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Some natural extensions of the parking space

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Abstract. We construct a family of S_n modules indexed by $c \in \{1, \ldots, n\}$ with the property that upon restriction to S_{n-1} they recover the classical parking function representation of Haiman. The construction of these modules relies on an S_n-action on a set that is closely related to the set of parking functions. We compute the characters of these modules and use the resulting description to classify them up to isomorphism. In particular, we show that the number of isomorphism classes is equal to the number of divisors d of n satisfying $d \neq 2 \pmod 4$. In the cases $c = n$ and $c = 1$, we compute the number of orbits. Based on empirical evidence, we conjecture that when $c = 1$, our representation is h-positive and is in fact the (ungraded) extension of the parking function representation constructed by Berget and Rhoades.

Keywords: parking functions, representations, symmetric group

1 Introduction

Parking functions were introduced by Konheim and Weiss [10] in their investigation of hashing functions in computer science. Since then, they, along with their various generalizations, have attracted plenty of attention and have proven to be a fertile source of interesting mathematics. This is reflected by their appearances in diverse areas such as hyperplane arrangements [3], representation theory [2], polytopes [18], the sandpile model [5], and Macdonald polynomials [9]. The last of these areas provides the context for our work and we detail our motivation next.

An integer sequence $(x_1, \ldots, x_n)$ is a *parking function* if its weakly increasing rearrangement $(z_1, \ldots, z_n)$ satisfies $0 \leq z_i \leq i - 1$ for $i = 1, \ldots, n$. This definition implies that rearranging the entries in one parking function results in another. Haiman [9] was the first to study the S_n action on the set of parking functions of length n. We denote the resulting S_n-representation by ρ_n. Two decades later, Berget-Rhoades [4] studied the following seemingly unrelated representation σ_n of S_n. Let K_n denote the complete

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graph with vertex set $[n] := \{1, \ldots, n\}$. Given a subgraph $G \subseteq K_n$, we attach to it the polynomial $p(G) := \prod_{ij \in E(G)} (x_i - x_j) \in \mathbb{C}[x_1, \ldots, x_n]$. Here $E(G)$ refers to the set of edges of G and we record those by listing the smaller number first. Define V_n to be the $\mathbb{C}$-linear span of $p(G)$ over all G for which the complement $\overline{G}$ is a connected graph. We remark here that V_n first appears in the work of Postnikov and Shapiro [14], where the graphs G with the property that $\overline{G}$ is connected are called *slim graphs*. The natural action of S_n on $\mathbb{C}[x_1, \ldots, x_n]$ that permutes variables gives an action on V_n because relabeling vertices preserves connectedness. Amongst various other interesting things, Berget and Rhoades [4, Theorem 2] establish the remarkable fact that the restriction of σ_n to S_{n-1} is isomorphic to ρ_{n-1}. The question of extending symmetric group representations in general has also received attention; see [13, 19].

The primary goal of this extended abstract is to construct a family of permutation representations $\widehat{\mathrm{PF}}_{n,c}$ of S_n with easy-to-compute characters, which all also restrict to ρ_{n-1}. Interestingly, the modular behavior of the sum of elements in a parking function (closely related to the area statistic on parking functions) plays a key role in our analysis, and our arguments rely on some subtle number-theoretic considerations. The authors in fact believe that the representation $\widehat{\mathrm{PF}}_{n,1}$ is isomorphic to the (ungraded) Berget-Rhoades representation mentioned above; see Conjecture 3.3.

The full version of this paper is [12].

2 Background

For any undefined terminology in the context of symmetric functions, we refer the reader to [17]. For $n \geq 1$, we denote by $\mathbb{Z}_n$ the set of integers modulo n. Typically, representatives from residue classes modulo n will be implicitly assumed to belong to $\{0, \ldots, n-1\}$. Throughout, S_n denotes the symmetric group consisting of permutations of $[n]$. We use both the cycle notation and the one-line notation for permutations depending on our needs. If we use the latter, then we let π_i denote the image of i under the permutation π for a positive integer i.

