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Minimal elements for the limit weak order on affine Weyl groups

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Abstract. The limit weak order on an affine Weyl group was introduced by Lam and Pylyavskyy [*Transform. Groups* **18** (2013), 179–231] in their study of total positivity for loop groups [*Adv. Math.* **230** (2012), 1222–1271]. They showed that in the case of the affine symmetric group the minimal elements of this poset coincide with the infinite fully commutative reduced words and with infinite powers of Coxeter elements. We answer several open problems raised there by classifying minimal elements in all affine types and relating these elements to the classes of fully commutative and Coxeter elements[3].

Keywords: Coxeter element, fully commutative, affine Weyl group, weak order, total positivity

1 Introduction

An infinite word $s_{i_1}s_{i_2}\cdots$ in the simple generators of an infinite Coxeter group $\widetilde{W}$ (see Section 2.1 for background) is called an *infinite reduced word* if all of its finite prefixes $s_{i_1}\cdots s_{i_k}$ are reduced words in $\widetilde{W}$; we will identify such a word with its sequence $\mathbf{i} = i_1 i_2 \cdots$ of indices. Associated to $\mathbf{i}$ is an *inversion set* $\mathrm{Inv}(\mathbf{i})$, a subset of the set of reflections of $\widetilde{W}$, which induces an equivalence relation on the set of infinite reduced words: $[\mathbf{i}] = [\mathbf{j}]$ if and only if $\mathrm{Inv}(\mathbf{i}) = \mathrm{Inv}(\mathbf{j})$. An equivalence class of infinite reduced words is called a *limit element* of $\widetilde{W}$.

The *limit weak order* for $\widetilde{W}$, introduced by Lam and Pylyavskyy [4] is the partial order $(\widetilde{\mathcal{W}}, \leq)$ on the set of limit elements with order given by containment of inversion sets. In [4] this order (conjecturally) encodes the containment relations between certain strata in the totally positive space studied there, while Lam and Thomas show in [5] that $\widetilde{\mathcal{W}}$ encodes the closure relations among components of the Tits boundary of $\widetilde{W}$. In both instances, understanding the minimal elements in the limit weak order is of significant interest. In the case when $\widetilde{W}$ is the affine symmetric group, Lam and Pylyavskyy show

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[3] See [3] for a full version of this work.

that the minimal elements coincide with two other important classes of elements: fully commutative limit elements and infinite powers of Coxeter elements.

Theorem 1.1 (Lam and Pylyavskyy [4]). *Let $\widetilde{W}$ be the affine symmetric group, then the following are equivalent for an infinite reduced word* $\mathbf{i}$*:*

1. $[\mathbf{i}]$ *is minimal in* $\widetilde{\mathcal{W}}$*,*
2. $[\mathbf{i}]$ *is fully commutative,*
3. $[\mathbf{i}] = [c^\infty]$ *for a Coxeter element* c *of* $\widetilde{W}$*.*

As natural extensions of Theorem 1.1, Lam and Pylyavskyy posed the following open problems:

Problem 1 (Lam and Pylyavskyy [4]). Describe, in terms of infinite reduced words, the minimal elements in limit weak order for all affine Weyl groups.

Problem 2 (Lam and Pylyavskyy [4]). Are all minimal elements in limit weak order fully commutative?

In this extended abstract we describe a complete resolution of Problems 1 and 2 and give further extensions of Theorem 1.1:

- In Section 2 we cover needed background material.
- In Section 3 we note that, when $\widetilde{W}$ is an affine Weyl group with corresponding finite Weyl group W, the minimal elements of $\widetilde{\mathcal{W}}$ coincide with infinite powers of translations by multiples of W-conjugates of fundamental coweights. We give a general, type-uniform procedure for generating infinite reduced words corresponding to these elements; computations in Appendix A using this procedure resolve Problem 1.
- Although none of the three equivalences in Theorem 1.1 continues to hold in general affine Weyl groups, we show that infinite fully commutative elements and infinite powers of Coxeter elements are still minimal in $\widetilde{\mathcal{W}}$. Therefore it makes sense to ask for which fundamental coweights $\omega_i^\vee$ the corresponding infinite translation element is fully commutative or is a power of a Coxeter element; the answer is depicted in Figure 1.
- In fact we show in Section 4 that, except in type A, there is a unique $\omega_i^\vee$ corresponding to the Coxeter elements, and we give a simple rule for identifying the corresponding node in the Dynkin diagram.

- Finally, in Section 5 we show that the fundamental coweights whose infinite translation elements are fully commutative are exactly the minuscule and cominuscule weights. In particular, Problem 2 has a negative answer except in type A. This also allows us to completely classify fully commutative infinite reduced words in affine Weyl groups.

