Séminaire Lotharingien de Combinatoire **85B** (2021)
Article #44, 12 pp.

Proceedings of the 33rd Conference on Formal Power Series and Algebraic Combinatorics (Ramat Gan)

F- and H-Triangles for ν-Associahedra

Cesar Ceballos*[1] and Henri Mühle†[2]

[1] *TU Graz, Institut für Geometrie, Kopernikusgasse 24, 8010 Graz, Austria.*
[2] *TU Dresden, Institut für Algebra, Zellescher Weg 12–14, 01069 Dresden, Germany.*

Abstract. For any northeast path ν, we define two bivariate polynomials associated with the ν-associahedron: the F- and the H-triangle. We prove combinatorially that we can obtain one from the other by an invertible transformation of variables. These polynomials generalize the classical F- and H-triangles of F. Chapoton in type A. Our proof is completely new and has the advantage of providing a combinatorial explanation of the nature of the relation between the F- and H-triangle.

Keywords: ν-Tamari lattice, ν-associahedron, F-triangle, H-triangle

1 Introduction

The ν-Tamari lattice is an intriguing object in combinatorics which was originally motivated by enumerative problems in the study of higher trivariate diagonal harmonics. Nowadays, it has applications and connections to other areas, including polytope theory, subword complexes, Hopf algebras, multivariate diagonal harmonics, and parabolic Catalan combinatorics, as well as to the enumeration of various combinatorial objects such as certain lattice walks in the quarter plane, non-crossing tree-like tableaux, and non-separable planar maps, see [2, 3, 4, 12] and the references therein. The ν-Tamari lattice depends on a fixed northeast path ν, and was defined in [12] as a certain rotation order on the set of *ν-paths*, i.e. northeast paths weakly above ν. Alternatively, it can be described in terms of certain binary trees, called *ν-trees* [4].

Motivated by an open problem of F. Bergeron about the geometry of m-Tamari lattices, the first author, together with A. Padrol and C. Sarmiento [3], showed that the *ν-Tamari lattice* $\mathsf{Tam}(\nu)$ has a nice underlying geometric structure. They proved that its Hasse diagram can be obtained as the edge graph of a polyhedral complex called the *ν-associahedron* $\mathsf{Asso}(\nu)$. This complex is dual to a certain triangulation of a certain polytope, which they used to exhibit explicit geometric realizations of the ν-associahedron using techniques from tropical geometry. The simplicial complex of faces of this triangulation is the *ν-Tamari complex* $\mathcal{TC}_\nu$.

*cesar.ceballos@tugraz.at.
†henri.muehle@tu-dresden.de.

If $\nu = (NE)^n$ is the staircase path with $2n$ steps, the corresponding three objects are the Tamari lattice, the associahedron and the cluster complex in linear type A from the theory of cluster algebras, respectively. The $(NE)^n$-paths are better known under the name *Dyck paths*, and we will simply write $\mathsf{Tam}(n)$, $\mathsf{Asso}(n)$, $\mathcal{TC}_n$ rather than $\mathsf{Tam}\big((NE)^n\big)$, $\mathsf{Asso}\big((NE)^n\big)$, $\mathcal{TC}_{(NE)^n}$.

F. Chapoton has observed a remarkable enumerative connection between the cluster complex $\mathcal{TC}_n$ and the set of Dyck paths [6]. More precisely, he defined the following two polynomials:

- the *F-triangle* $F_n(x,y)$ is the bivariate generating function of the faces of $\mathcal{TC}_n$, where the variable x accounts for so-called positive roots per face and y accounts for so-called negative simple roots;
- the *H-triangle* $H_n(x,y)$ is the bivariate generating function of Dyck paths, where the variable x accounts for the valleys of this path and y accounts for the returns.

He then conjectured that these polynomials are related by the following invertible transformation:

$$F_n(x,y) = x^{n-1} H_n\left(\frac{x+1}{x}, \frac{y+1}{x+1}\right). \tag{1.1}$$

This conjecture was generalized for Fuß–Catalan families by D. Armstrong [1], and was proven in this general setting by M. Thiel [13, Theorem 2]. Thiel's proof makes clever use of a combinatorial bijection on so-called k-generalized nonnesting partitions which leads to a differential equation involving the H-triangle. Using a differential equation by C. Krattenthaler involving the F-triangle, he then proves (1.1) by induction.