2.1 Symmetric functions

A *partition* $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a weakly decreasing sequence of positive integers. The λ_i's' are the *parts* of λ, their sum its *size*, and their number its *length*, which is denoted by $\ell(\lambda)$. If λ has size n, then we denote this by $\lambda \vdash n$. Furthermore, letting m_i denote the multiplicity of the part i in λ for $i \geq 1$, we set $z_\lambda := \prod_{i \geq 1} i^{m_i} m_i!$. The cycle type of a permutation π is a partition that we denote $\lambda(\pi)$.

We consider the following distinguished bases for the ring of symmetric functions Λ: the *power sum* symmetric functions $\{p_\lambda \colon \lambda \vdash n\}$, the *complete homogeneous* symmetric functions $\{h_\lambda \colon \lambda \vdash n\}$, and the *Schur* symmetric functions $\{s_\lambda \colon \lambda \vdash n\}$.

The representation theory of the symmetric group is intimately tied to Λ and the connection is made explicit by the *Frobenius characteristic*. Given a representation ρ of S_n, denote the corresponding character by χ_ρ. Then

$$\mathrm{Frob}(\rho) = \frac{1}{n!}\sum_{\pi \in S_n} \chi_\rho(\pi) p_{\lambda(\pi)} = \sum_{\lambda \vdash n} \chi_\rho(\lambda)\frac{p_\lambda}{z_\lambda}.$$

Under Frob, the irreducible representation of S_n corresponding to the partition $\mu \vdash n$ gets mapped to the Schur function s_μ. We proceed to define parking functions and an associated representation whose study has substantially motivated algebraic combinatorics in the last two decades.

2.2 Parking functions

As mentioned earlier, an integer sequence $(x_1, \ldots, x_n)$ is a parking function if its weakly increasing rearrangement $(z_1, \ldots, z_n)$ satisfies $0 \le z_i \le i-1$ for $i = 1, \ldots, n$. We denote by PF_n the set of all parking functions of length n. For example, one can check that

$$\mathrm{PF}_3 = \{000, 001, 010, 100, 002, 020, 200, 011, 101, 110, 012, 021, 102, 120, 201, 210\}.$$

In the preceding example, we have omitted commas and parentheses in writing our parking functions for the sake of clarity, and we will do this throughout without explicit mention.

It is well known that $|\,\mathrm{PF}_n\,| = (n+1)^{n-1}$. One way to see this is through the following result present in [6] (where it is attributed to H. O. Pollak) that will also be crucial in the sequel.

Theorem 2.1 (Pollak). *The map* $\mathrm{PF}_n \to \mathbb{Z}_{n+1}^{n-1}$*, given by*

$$(x_1, \ldots, x_n) \mapsto (x_2 - x_1, \ldots, x_n - x_{n-1}),$$

where subtraction is performed modulo $n+1$*, is a bijection.*

Note that in particular Theorem 2.1 says that for an arbitrary sequence $(\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{Z}_{n+1}^{n-1}$, exactly one of the sequences $(y, y+\alpha_1, y+\alpha_1+\alpha_2, \ldots, y+\alpha_1+\cdots+\alpha_{n-1})$, $y \in \mathbb{Z}_{n+1}$, is in PF_n.

Recall the natural action ρ_n of S_n on PF_n defined by

$$\pi \cdot (x_1, \ldots, x_n) = (x_{\pi_1}, \ldots, x_{\pi_n}).$$

For a partition $\lambda = (\lambda_1, \ldots \lambda_\ell) \vdash n$, the number of fixed points of the action of the permutation with cycle decomposition $(1, \ldots, \lambda_1)(\lambda_1+1, \ldots, \lambda_1+\lambda_2)\cdots$ is equal to the number of sequences $(\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{Z}_{n+1}^{n-1}$ satisfying $\alpha_i = 0$ for $i \in [n-1] \setminus \{\lambda_1, \lambda_1 + \lambda_2, \ldots, \lambda_1 + \cdots + \lambda_{\ell-1}\}$. It follows that the character χ_{ρ_n} of ρ_n satisfies

$$\chi_{\rho_n}(\pi) = (n+1)^{\ell-1},$$

where $\ell := \ell(\lambda(\pi))$.