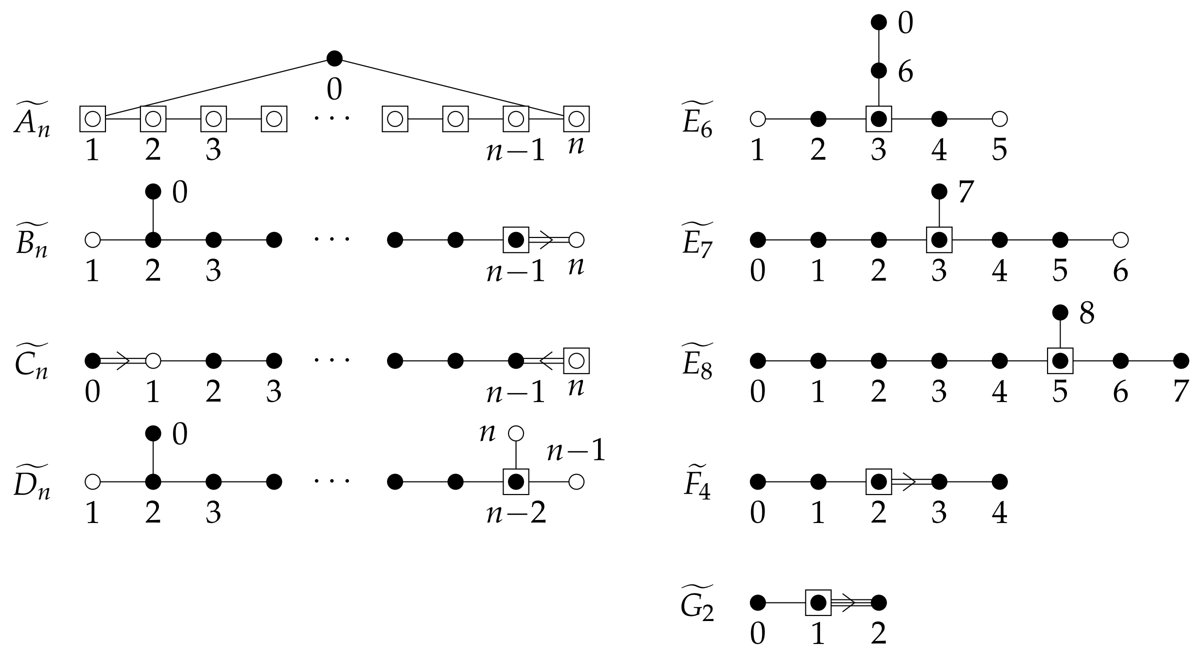

Figure 1: The Dynkin diagrams for the affine Weyl groups. In each case the affine node is labelled 0, the Coxeter nodes are boxed, and the fully commutative nodes are unfilled.

2 Background

2.1 Coxeter groups

We refer the reader to Björner–Brenti [1] for basics on Coxeter groups. Let W be a Coxeter group with simple reflections $S = \{s_1, \cdots, s_n\}$. Any element $c \in W$ which is the product of the n simple reflections in some order is called a *Coxeter element*.

Given $w \in W$, an expression

$$w = s_{i_1} \cdots s_{i_\ell}$$

of minimal length is called a *reduced word* for w, and in this case $\ell = \ell(w)$ is called the *length* of w. The (right) weak order $\leq_R$ on W is the partial order with cover relations

$w \lessdot_R ws_i$ whenever $\ell(ws_i) = \ell(w) + 1$.

A well known theorem of Tits [8] states that all reduced words for w are connected via the defining relations $s_i s_j s_i \cdots = s_j s_i s_j \cdots$ with $m_{ij} \in \{2, 3, \ldots\}$ factors on each side (called a *commutation move* if $m_{ij} = 2$ and a *braid move* if $m_{ij} \geq 3$). If no reduced word for w admits the application of a braid move then w is called *fully commutative* [7]. We say an infinite reduced word $\mathbf{i} = i_1 i_2 \cdots$ is fully commutative if all elements $w = s_{i_1} \cdots s_{i_k}$ are fully commutative for $k = 1, 2, \ldots$.

For $J \subseteq S$, the *parabolic subgroup* W_J is the subgroup of W generated by J, viewed as a Coxeter group with simple reflections J. Each left coset wW_J of W_J in W contains a unique element w^J of minimal length, and the set $\{w^J \mid w \in W\}$ of these minimal coset representatives is called the *parabolic quotient* W^J. Letting $w_J \in W_J$ be the unique element such that $w^J w_J = w$, we have $\ell(w^J) + \ell(w_J) = \ell(w)$. If W^J is finite it contains a unique element w_0^J of maximum length.

2.2 Affine Weyl groups

We refer the reader to Bourbaki [2] for more details on affine Weyl groups. For the remainder of the extended abstract, we let $\widetilde{W}$ denote an affine Weyl group with associated irreducible finite Weyl group W. We number the simple reflections so that W has simple reflections $S = \{s_1, \ldots, s_n\}$ while $\widetilde{W}$ has $\widetilde{S} = \{s_0\} \sqcup S$. We write J_i for $S \setminus \{s_i\}$.