Unfortunately, the combinatorial nature of the relation between the F- and the H-triangle is obscured in Thiel's proof. The main result of the present article is a combinatorial proof of a generalization of (1.1) to ν-paths and the ν-associahedron.

Given any northeast path ν, we denote by $\mathsf{deg}(\nu)$ the maximal number of valleys that a northeast path weakly above ν can have. In other words, $\mathsf{deg}(\nu)$ describes the size of the largest staircase shape that fits above ν in the rectangle enclosing ν. The H-triangle associated with ν is simply the bivariate generating function of ν-paths, denoted by $H_\nu(x,y)$, where the variable x accounts for valleys and the variable y accounts for returns. The F-triangle is the bivariate generating function $F_\nu(x,y)$ of the faces of $\mathsf{Asso}(\nu)$, where x and y account for a new pair of statistics that we introduce in this paper. Our main result shows that these polynomials satisfy (1.1).

Theorem 1.1. *For every northeast path ν, the following holds:*

$$F_\nu(x,y) = x^{\mathsf{deg}(\nu)} H_\nu\left(\frac{x+1}{x}, \frac{y+1}{x+1}\right). \tag{1.2}$$

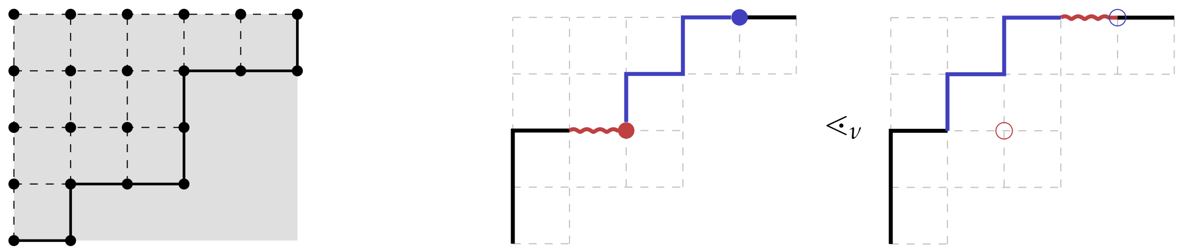

Figure 1: (left) A northeast path $\nu = ENEENNEEN$, its bounding rectangle (shaded in gray), the associated Ferrers diagram (indicated by the dashed lines), the associated set of lattice points (indicated by the black dots). (right) An example of a rotation of a ν-path by the valley marked in red.

Our proof of Theorem 1.1 is completely combinatorial. It relies on the geometry of the ν-associahedron and exploits a bijection of [4] which sends ν-paths to ν-trees.

If $\nu = N^{a_1}E^{a_1}N^{a_2}E^{a_2}\cdots N^{a_r}E^{a_r}$ for positive integers $a_1, a_2, \ldots, a_r$, then Theorem 1.1 sheds quite some light on the constructions from [11, Section 6] and [9, Section 5]. If $a_2 = a_3 = \cdots = a_r$, then our F-triangle combinatorially realizes the case $m = 1$ of the F-triangle computed abstractly in [8, Theorem 4.3].

We wish to remark that analogues of F- and H-triangles arising in different (geometric) contexts but satisfying (1.1), too, were for instance considered in [7, 10].

2 Basics

2.1 Northeast paths

A *northeast path* is a lattice path in $\mathbb{N}^2$ starting at the origin, and consisting of finitely many steps of the form $(0, 1)$ (*north steps*) and $(1, 0)$ (*east steps*). We write such a path as a word over the alphabet $\{N, E\}$, where each N represents a north step and each E an east step. Throughout this paper, we let ν denote (a fixed) such northeast path. Let F_ν denote the Ferrers diagram that lies weakly above ν in the smallest rectangle containing ν. Let A_ν denote the set of lattice points inside F_ν. See the left part of Figure 1 for an illustration.

2.2 The ν-Tamari lattice

Let us denote by $\mathcal{D}_\nu$ the set of all *ν-paths*, i.e. northeast paths that live entirely inside F_ν sharing start and end points with ν and lie weakly above ν. For $\mu \in \mathcal{D}_\nu$, a *valley* is a point $p \in A_\nu$ which lies on μ and is preceded by an east step and followed by a north step.