3 Main results

For $n \in \mathbb{N}$ and $1 \le c \le n$, define the set

$$\widehat{\mathrm{PF}}_{n,c} = \{(x_1, \ldots, x_n) \in \mathbb{Z}_n^n \colon (x_1, \ldots, x_{n-1}) \in \mathrm{PF}_{n-1},\ x_n = c - \sum_{1 \le i \le n-1} x_i \,(\mathrm{mod}\, n)\}.$$

In other words, given a parking function $(x_1, \ldots, x_{n-1})$, x_n is uniquely determined by the constraint $\sum_{i=1}^n x_i = c \,(\mathrm{mod}\, n)$. For example, the reader may check that

$$\widehat{\mathrm{PF}}_{3,1} = \{001, 010, 100\}, \quad \widehat{\mathrm{PF}}_{3,2} = \{002, 011, 101\}, \quad \widehat{\mathrm{PF}}_{3,3} = \{000, 012, 102\}.$$

It is obvious that, for every $1 \le c \le n$, the projection $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1})$ is a bijection $\widehat{\mathrm{PF}}_{n,c} \to \mathrm{PF}_{n-1}$. In particular, we have $|\widehat{\mathrm{PF}}_{n,c}| = n^{n-2}$. Again, we can construct an action $\tau_{n,c}$ of S_n on $\widehat{\mathrm{PF}}_{n,c}$. Take $\pi \in S_n$ and $(x_1, \ldots, x_n) \in \widehat{\mathrm{PF}}_{n,c}$. Note that $(x_{\pi_1}, \ldots, x_{\pi_{n-1}})$ is not necessarily in PF_{n-1}, and therefore $(x_{\pi_1}, \ldots, x_{\pi_n})$ is not necessarily in $\widehat{\mathrm{PF}}_{n,c}$. However, by Pollak's theorem, exactly one of the sequences $(y + x_{\pi_1}, \ldots, y + x_{\pi_{n-1}})$ is in PF_{n-1}, and therefore $(y + x_{\pi_1}, \ldots, y + x_{\pi_n}) \in \widehat{\mathrm{PF}}_{n,c}$. This element is the action of π on $(x_1, \ldots, x_n)$. For example, consider the action of $\pi = 1432 \in S_4$ on $0003 \in \widehat{\mathrm{PF}}_{4,3}$. Naïvely permuting elements of the sequence 0003 according to π leads to 0300. Note that $030 \notin \mathrm{PF}_3$, but adding 1 to each coordinate gives $101 \in \mathrm{PF}_3$. Thus $1432 \cdot 0003 = 1011$.

The following is our first main result.

Theorem 3.1. *The map $\tau_{n,c}$ is an action of S_n on $\widehat{\mathrm{PF}}_{n,c}$ whose restriction to S_{n-1} is isomorphic to ρ_{n-1}. Furthermore, the character $\chi_{n,c} := \chi_{\tau_{n,c}}$ can be computed as follows. Choose a permutation $\pi \in S_n$ with cycle type $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, and write $d := \mathrm{GCD}(\lambda_1, \ldots, \lambda_\ell)$. Then*

$$\chi_{n,c}(\pi) = \begin{cases} \frac{d^2 n^{\ell-2}}{2} & d \text{ even}, \frac{n}{d} \text{ odd, and } d | 2c \\ d^2 n^{\ell-2} & d \text{ even}, \frac{n}{d} \text{ even, and } d | c \\ d^2 n^{\ell-2} & d \text{ odd and } d | c \\ 0 & \text{otherwise.} \end{cases}$$

As a corollary, we completely classify the representations $\tau_{n,c}$ up to isomorphism, and show in particular that the number of non-isomorphic representations is equal to the number of divisors of n that are not $2 \pmod 4$. We refer the reader to Section 4 for further details, in particular to Theorem 4.3 and Corollary 4.4.

Subsequently we focus on the cases where c equals n (equivalently, 0) and 1. In both cases we compute the multiplicity of the trivial representation in $\widehat{\mathrm{PF}}_{n,c}$, or equivalently, the number of orbits under $\tau_{n,c}$. As our second main result, we state below the character in the case $c = 1$ as well as the number of orbits.