We let Φ denote the finite root system associated to W, Φ^+ denote a choice of positive roots, and $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ denote the corresponding set of simple roots. We write ξ for the highest root of Φ^+ and make the notational convention that $\alpha_0 = -\xi$. Write $\widetilde{\Delta} = \{\alpha_0, \ldots, \alpha_n\}$.

We write V for the Euclidean space containing Φ and $\langle , \rangle$ for the inner product. The group $\widetilde{W}$ acts faithfully on V by affine linear transformations, and the action of $W \subset \widetilde{W}$ is linear and preserves the inner product.

The *fundamental coweights* $\omega_1^\vee, \ldots, \omega_n^\vee$ are determined by the formula $\langle \alpha_i, \omega_j^\vee \rangle = \delta_{ij}$. For $i = 1, \ldots, n$ the *simple coroot* $\alpha_i^\vee$ is defined by $\alpha_i^\vee = \frac{2}{\langle \alpha_i, \alpha_i \rangle} \alpha_i$. The *coroot lattice* is $Q^\vee = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i^\vee$. For $i = 1, \ldots, n$ we let k_i denote the smallest positive integer (necessarily finite) such that $k_i \omega_i^\vee \in Q^\vee$. For each $\lambda \in Q^\vee$ there is a unique element t_λ in $\widetilde{W}$ which acts on V via translation by λ. This realizes $\widetilde{W}$ as the semidirect product $W \ltimes Q^\vee$ where $wt_\lambda w^{-1} = t_{w\lambda}$ for $w \in W$.

For $w \in \widetilde{W}$ the *inversion set* is defined to be

$$\mathrm{Inv}(w) = \{\alpha_{i_1}, s_{i_1}\alpha_{i_2}, s_{i_1}s_{i_2}\alpha_{i_3}, \cdots, s_{i_1} \cdots s_{i_{k-1}}\alpha_{i_k}\}$$

where $w = s_{i_1} \cdots s_{i_k}$ is any reduced word for w (it is an important fact that the inversion set does not depend on the reduced word chosen). It is clear from the definition that if $w \leq_R w'$ then $\mathrm{Inv}(w) \subseteq \mathrm{Inv}(w')$; in fact, the converse holds as well: weak order is equivalent to containment of inversion sets. If $\mathbf{i} = i_1 i_2 \cdots$ is an infinite reduced word, then the prefixes $w^{(k)} = s_{i_1} \cdots s_{i_k} \in \widetilde{W}$ clearly satisfy $w^{(k)} \leq_R w^{(k')}$ whenever $k \leq k'$. The inversion set of $\mathbf{i}$ is defined to be the increasing union

$$\mathrm{Inv}(\mathbf{i}) = \bigcup_{k=1}^{\infty} \mathrm{Inv}(w^{(k)}).$$

The limit weak order $\widetilde{\mathcal{W}}$ on the limit elements $[\mathbf{i}]$ is determined by containment of these inversion sets.

Associated to $\widetilde{W}$ is an affine hyperplane arrangement $\mathcal{H}$ in V, with hyperplanes $H_{\alpha,k} = \{x \in V \mid \langle x, \alpha \rangle = k\}$ for $\alpha \in \Phi^+$, $k \in \mathbb{Z}$. The connected components of the complement of $\mathcal{H}$ are called *alcoves*, and $\widetilde{W}$ acts simply transitively on the set of alcoves. Fixing the *fundamental alcove* $\mathcal{A}_{\mathrm{id}}$ to be that bounded by $H_{\alpha_i,0}$ for $i = 1, \ldots, n$ and $H_{\alpha_0,-1}$, this action determines a labelling of the alcoves $\mathcal{A}_w$ by elements $w \in \widetilde{W}$. The inversions of w are in natural bijection with the hyperplanes $H_{\alpha,k}$ separating $\mathcal{A}_w$ from $\mathcal{A}_{\mathrm{id}}$.

The Dynkin diagram of $(\widetilde{W}, \widetilde{S})$ is a directed graph with nodes $\widetilde{\Delta}$ such that there are $-2\langle \alpha_i, \alpha_j \rangle / \langle \alpha_j, \alpha_j \rangle$ directed edges from α_i to α_j, for $i \neq j$. Finite and affine Weyl groups are completely classified by their Dynkin diagrams. The irreducible affine groups consist of four infinite families $\widetilde{A_n}(n \geq 1), \widetilde{B_n}(n \geq 2), \widetilde{C_n}(n \geq 2), \widetilde{D_n}(n \geq 4)$ and the exceptional types $\widetilde{E_6}, \widetilde{E_7}, \widetilde{E_8}, \widetilde{F_4}$, and $\widetilde{G_2}$. See the corresponding Dynkin diagrams in Figure 1.