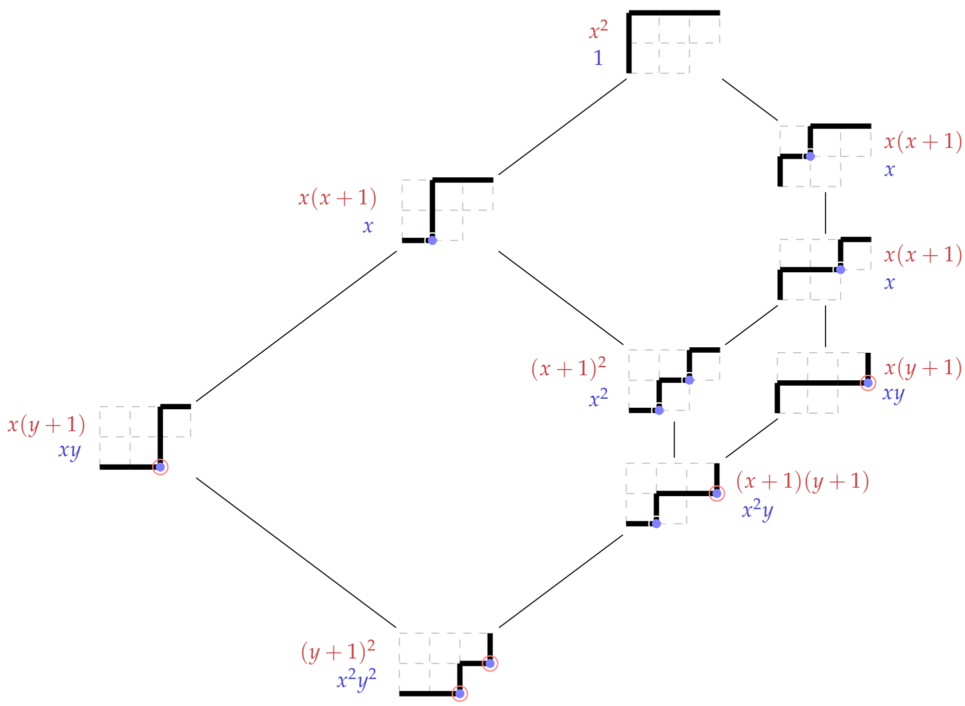

Figure 2: The ν-Tamari lattice labeled by ν-paths for $\nu = EENEN$. Each path is additionally labeled by the term it contributes to $F_\nu(x,y)$ (top expression in red) and to $H_\nu(x,y)$ (bottom expression in blue).

We denote by $\mathsf{val}(\mu)$ the number of valleys of μ. A valley p of μ is a *return*, if p is also a valley of ν. We denote by $\mathsf{ret}(\mu)$ the number of returns of μ. The *degree* of ν is defined as the maximum number of valleys that a ν-path can have: $\mathsf{deg}(\nu) \overset{\mathsf{def}}{=} \max\{\mathsf{val}(\mu) \mid \mu \in \mathcal{D}_\nu\}$.

If p is a valley of μ, then we denote by $\mathsf{horiz}_\nu(p)$ the maximal number of east steps that we can append to p without going beyond ν. In other words, if $p = (i,j)$, then we look for the rightmost point in row j that lies on ν; say that this point is (k,j). Then $\mathsf{horiz}_\nu(p) \overset{\mathsf{def}}{=} k - i$.

Let p' denote the first point on μ after p with $\mathsf{horiz}_\nu(p') = \mathsf{horiz}_\nu(p)$. Let $\mu[p,p']$ denote the subpath of μ which lies between p and p'. The *rotation* of μ by p is the unique northeast path which arises from μ by swapping the east step before p with $\mu[p,p']$. If μ' is the path arising from μ in this manner, then we write $\mu \lessdot_\nu \mu'$. It is quickly verified that $\lessdot_\nu$ is an acyclic binary relation on $\mathcal{D}_\nu$, and we denote its reflexive and transitive closure by $\leq_\nu$. See the right part of Figure 1 for an illustration.

The partially ordered set $\mathsf{Tam}(\nu) \overset{\mathsf{def}}{=} (\mathcal{D}_\nu, \leq_\nu)$ is a lattice; the *ν-Tamari lattice*; see [12, Theorem 1.1]. Figure 2 shows the ν-Tamari lattice for the path $\nu = EENEN$, which has degree 2.