Theorem 3.2. *The character $\chi_{n,1}$ can be computed as follows. Choose a permutation $\pi \in S_n$ with cycle type $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, and write $d := \mathrm{GCD}(\lambda_1, \ldots, \lambda_\ell)$. Then*

$$\chi_{n,1}(\pi) = \begin{cases} n^{\ell-2} & d = 1 \\ 2n^{\ell-2} & d = 2,\ n = 2 \pmod 4 \\ 0 & \textit{otherwise.} \end{cases}.$$

As a consequence, the number of orbits of the action $\tau_{n,1}$ is given by

$$o_{n,1} = \frac{1}{n^2} \sum_{d|n} (-1)^{n+d} \mu(n/d) \binom{2d-1}{d},$$

where μ is the classical Möbius function.

Note that the sequence $(o_{n,1})_{n\in\mathbb{N}}$ starts with $1, 1, 1, 2, 5, 13, 35, 100, 300$ (see [16, A131868]). The computation of $\chi_{n,1}$ obviously follows from 3.1. The proof of the formula for the number of orbits can be found in the full version of the paper [12].

Recall from the introduction that understanding the Berget-Rhoades extension was our main motivation. In this context, we offer the following conjecture to close this section.

Conjecture 3.3. *The representation $\tau_{n,1}$ is isomorphic to σ_n. Furthermore,* $\mathrm{Frob}(\tau_{n,1})$ *expands positively in the basis of homogeneous symmetric functions, i.e., it is h-*positive.

It is worth noting that from the original definition of σ_n in terms of slim graphs, it is not straightforward to compute its character. In this regard, assuming the validity of Conjecture 3.3, one could say that $\tau_{n,1}$ is the computationally more amenable representation. See also the first final remark.

4 Characters and classification of the $\widehat{\mathrm{PF}}_{n,c}$

Before providing a proof to our main result stated earlier, we state a lemma that will be useful in compute the character $\chi_{n,c}$ of the S_n action on $\widehat{\mathrm{PF}}_{n,c}$.

Lemma 4.1. *For $a_1, \ldots, a_k, c \in \mathbb{Z}$, $m \in \mathbb{N}$ the number of tuples $(x_1, \ldots, x_k) \in \{0, \ldots, m-1\}^k$ that satisfy*

$$a_1 x_1 + \cdots + a_k x_k = c \pmod m$$

is equal to dm^{k-1} if $d|c$, and 0 otherwise. Here $d = \mathrm{GCD}(a_1, \ldots, a_k, m)$.

4.1 Proof of Theorem 3.1

Since the maps $(x_1,\dots,x_n) \mapsto (x_{\pi_1},\dots,x_{\pi_n})$ and $(x_1,\dots,x_n) \mapsto (y+x_1,\dots,y+x_n)$ commute, we conclude that $\tau_{n,c}$ is an action. It is also clear that the restriction of τ_n to S_{n-1} is ρ_{n-1}. It remains to compute the character $\chi_{n,c}$.

Without loss of generality, assume that $\pi = (1,\dots,\lambda_1)(\lambda_1+1,\dots,\lambda_1+\lambda_2)\cdots$, and set $d := \mathrm{GCD}(\lambda)$. Also, following [17, Equation 7.103], define

$$b(\lambda') := \sum_{i=1}^{\ell} \binom{\lambda_i}{2} \tag{4.1}$$

where λ' denotes the transpose of λ. As λ is fixed, we set $b := b(\lambda')$ for convenience. We want to count the number of fixed points of π.

Suppose that $\pi \cdot (x_1,\dots,x_n) = (x_1,\dots,x_n)$. We have

$$\pi \cdot (x_1,\dots,x_n) = (x_2+y,\dots,x_{\lambda_1}+y, x_1+y, x_{\lambda_1+2}+y,\dots,x_{\lambda_1+\lambda_2}+y, x_{\lambda_1+1}+y,\dots)$$

for some $y \in \mathbb{Z}_n$, so $x_1 = x_2+y, x_2 = x_3+y,\dots,x_{\lambda_1-1} = x_{\lambda_1}+y, x_{\lambda_1} = x_1+y$, $x_{\lambda_1+1} = x_{\lambda_1+2}+y, x_{\lambda_1+2} = x_{\lambda_1+3}+y,\dots x_{\lambda_1+\lambda_2-1} = x_{\lambda_1+\lambda_2}+y, x_{\lambda_1+\lambda_2} = x_{\lambda_1+1}+y$ etc.