3 Words for infinite translation elements

3.1 Translations by fundamental coweights

The following proposition, implicit in [4] and [5], describes the minimal elements of $\widetilde{\mathcal{W}}$ geometrically: they are the infinite translations in the directions of the rays of the corresponding reflection arrangement.

Proposition 3.1. *The minimal elements of $\widetilde{\mathcal{W}}$ are precisely*

$$\{[t^{\infty}_{wk_i\omega_i^{\vee}}] \mid 1 \leq i \leq n, w \in W^{J_i}\}$$

where $J_i = \{s_j \mid j \neq i\}$.

In Section 3.2 we give a method for constructing infinite reduced words for these and other infinite translation elements. Understanding these reduced words is necessary for resolving Problems 1 and 2 and understanding the limit Coxeter elements, for the characterization of minimal elements in Proposition 3.1 is not immediately applicable to any of these problems.

3.2 Explicit reduced words

In this section, we explain how to write down explicit infinite reduced words that correspond to open faces of the reflection arrangement of W. The content of this section generalizes that of Section 4.7 of [4], which is specific to type A. Our formulation and arguments are type-uniform and the proof ideas will be different from that of [4].

Recall that the set of simple roots for W is $\Delta = \{\alpha_1, \dots, \alpha_n\}$ while the set of simple roots for $\widetilde{W}$ is $\widetilde{\Delta} = \{\alpha_0, \dots, \alpha_n\}$, where $\alpha_0 = -\xi$ with ξ being the highest root of W.

Let $\lambda \neq 0 \in Q^\vee$. We now explicitly write down an infinite reduced word $\mathbf{i} = s_{i_1} s_{i_2} \cdots$ such that $[\mathbf{i}] = [t_\lambda^\infty]$. The construction is inductive. Let $\lambda^{(0)} = \lambda$. For $j \geq 1$, we choose $i_j \in \{0, 1, \dots, n\}$ such that $\langle \lambda^{(j-1)}, \alpha_{i_j} \rangle < 0$ and then let $\lambda^{(j)} = s_{i_j} \lambda^{(j-1)}$.

Notice that if $\langle \lambda, \alpha_k \rangle \geq 0$ for all $k = 0, \dots, n$, then we must have $\langle \lambda, \alpha_k \rangle = 0$ for all $k = 0, \dots, n$ since $-\alpha_0 = \xi$ is a positive linear combination of $\alpha_1, \dots, \alpha_n$. And since $\alpha_1, \dots, \alpha_n$ spans, the equalities imply $\lambda = 0$. Therefore, as long as $\lambda \neq 0$, none of its Weyl group translates will be 0 so the above procedure will continue indefinitely.

Proposition 3.2. *Let $\lambda \neq 0 \in Q^\vee$ and construct the infinite word $\mathbf{i}$ as above. Then $\mathbf{i}$ is reduced and $[\mathbf{i}] = [t_\lambda^\infty]$.*

Proof sketch. Recall that the fundamental alcove of the affine hyperplane arrangement of $\widetilde{W}$ is given by

$$\mathcal{A}_{\mathrm{id}} = \{x \in V \mid \langle x, \alpha_i \rangle > 0 \text{ for } i = 1, \dots, n \text{ and } \langle x, \alpha_0 \rangle > -1\}.$$

For any α_i such that $\langle \lambda, \alpha_i \rangle < 0$, we can choose $\mu \in \mathcal{A}_{\mathrm{id}}$ close to the hyperplane $H_{\alpha_i,k}$ ($k = 0$ if $i = 1, \dots, n$ and $k = -1$ if $i = 0$ where we identify $H_{\alpha_0,-1}$ with $H_{\xi,1}$) that bounds $\mathcal{A}_{\mathrm{id}}$ such that moving in the direction of λ from μ intersects $H_{\alpha_i,k}$ first, among all $n+1$ hyperplanes that bound $\mathcal{A}_{\mathrm{id}}$. Conversely, if $\langle \lambda, \alpha_i \rangle \geq 0$, then from any $\mu \in \mathcal{A}_{\mathrm{id}}$ and moving in the direction of λ, we are only getting further away from $H_{\alpha_i,k}$ and can never encounter this hyperplane.

After such an α_{i_1} is chosen with $\langle \lambda, \alpha_{i_1} \rangle < 0$, we move into the alcove $\mathcal{A}_{s_{i_1}}$, which is the alcove reflected across the hyperplane $H_{\alpha_{i_1},k}$ from the fundamental alcove, and is also the alcove that can be reached from some point in $\mathcal{A}_{\mathrm{id}}$ by moving in the direction of λ. Reflecting by s_{i_1}, choosing α_{i_2} such that $\langle s_{i_1}\lambda, \alpha_{i_2} \rangle < 0$ is exactly the same as choosing an alcove $\mathcal{A}_{s_{i_2}s_{i_1}}$ which can be reached from some point $\mu \in \mathcal{A}_{s_{i_1}}$ by translating in the direction of λ.