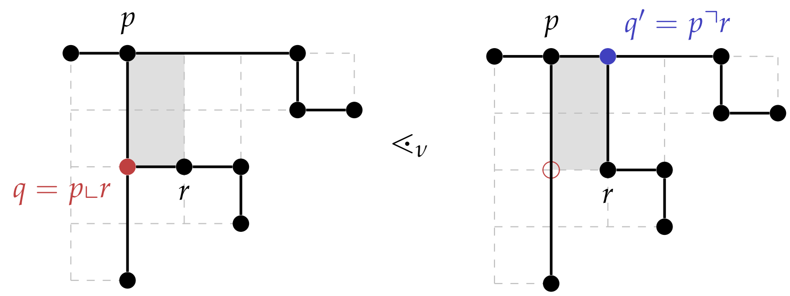

Figure 3: The rotation operation of a ν-tree by the ascent node q. The rectangle $p\square r$ is highlighted.

2.3 The ν-Tamari lattice via trees

As shown in [4], we can alternatively define the ν-Tamari lattice in terms of a special family of trees.

We say that two points $p, q \in A_\nu$ are *ν-incompatible* if p is strictly southwest or strictly northeast of q and the smallest rectangle containing p and q lies entirely in F_ν. Otherwise, p and q are *ν-compatible*; we write $p \sim_\nu q$ in this case, and drop the reference to the path if no confusion may arise. A *ν-tree* is a maximal collection of pairwise ν-compatible elements of A_ν. We denote by $\mathcal{T}_\nu$ the set of all ν-trees.

If T is a ν-tree, we can connect two distinct elements $p, q \in T$ if p and q either lie in the same row or in the same column, and there is no element of T on the line segment connecting p and q. In particular, this allows us to visualize ν-trees as classical rooted binary trees [4, Lemma 2.4]. Let $T \in \mathcal{T}_\nu$ and let $p, q \in T$ be two elements which do not lie in the same row or same column. Let $p\square r$ denote the smallest rectangle containing p and r. We write $p \llcorner r$ (resp. $p \urcorner r$) for the lower left corner (resp. upper right corner) of $p\square r$.

An element $q \in T$ is an *ascent* of T if $q = p \llcorner r$ for some elements $p, r \in T$. In such a case, we choose p, r canonically so that no other elements besides q, p, r lie in $p\square r$. Let $\mathsf{Asc}(T)$ denote the set of ascents of T, and write $\mathsf{asc}(T) \stackrel{\text{def}}{=} \big|\mathsf{Asc}(T)\big|$.

The *rotation* of T by the ascent q is $T' = \big(T \setminus \{q\}\big) \cup \{q'\}$, where $q' = p \urcorner r$. Figure 3 illustrates this rotation operation. As proven in [4, Lemma 2.10], the rotation of a ν-tree is also a ν-tree. By abuse of notation, we write $T \lessdot_\nu T'$ if T' is a rotation of T, and denote by $\leq_\nu$ the reflexive and transitive closure of $\lessdot_\nu$. The partial order $(\mathcal{T}_\nu, \leq_\nu)$ is a lattice, which is isomorphic to the ν-Tamari lattice [4, Theorem 3.3].

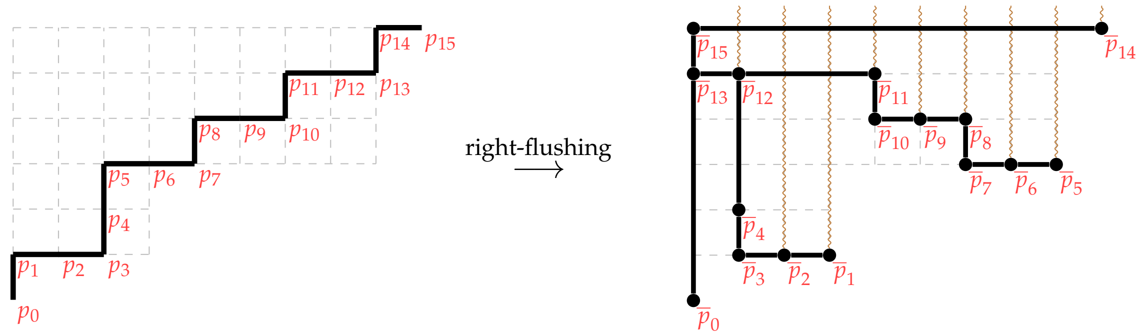

Figure 4: Illustrating the bijection from ν-paths to ν-trees.