The equalities immediately imply that $\lambda_i \cdot y = 0 \pmod n$, and consequently $d \cdot y = 0 \pmod n$. In other words, $y = k \cdot \frac{n}{d}$ for some $k \in \mathbb{Z}$, $0 \le k < d$. Furthermore, the sum of the coordinates of $\pi \cdot (x_1,\dots,x_n)$ is, modulo n, equal to c, and therefore

$$\lambda_1 x_1 + \binom{\lambda_1}{2} y + \lambda_2 x_{\lambda_1+1} + \binom{\lambda_2}{2} y + \cdots + \lambda_\ell x_{\lambda_1+\cdots+\lambda_{\ell-1}+1} + \binom{\lambda_\ell}{2} y = c \pmod n. \tag{4.2}$$

Set $f_1 := x_1$, $f_2 := x_{\lambda_1+1},\dots,$ $f_\ell = x_{\lambda_1+\cdots+\lambda_{\ell-1}+1}$. Then counting fixed points of π is tantamount to counting tuples $(f_1,\dots,f_l) \in \mathbb{Z}_n^\ell$ (up to translation by $(1,\dots,1) \in \mathbb{Z}_n^\ell$) that satisfy

$$\sum_{i=1}^{\ell} \lambda_i f_i + yb = c \pmod n. \tag{4.3}$$

Assume first that d is odd. Then $d|\lambda_i$ implies $d|\binom{\lambda_i}{2}$, and therefore $d|b$. It follows that

$$yb = \frac{b}{d} \cdot k \cdot n = 0 \pmod n,$$

which in turn implies that (4.3) reduces to

$$\sum_{i=1}^{\ell} \lambda_i f_i = c \pmod n. \tag{4.4}$$

Using Lemma 4.1 and recalling that we have d choices for y, we infer that there are $d^2 n^{\ell-2}$ (we have power of $\ell - 2$ instead of $\ell - 1$ because we look at tuples up to translation by $(1, \ldots, 1)$, i.e. we can fix one of f_i's to be, say, 0) elements in $\widehat{\mathrm{PF}}_{n,c}$ fixed by π if $d|c$, and 0 otherwise.

Now assume that d (and consequently n) is even. Then $\frac{d}{2}|\frac{\lambda_i}{2}$ and $\frac{d}{2}|\binom{\lambda_i}{2}$, thereby implying $d|2b$. It follows that

$$yb = \frac{2b}{d} \cdot k \cdot \frac{n}{2}. \tag{4.5}$$

We are naturally led to consider two scenarios based on the parity of $2b/d$. First note that $n/d = \lambda_1/d + \cdots + \lambda_\ell/d$ is odd if and only if the number of odd numbers among $\lambda_1/d, \ldots, \lambda_\ell/d$ is odd. On the other hand $2b/d = \lambda_1(\lambda_1 - 1)/d + \cdots + \lambda_\ell(\lambda_\ell - 1)/d$, and $\lambda_1 - 1, \ldots, \lambda_\ell - 1$ are all odd, so $2b/d$ is also odd if and only if the number of odd numbers among $\lambda_1/d, \ldots, \lambda_\ell/d$ is odd. In other words, $2b/d$ and n/d have the same parity.

Suppose that $2b/d$ and n/d are even. In view of the equality in (4.5), we may rewrite (4.3) as

$$\sum_{i=1}^{\ell} \lambda_i f_i = c \pmod{n}. \tag{4.6}$$

Like before, we infer that $d^2 n^{\ell-2}$ elements in $\widehat{\mathrm{PF}}_{n,c}$ are fixed by π if $d|c$, and 0 otherwise.