Continue this procedure described above, the alcove path described by $\mathbf{i}$ is a sequence of alcoves starting at $\mathcal{A}_{\mathrm{id}}$ such that the next one can be obtained by moving from some point inside the previous alcove in direction λ. As a result, we see that no hyperplanes can be crossed twice by this alcove path $\mathbf{i}$ and that the hyperplanes crossed are exactly those crossed by t_λ^∞. Therefore, $\mathbf{i}$ is reduced and $[\mathbf{i}] = [t_\lambda^\infty]$. □

Remark 3.3. Notice that if after some number of iterations we have $\lambda^{(k)} = \lambda$, the construction will repeat itself, so in this case $[t_\lambda^\infty] = [(s_{i_1} \cdots s_{i_k})^\infty]$.

4 Limit Coxeter elements

Proposition 4.1. *Let $\widetilde{W}$ be an affine Weyl group other than the affine symmetric group, and let c, c' be any two Coxeter elements for $\widetilde{W}$, then c and c' are W-conjugate.*

Proof. It is well-known (and easy to verify) that the distinct Coxeter elements c for any Coxeter group correspond naturally to the acyclic orientations $\mathcal{O}$ of the edges of the Dynkin diagram, with a directed edge from α_i to an adjacent node α_j indicating that s_i precedes s_j in the product defining c.

Let c, c' be two Coxeter elements of $\widetilde{W}$ with corresponding orientations $\mathcal{O}, \mathcal{O}'$. Conjugating c by s_i corresponds to reversing the orientation of all edges incident to the node α_i. Since the Dynkin diagram is a tree, it is not hard to see that we may move from $\mathcal{O}$ to $\mathcal{O}'$ by a sequence of such moves, so c, c' are $\widetilde{W}$-conjugate. To see that they are in fact W-conjugate, note that reversing orientations at a single node α_i has the same effect as reversing at every node except α_i. Therefore we can connect c and c' without ever conjugating by s_0. □

It is a theorem of Speyer [6] that infinite powers of Coxeter elements c in $\widetilde{W}$ are always reduced. Since W is finite, this implies that some power of c lies in $Q^\vee$.

Corollary 4.2. *If $\widetilde{W}$ is an affine Weyl group other than the affine symmetric group and if we have $[c^\infty] = [t^\infty_{wk_i\omega_i^\vee}]$ for some Coxeter element c for $\widetilde{W}$ and some $w \in W$, then for every Coxeter element c' we have*

$$[(c')^\infty] = [t^\infty_{uk_i\omega_i^\vee}]$$

for some $u \in W$.

Proof. We must have $c^a = t^b_{wk_i\omega_i^\vee}$ for some positive integers a, b. Let $v \in W$ be such that $vcv^{-1} = c'$ (guaranteed to exist by Proposition 4.1), then we have

$$(c')^a = (vcv^{-1})^a = vc^a v^{-1} = vt^b_{wk_i\omega_i^\vee}v^{-1} = vt_{wbk_i\omega_i^\vee}v^{-1} = t^b_{vwk_i\omega_i^\vee}.$$

Thus we can take $u = vw$. □

In light of Corollary 4.2, we say α_i is a *Coxeter node* for $\widetilde{W}$ if $[c^\infty] = [t^\infty_{wk_i\omega_i^\vee}]$ for some $w \in W$ and some Coxeter element c. By Corollary 4.2 (except when $\widetilde{W}$ is the affine symmetric group, where all nodes are Coxeter nodes by Theorem 1.1) the Coxeter node is unique if it exists.

In the classification of irreducible finite root systems a standard reduction uses the fact that (except in type A) every Dynkin diagram contains either a unique node α_i adjacent to three other nodes or a unique multiple edge. This multiple edge, if it exists, connects two nodes whose corresponding simple roots have different lengths; call the longer one α_i. In either case, we say α_i is the *heavy node*.

Theorem 4.3. *Let $\widetilde{W}$ be an affine Weyl group other than the affine symmetric group. Then the Coxeter node exists and is equal to the heavy node.*

The proof of Theorem 4.3 is too long to include here, but it makes use of the technique introduced in Proposition 3.2.

5 Infinite fully commutative elements

5.1 Fully commutative nodes

Given infinite reduced words $\mathbf{i}$ and $\mathbf{j}$, we say there is a *braid limit* from $\mathbf{i}$ to $\mathbf{j}$ (written $\mathbf{i} \to \mathbf{j}$) if there is a (possibly infinite) sequence of braid and commutation moves taking $\mathbf{i}$ to $\mathbf{j}$. Note that $\mathbf{i} \to \mathbf{j}$ does not imply $\mathbf{j} \to \mathbf{i}$ since an infinite sequence of moves might irreversibly send a letter of $\mathbf{i}$ "to infinity" (see Example 3 of [4]).

