of exactly $(0,0)$, $(n,0)$, $(0,n)$, $(n,n-1)$, and $(n-1,n)$. This forms an $n \times n$ square with one corner "cut off" by the line segment connecting $(n, n-1)$ to $(n-1, n)$. We can explicitly calculate the area of this region to be $n^2 - \frac{1}{2}$. To obtain the normalized volume we multiply by $\dim(\mathcal{P}(2,n))! = 2!$ giving us $2n^2 - 1$. $\square$

Conjecture 4.25. *$\mathcal{P}(m, 2)$ has normalized volume equal to $3^m - m$.*

Using SageMath, this conjecture has been confirmed for $m \leq 50$. We have also used SageMath to compute the Ehrhart polynomials for $\mathcal{P}(m, n)$ for $m, n \leq 7$ and note that in all of these cases their coefficients are positive.

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