Finally consider the case where $2b/d$ and n/d are odd. We need to count solutions to

$$\sum_{i=1}^{\ell} \lambda_i f_i = c + \frac{kn}{2} \pmod{n}. \tag{4.7}$$

Note crucially that since $\frac{n}{d}$ is odd, it cannot be that d divides both c and $c + \frac{n}{2}$. From the odd $k \in \{0, \ldots, d-1\}$, we get a contribution of $\frac{d^2 n^{\ell-2}}{2}$ if $d|(c + \frac{n}{2})$, and 0 otherwise. From the even $k \in \{0, \ldots, d-1\}$, we get a contribution of $\frac{d^2 n^{\ell-2}}{2}$ if $d|c$, and 0 otherwise. We leave it to the reader to check that in the case under consideration we have

$$d|c \text{ or } d|(c + \frac{n}{2}) \Leftrightarrow d|2c. \tag{4.8}$$

This concludes our proof.

4.2 Number of non-isomorphic $\widehat{\mathrm{PF}}_{n,c}$

Given a positive integer n, let $v_2(n)$ denote the 2-adic valuation of n, i.e., the highest power of 2 that divides n. Define D_n to be the following subset of the set of divisors of n:

$$D_n := \{k|n : n/k = n \pmod{2}\}. \tag{4.9}$$

For instance, we have $D_{12} = \{1, 2, 3, 6\}$. We will show that D_n indexes the isomorphism classes of the representations $\tau_{n,c}$. Prior to that we establish a straightforward lemma on the cardinality of D_n.

Lemma 4.2. *The cardinality of D_n equals the number of divisors of n that are not 2 modulo 4.*

Proof. Let $d(n)$ denote the number of divisors of n. Then $|D_n|$ equals $d(n)$ if $v_2(n) = 0$, and $v_2(n) \cdot d\left(\frac{n}{2^{v_2(n)}}\right)$ otherwise. It is easily checked the number of divisors of n that are not 2 modulo 4 satisfies the same recursion: such a divisor d must satisfy $v_2(d) \neq 1$. $\square$

For $k \in D_n$, consider the set

$$C_{n,k} := \{m \in [n] : \mathrm{GCD}(n,m) \in \{k, 2k\} \text{ if } \frac{n}{k} = 2 \,(\mathrm{mod}\, 4) \text{ and } \mathrm{GCD}(n,m) = k \text{ otherwise}\}. \tag{4.10}$$

As an example, consider $n = 12$, in which case we have

$$C_{12,1} = \{1, 5, 7, 11\}, \quad C_{12,2} = \{2, 4, 8, 10\}, \quad C_{12,3} = \{3, 9\}, \quad C_{12,6} = \{6, 12\}.$$

Note that the sets $C_{12,1}$, $C_{12,2}$, $C_{12,3}$, and $C_{12,6}$ form a partition of [12]. In fact the following more general statement is established in [12, Lemma 4.3]. We have

$$\coprod_{k \in D_n} C_{n,k} = [n], \tag{4.11}$$

where $\coprod$ denotes disjoint union. We are now ready for the classification.

Theorem 4.3. *For $k \in D_n$, the representations $\tau_{n,c}$ are isomorphic for all $c \in C_{n,k}$. Furthermore, for distinct $k, k' \in D_n$, we have that $\tau_{n,c}$ and $\tau_{n,c'}$ are non-isomorphic for every $c \in C_{n,k}$ and $c' \in C_{n,k'}$.*

As an immediate consequence of Theorem 4.3, we have:

Corollary 4.4. *There are $|D_n|$ many non-isomorphic representations among the $\tau_{n,c}$.*

In view of Lemma 4.2, we have that $|D_n|$ is given by [16, A320111]. Observe also the curious fact that the sequence $\{|D_n|\}_{n \geq 1}$ gives a multiplicative arithmetic function.