The following proposition is a generalization of Lemma 4.6 from [4].

Proposition 5.1. *Let $\mathbf{i}$ and $\mathbf{j}$ be infinite reduced words. Then $[\mathbf{j}] \leq [\mathbf{i}]$ if and only if $\mathbf{i} \to \mathbf{j}$.*

We omit the proof of Proposition 5.1 since the arguments in [4] carry over to all types.

Corollary 5.2. *If $\mathbf{i}$ is a fully commutative infinite reduced word, then $[\mathbf{i}]$ is a minimal element in $\widetilde{\mathcal{W}}$.*

Proof. Since $\mathbf{i}$ is fully commutative, any braid limit $\mathbf{i} \to \mathbf{j}$ uses only commutation moves. Since $\mathbf{i}$ is reduced, and since all parabolic subgroups of $\widetilde{W}$ are finite, no single letter of $\mathbf{i}$ can move off to infinity, as it would eventually encounter another letter of the same kind. This implies that $[\mathbf{i}] = [\mathbf{j}]$, since any finite sequence of these moves does not change the inversion set. Therefore, by Proposition 5.1, there does not exist $[\mathbf{j}]$ strictly smaller than $[\mathbf{i}]$ in $\widetilde{\mathcal{W}}$. □

Lemma 5.3. *Let $\lambda \in Q^\vee$. Then $[t_\lambda^\infty]$ is fully commutative if and only if $[t_{w\lambda}^\infty]$ is fully commutative for any $w \in W$.*

Proof. We first investigate relations between explicit words of $t^\infty_{k_i\omega_i^\vee}$ and $t^\infty_{wk_i\omega_i^\vee}$. Recall that $J_i = \{s_j \mid j \neq i, 0\}$ and let $w = w^{J_i} w_{J_i}$ be the parabolic decomposition. Since w_{J_i} fixes $\omega_i^\vee$, $wk_i\omega_i^\vee = w^{J_i}k_i\omega_i^\vee$. The inversion set $\mathrm{Inv}(w^{J_i}) \subseteq \{\alpha \in \Phi^+ \mid \alpha_i \leq \alpha\}$ is contained in the set of all positive roots of W supported on α_i. The corresponding hyperplanes of $\mathrm{Inv}(w^{J_i})$ must all be crossed if we move in the direction of $\omega_i^\vee$. Thus, we have $\mathrm{Inv}(w^{J_i}) \subset \mathrm{Inv}(t_{-k_i\omega_i^\vee})$. As $t_{k_i\omega_i^\vee} = t^{-1}_{-k_i\omega_i^\vee}$, we can then recognize $(w^{J_i})^{-1}$ as a prefix for $t_{k_i\omega_i^\vee}$ and write $u = w^{J_i} t_{k_i\omega_i^\vee}$. In this way, we can choose reduced words for $(w^{J_i})^{-1}$ and u so that $t_{k_i\omega_i^\vee} = (w^{J_i})^{-1}u$ and $t_{wk_i\omega_i^\vee} = t_{w^{J_i}k_i\omega_i^\vee} = w^{J_i} t_{k_i\omega_i^\vee} (w^{J_i})^{-1} = u(w^{J_i})^{-1}$. Therefore, both

$t^{\infty}_{k_i\omega_i^\vee}$ and $t^{\infty}_{wk_i\omega_i^\vee}$ are consecutive subwords of each other. Thus one is fully commutative if and only if the other is.

For the purpose of this lemma, we can without loss of generality assume that $[t^{\infty}_{\lambda}]$ is fully commutative. By Corollary 5.2 and Proposition 3.1, we have that $\lambda = uk_i\omega_i^\vee$ for some $u \in W$ and fundamental weight $\omega_i^\vee$. By our argument above, $[t^{\infty}_{w\lambda}] = [t^{\infty}_{wu\omega_i^\vee}]$ is fully commutative as well. $\square$

Building up from Corollary 5.2, Proposition 3.1 and Lemma 5.3, we see that an infinite fully commutative reduced word $[\mathbf{i}]$ must be $[t^{\infty}_{wk_i\omega_i}]$ for some $w \in W$ and some particular fundamental weight ω_i.

Definition 5.4. We say that a node α_i of the Dynkin diagram of W is *fully commutative* if $[t^{\infty}_{k_i\omega_i^\vee}]$ (or equivalently, $[t^{\infty}_{wk_i\omega_i^\vee}]$ for any $w \in W$) is fully commutative.

A weight λ is *minuscule* if all weights in the associated irreducible representation of the corresponding simple Lie algebra lie in the W-orbit of λ, and *cominuscule* if $\lambda^\vee$ is a minuscule weight for the dual root system. The classification of minuscule weights is well known (see, e.g. [2]). We say that a node of the Dynkin diagram is *minuscule* if the corresponding fundamental weight is minuscule or cominuscule.