2.4 The right flushing bijection

The isomorphism between the ν-Tamari lattice and the rotation lattice of ν-trees is given by a simple bijection between the set of ν-paths and the set of ν-trees which we now recall. Given a ν-path μ, let a_i be the number of lattice points in μ at height i, for $i \geq 0$. There exists exactly one ν-tree T containing a_i nodes at height i for each $i \geq 0$. Vice-versa, given a ν-tree with "height sequence" $a_0, a_1, a_2, \ldots$, there is a unique ν-path with the same height sequence. We denote by $\Phi\colon \mathcal{D}_\nu \to \mathcal{T}_\nu$ the map that sends μ to T. This map is a bijection between the set of ν-paths and the set of ν-trees. Moreover, it is an isomorphism between the ν-Tamari lattice and the rotation lattice of ν-trees [4, Proposition 16]. The map Φ is called the *right flushing bijection* [4], and is illustrated in Figure 4.

The reason why this is called "right flushing" is because it can be described as follows. Let μ be a ν-path with height sequence $a_0, a_1, a_2, \ldots$. We build the ν-tree $T = \Phi(\mu)$ with the same height sequence by recursively adding a_i nodes at height i from bottom to top, from right to left, avoiding forbidden positions. The forbidden positions are those above a node that is not the left most node in a row. In Figure 4, the forbidden positions are the ones that belong to the wiggly lines. Note that the order of the nodes per row is reversed.

2.5 The ν-Tamari complex and the ν-associahedron

Generalizing the ν-trees mentioned above, we define a *ν-face* as a collection of pairwise ν-compatible elements of A_ν (not necessarily maximal as in the case of ν-trees). The collection of ν-faces forms a simplicial complex, which we call the *ν-Tamari complex* and denote by $\mathcal{TC}_\nu$. This complex was originally defined using a different language in [3], and we use the terminology introduced in [4].

The ν-Tamari complex is the simplicial complex of faces of a triangulation of a polytope studied in [3]. The dual of this triangulation is a polyhedral complex called the

ν-associahedron $\mathsf{Asso}(\nu)$, whose faces are in correspondence (via duality) with the interior faces of the triangulation. Such interior faces were classified in [3] as *covering ν-faces*, which are defined as those ν-faces containing the top-left corner of A_ν and at least one point in each row and column in F_ν [3, 4].

Under this correspondence, the *ν-associahedron* is defined as the polyhedral complex whose cells are covering ν-faces ordered by reversed inclusion.

$$\mathsf{Asso}(\nu) \stackrel{\mathsf{def}}{=} \{C \mid C \text{ is a covering } \nu\text{-face}\}.$$

If ν is a northeast path from $(0,0)$ to (m,n), the *dimension* of a covering ν-face C is:

$$\mathsf{dim}(C) \stackrel{\mathsf{def}}{=} m+n+1-|C|.$$

In particular, one can check that the number of elements of every ν-tree is constant, and equal to $m+n+1$. So, the ν-trees correspond to the zero-dimensional faces (vertices) of the ν-associahedron. Every time we remove a node (when possible), we increase the dimension of the resulting face by one.

An example of the ν-associahedron for $\nu = EENEN$ is illustrated in Figure 5. The faces of this figure are labeled by covering ν-faces and the vertices by ν-trees. Its edge graph coincides with the Hasse diagram of the ν-Tamari lattice in Figure 2. The advantage of working with the ν-associahedron is that it captures the full geometric information behind the ν-Tamari lattice.

3 The F- and the H-triangle associated with ν

Let $C \in \mathsf{Asso}(\nu)$ be a covering ν-face. We say that $p \in C$ is *relevant* if:

- it is in the first column,
- there is another point $q \neq p$ in C that is in the same row, and
- its row contains a valley of ν.

We denote by $\mathsf{Rel}(C)$ the set of relevant nodes in C, and we let $\mathsf{rel}(C) \stackrel{\mathsf{def}}{=} \big|\mathsf{Rel}(C)\big|$.