Example 4.5. Consider $n = 6$. Then $D_6 = \{1, 3\}$. Here are the power sum expansions for the two non-isomorphic representations amongst the $\tau_{6,c}$ for $c \in [6]$:

$$\begin{aligned}
\mathrm{Frob}(\tau_{6,1}) &= \frac{9}{5}p_{1^6} + \frac{9}{2}p_{21^4} + \frac{9}{4}p_{221^2} + \frac{1}{4}p_{222} + 2p_{31^3} + p_{321} + \frac{3}{4}p_{41^2} + \frac{1}{4}p_{42} + \frac{1}{5}p_{51},\\
\mathrm{Frob}(\tau_{6,3}) &= \frac{9}{5}p_{1^6} + \frac{9}{2}p_{21^4} + \frac{9}{4}p_{221^2} + \frac{1}{4}p_{222} + 2p_{31^3} + p_{321}\\
&\quad + \frac{1}{2}p_{33} + \frac{3}{4}p_{41^2} + \frac{1}{4}p_{42} + \frac{1}{5}p_{51} + \frac{1}{2}p_{6}.
\end{aligned}$$

5 Final remarks

Further work with Sulzgruber

Berget and Rhoades asked whether the permutation representation obtained by the action of S_{n-1} on parking functions of length $n-1$ can be extended to a permutation action of S_n. In joint work with Robin Sulzgruber [11], we answer this question in the affirmative. We realize our module in two different ways. The first description involves binary Lyndon words and the second involves the action of the symmetric group on the lattice points of the trimmed standard permutahedron. Note that this proves the second part of Conjecture 3.3.

A generalization to certain families of rational parking functions

We can broaden the scope of our results by applying our techniques to a subclass of the set of *rational parking functions*. These functions are a generalization of usual parking functions and their study is an active field of research in recent years [1, 8].

Consider coprime positive integers a and b. Define an (a,b)*-parking function* to be a sequence $(x_1,\ldots,x_a)$ of nonnegative integers with the property that the weakly increasing arrangement $(z_1,\ldots,z_a)$ satisfies $z_i \leq \frac{(i-1)b}{a}$. We denote the set of (a,b)-parking functions by $\mathrm{PF}_{a,b}$. It is clear that the set $\mathrm{PF}_{n,n+1}$ is the set PF_n from before.

We denote the natural action of S_a on $\mathrm{PF}_{a,b}$ by $\rho_{a,b}$. A generalization of Pollak's proof implies that the map from $\mathrm{PF}_{a,b} \to \mathbb{Z}_b^{a-1}$ given by mapping $(x_1,\ldots,x_a) \mapsto (x_2 - x_1,\ldots,x_a - x_{a-1})$, where subtraction is performed modulo b, is a bijection. This implies that $|\mathrm{PF}_{a,b}| = b^{a-1}$.

Mimicking our ideas from before, we construct a new set that is equinumerous with $\mathrm{PF}_{a,b}$. For $c \in [b]$, define the set

$$\widehat{\mathrm{PF}}_{a,b,c} := \{(x_1,\ldots,x_{a+1})\colon (x_1,\ldots,x_a) \in \mathrm{PF}_{a,b}, \sum_{1\leq i\leq a+1} x_i = c \pmod{b}\}.$$

As usual, we take x_{a+1} to lie in $\{0,\ldots,b-1\}$. Clearly, we have $|\widehat{\mathrm{PF}}_{a,b,c}| = b^{a-1}$.

In order to mimic our action from Section 3, we need to impose the constraint that $b|(a+1)$. Henceforth, assume that this is indeed the case. This given, we can construct an action $\tau_{a,b,c}$ of the symmetric group S_{a+1} on $\widehat{\mathrm{PF}}_{a,b,c}$. Take $\pi \in S_{a+1}$ and $(x_1,\ldots,x_{a+1}) \in \widehat{\mathrm{PF}}_{a,b,c}$. Like before, $(x_{\pi_1},\ldots,x_{\pi_a})$ is not necessarily in $\mathrm{PF}_{a,b}$, and therefore $(x_{\pi_1},\ldots,x_{\pi_{a+1}})$ is not necessarily in $\widehat{\mathrm{PF}}_{a,b,c}$. However, by the generalized Pollak's theorem, exactly one of the sequences $(y + x_{\pi_1},\ldots,y + x_{\pi_a})$ is in $\mathrm{PF}_{a,b}$, and therefore $(y + x_{\pi_1},\ldots,y + x_{\pi_{a+1}}) \in \widehat{\mathrm{PF}}_{a,b,c}$. This element is the action of π on $(x_1,\ldots,x_{a+1})$. The careful reader should note that we made use of the fact $b|(a+1)$ in obtaining an action.