The following is our main result of the section, completely answering Problem 2.

Theorem 5.5. *Let $\widetilde{W}$ be any affine Weyl group, then the fully commutative nodes are exactly the minuscule nodes.*

Proof sketch. If a node α_i is fully commutative, then $t_{k_i\omega_i^\vee}$ is fully commutative and so $w_0^{J_i}$ must be fully commutative as well, since $(w_0^{J_i})^{-1} \leq_R t_{k_i\omega_i^\vee}$ (see the proof of Lemma 5.3). By Theorem 6.1 of [7], this implies that α_i is a minuscule node.

For the converse, it is possible to apply Proposition 3.2 to verify that the infinite translation elements corresponding to any minuscule node are indeed fully commutative. $\square$

5.2 Direct classification of fully commutative infinite reduced words

Corollary 5.2, Proposition 3.1, Proposition 3.2 and Lemma 5.3 together allow us to explicitly produce all fully commutative infinite reduced words. In this section we sketch an alternative, more direct, "density" approach to the same problem of classifying infinite fully commutative words.

Let $\widetilde{W}$ be an affine Weyl group not of type A and let v be a node of the Dynkin diagram such that v connects to its neighbors by simple edges (i.e. $(s_v s_i)^3 = \mathrm{id}$ if v is adjacent to i, and v connects to each connected component of $\widetilde{S} \setminus \{s_v\}$ in one of the three ways shown in Figure 2). We call them type A_m branch, type B_m branch and type D_m branch respectively and say that such v is a *branch node*. In type $\widetilde{B_3}$, $\widetilde{C_3}$, $\widetilde{F_4}$, $\widetilde{G_2}$, such a branch node v does not exist, but computation by hand is possible.

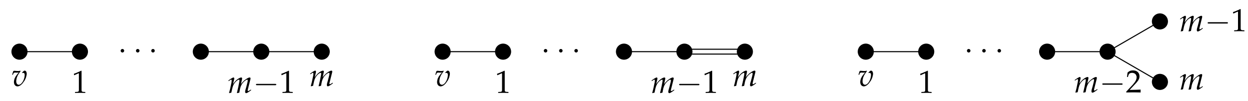

Figure 2: A type A_m branch, type B_m branch and type D_m branch connected to v.

Let $J_1, \ldots, J_b$ be the connected components of $\widetilde{S} \setminus \{s_v\}$ and let $\mathbf{i}$ be a fully commutative reduced word. By identifying the positions of s_v in $\mathbf{i}$, we can write

$$\mathbf{i} = (w_{J_1}^{(0)} w_{J_2}^{(0)} \cdots w_{J_b}^{(0)}) s_v (w_{J_1}^{(1)} \cdots w_{J_b}^{(1)}) s_v \cdots s_v (w_{J_1}^{(p)} \cdots w_{J_b}^{(p)}) s_v \cdots$$

where $w_{J_k}^{(p)} \in W_{J_k}$. Note that $w_J^{(p)}$ commutes with $w_{J'}^{(p')}$ for $J \neq J'$. Also note that each $w_{J_k}^{(p)}$ is not well-defined for a class $[\mathbf{i}]$. Regardless, for $J \in \{J_1, \ldots, J_b\}$ and $p \geq 0$, let

$$d([\mathbf{i}], J)_p = \begin{cases} 0, & \text{if commutation moves can be applied so that } w_J^{(p)} = \mathrm{id} \\ 2, & \text{if commutation moves can be applied so that } w_{J'}^{(p)} = \mathrm{id} \text{ for all } J' \neq J \\ 1, & \text{otherwise} \end{cases}.$$

The following lemma says that we need a total "density" of at least 2 from $J_1, \ldots, J_b$.

Lemma 5.6. *For every $p \geq 1$, $d([\mathbf{i}], J_1)_p + \cdots + d([\mathbf{i}], J_b)_p \geq 2$.*

The following main technical lemma of the section provides an upper bound of the "density" of the branches.

Lemma 5.7. *With the above notation,*

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{p=1}^{N} d_p([\mathbf{i}], J) \leq \begin{cases} \frac{m}{m+1}, & \text{if } J \text{ is a type } A_m \text{ branch} \\ 1, & \text{if } J \text{ is a type } B_m \text{ or } D_m \text{ branch} \end{cases}.$$

Moreover, we can determine explicit words for each $w_J^{(p)}$ (for $p \gg 0$) when equality is achieved.

Surprisingly, for affine types the bounds from Lemma 5.6 and Lemma 5.7 coincide exactly. For example, in type $\widetilde{E_7}$, we choose $v = 3$ and obtain three branches of type A_1, A_3 and A_3 respectively. The bounds in Lemma 5.7 are 1/2, 3/4 and 3/4 which sum up to 2. Write down the explicit words provided by Lemma 5.7 when equality is achieved and with simple analysis of how different branches interact (see Table 1), we obtain all fully commutative infinite reduced words in $\widetilde{E_7}$: $w(s_3s_7s_5s_4s_3s_2s_6s_5s_4s_3s_7s_1s_2s_3s_0s_1s_2s_4)^{\infty}$.

A Computations for minimal elements

We provide a list of all minimal elements of $\widetilde{\mathcal{W}}$ in Table 2, computed using Proposition 3.2, answering Problem 1. We omit the cases $\widetilde{E_7}$ and $\widetilde{E_8}$ for the sake of space.

	$p+1$	$p+2$	$p+3$	$p+4$	$\cdots$
$w_{J_1}^{(\cdot)}$, $J_1 = \{s_7\}$	s_7	id	s_7	id	s_7
$w_{J_2}^{(\cdot)}$, $J_2 = \{s_0, s_1, s_2\}$	$s_1 s_2$	$s_0 s_1 s_2$	id	s_2	$s_1 s_2$
$w_{J_3}^{(\cdot)}$, $J_3 = \{s_4, s_5, s_6\}$	id	s_4	$s_5 s_4$	$s_6 s_5 s_4$	id

Table 1: Fully commutative infinite words for $\widetilde{E_7}$.

Type	Coweight	Reduced Word	Note
$\widetilde{B}_n$	$\omega_k^\vee$, $1 \le k \le n-1$	$w(s_0 s_1 s_2 \cdots s_{n-1} s_n s_{n-1} \cdots s_{k+1})^\infty$	$\omega_{n-1}^\vee$ Coxeter
	$\omega_n^\vee$	$w(s_0 s_2 s_3 \cdots s_n s_1 s_2 \cdots s_n)^\infty$	fully commutative
$\widetilde{C}_n$	$\omega_k^\vee$, $1 \le k \le n$	$w(s_0 s_1 s_2 \cdots s_n s_{n-1} \cdots s_k)^\infty$	$\omega_n^\vee$ Coxeter; $\omega_1^\vee, \omega_n^\vee$ f.c.
$\widetilde{D}_n$	$\omega_k^\vee$, $1 \le k \le n-2$	$w(s_0 s_1 \cdots s_n s_{n-2} s_{n-3} \cdots s_{k+1})^\infty$	$\omega_{n-2}^\vee$ Coxeter; $\omega_1^\vee$ f.c.
	$\omega_{n-1}^\vee$	$w(s_0 s_2 s_3 \cdots s_{n-2} s_n s_1 s_2 \cdots s_{n-2} s_{n-1})^\infty$	fully commutative
	$\omega_n^\vee$	$w(s_0 s_2 s_3 \cdots s_{n-1} s_1 s_2 \cdots s_{n-2} s_n)^\infty$	fully commutative
$\widetilde{G}_2$	$\omega_1^\vee$	$w(s_0 s_1 s_2)^\infty$	Coxeter
	$\omega_2^\vee$	$w(s_0 s_1 s_2 s_1 s_2)^\infty$	
$\widetilde{F}_4$	$\omega_1^\vee$	$w(s_0 s_1 s_2 s_3 s_4 s_2 s_3 s_2)^\infty$	
	$\omega_2^\vee$	$w(s_0 s_1 s_2 s_3 s_4)^\infty$	Coxeter
	$\omega_3^\vee$	$w(s_0 s_1 s_2 s_3 s_4 s_2 s_3)^\infty$	
	$\omega_4^\vee$	$w(s_0 s_1 s_2 s_3 s_4 s_1 s_2 s_3)^\infty$	
$\widetilde{E}_6$	$\omega_1^\vee$	$w(s_0 s_6 s_3 s_4 s_5 s_2 s_3 s_4 s_6 s_3 s_2 s_1)^\infty$	fully commutative
	$\omega_2^\vee$	$w(s_0 s_6 s_3 s_4 s_5 s_2 s_3 s_4 s_1)^\infty$	
	$\omega_3^\vee$	$w(s_0 s_1 s_2 s_3 s_4 s_5 s_6)^\infty$	Coxeter
	$\omega_4^\vee$	$w(s_0 s_6 s_3 s_2 s_1 s_4 s_3 s_2 s_5)^\infty$	
	$\omega_5^\vee$	$w(s_0 s_6 s_3 s_2 s_1 s_4 s_3 s_2 s_6 s_3 s_4 s_5)^\infty$	fully commutative
	$\omega_6^\vee$	$w(s_0 s_6 s_3 s_4 s_5 s_2 s_3 s_4 s_1 s_2 s_3)^\infty$	

Table 2: A list of minimal elements of $\widetilde{\mathcal{W}}$, associated to fundamental coweights as in Proposition 3.1. In each case w ranges over W^{J_k} for the words corresponding to $\omega_k^\vee$. We indicate the Coxeter and fully commutative nodes (see Theorems 4.3 and 5.5).

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