Rather than repeating the analysis from before, we simply state our result for $c = 1$.

Theorem 5.1. *Take $a = kb-1$ for $b,k \in \mathbb{N}$. The map $\tau_{a,b,1}$ is an action of S_{a+1} on $\widehat{\mathrm{PF}}_{a,b,1}$ whose restriction to S_a is isomorphic to $\rho_{a,b}$. Furthermore, the character $\chi_{\tau_{a,b,1}}$ can be computed as follows. Choose a permutation $\pi \in S_{a+1}$ with cycle type $(\lambda_1,\ldots,\lambda_\ell)$, and set $d := \mathrm{GCD}(\lambda_1,\ldots,\lambda_\ell,b)$. Then*

$$\chi_{\tau_{a,b,1}(\pi)} = \begin{cases} b^{\ell-2} & d=1 \\ 2b^{\ell-2} & d=2,\ b=2\,(\mathrm{mod}\,4),\ k \text{ odd} \\ 0 & \text{otherwise} \end{cases}.$$

Letting $o_{a,b,1}$ denote this number of orbits under $\tau_{a,b,1}$, we have the following equality:

$$o_{a,b,1} = \frac{1}{b^2}\sum_{d|b}(-1)^{k(b+d)}\mu(b/d)\binom{(k+1)d-1}{kd}.$$

A plausible approach to establishing Conjecture 3.3

One way to prove the conjecture would be to find an explicit action-preserving map between $\widehat{\mathrm{PF}}_{n,1}$ and a particular basis of the space V_n. The following table shows the construction (for a representative of each orbit) for $n = 3,4,5$. Consider the case $n = 3$ for instance. By its definition, V_3 would be spanned by elements of $\{1, x_1-x_2, x_2-x_3, x_1-x_3\}$, and one can extract a basis from this, say $\{1, x_1-x_2, x_2-x_3\}$. In fact, one can read from the table the following S_3-invariant basis of V_3:

$$\{1-2x_1+x_2+x_3, 1-2x_2+x_1+x_3, 1-2x_3+x_1+x_2\}.$$

The map

$$100 \mapsto 1-2x_1+x_2+x_3, \quad 010 \mapsto 1-2x_2+x_1+x_3, \quad 001 \mapsto 1-2x_3+x_1+x_2$$

commutes with the action. We were not able to find an appropriate basis for $n \geq 6$, but we did check the conjecture (via character computations) for $n = 6$ as well. Note further that V_n is naturally graded by the number of edges of a slim graph. We do not see a compatible grading in our $\widehat{\mathrm{PF}}_n$.

3	001	$1-2x_3+x_1+x_2$
4	0003	$1-3x_4+x_1+x_2+x_3$
	0012	$(x_4-x_1)(x_4-x_2)(1-2x_3+x_1+x_2)$
5	00001	$1-4x_5+x_1+x_2+x_3+x_4$
	00033	$(-3x_4+x_1+x_2+x_3)(-3x_5+x_1+x_2+x_3)$
	01113	$(x_1-x_2)(x_1-x_3)(x_1-x_4)$
	00114	$(x_5-x_3)(x_5-x_4)(-2x_3+x_1+x_2)(-2x_4+x_1+x_2)$
	00123	$(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_4-x_1)(x_4-x_2)(1-2x_3+x_1+x_2)$

The number of orbits

Our last remark concerns the number of orbits $o_{n,1}$ (respectively $o_{a,b,1}$) under the action $\tau_{n,1}$ (respectively $\tau_{a,b,1}$). Observe that the integrality of these quantities is not obvious from their explicit formula, and hence the action appears to be encoding a subtle number-theoretic fact. According to [16, A131868], $no_{n,1}$ is equal to the number of n-element subsets of $\{1,\ldots,2n-1\}$ that sum to 1 modulo n and we establish this in [11, Corollary 2.4]. The numbers $o_{a,b,1}$ show up in a topological setting as Betti numbers as described in [15, Section 5]. Again the counting problem considered in the aforementioned article is different from ours. Finally, note that certain special cases of the $o_{a,b,1}$ equal the extremal BPS invariants of twist knots [7, Proposition 1.2]. We intend to explore these intriguing connections further.

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