The *F-triangle* of $\mathsf{Asso}(\nu)$ is a generating function of the faces of $\mathsf{Asso}(\nu)$ defined by:

$$F_\nu(x,y) \stackrel{\mathsf{def}}{=} \sum_{C \in \mathsf{Asso}(\nu)} x^{\mathsf{deg}(\nu)-\mathsf{dim}(C)-\mathsf{rel}(C)} y^{\mathsf{rel}(C)}. \tag{3.1}$$

In Figure 5, the positions of the relevant nodes are circled in red in order to easily visualize the value of the statistic $\mathsf{rel}(C)$ on each face. The degree is $\mathsf{deg}(\nu) = 2$. In

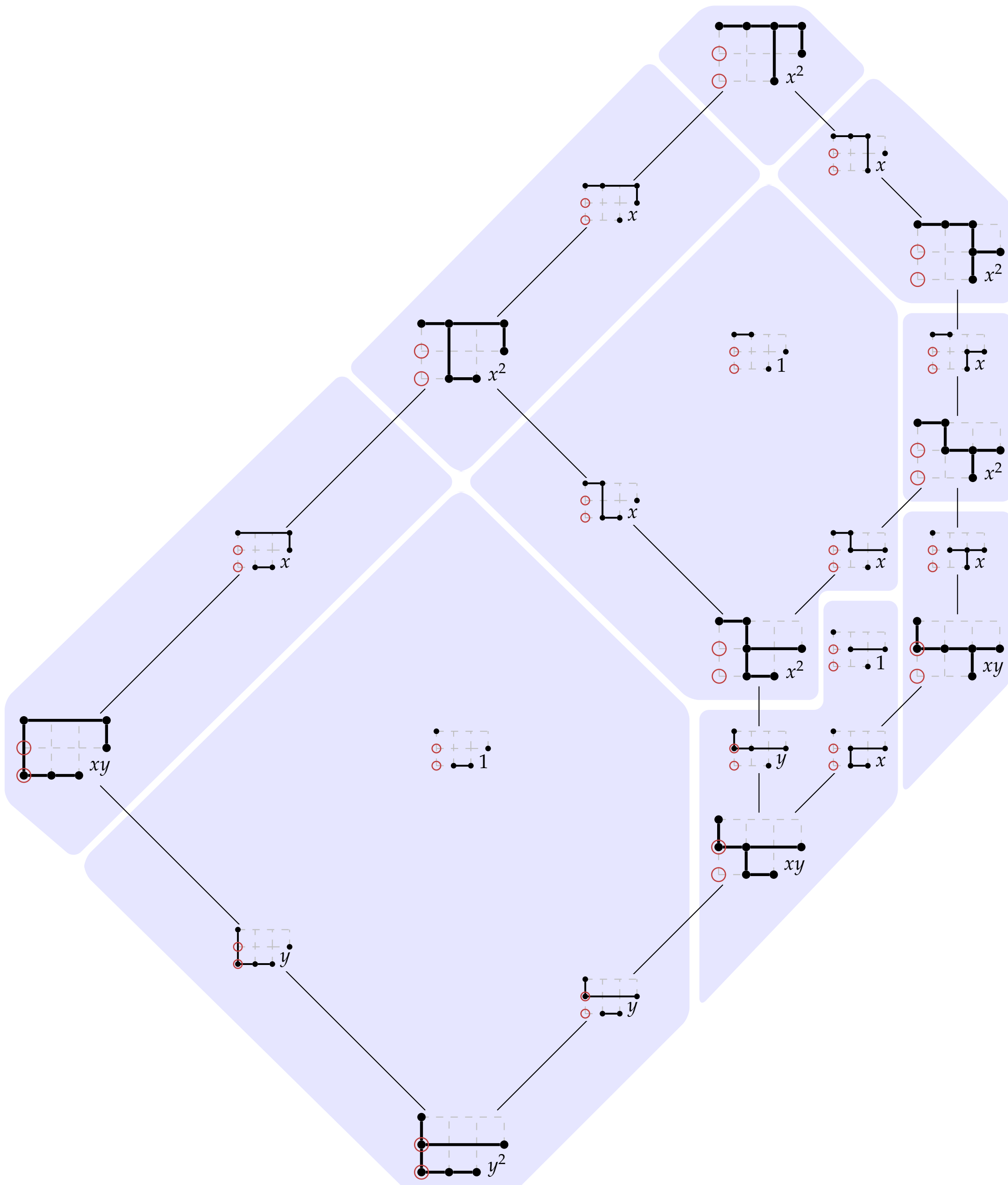

Figure 5: The ν-associahedron $\mathsf{Asso}(\nu)$ for $\nu = EENEN$, whose faces are labeled by covering ν-faces. Each face is additionally labeled by the term it contributes to $F_\nu(x, y)$.

addition, each face is labeled by the term it contributes to the F-triangle. Adding up, we obtain

$$F_{EENEN}(x,y) = 5x^2 + 3xy + y^2 + 8x + 3y + 3.$$

The *H-triangle* of ν is the generating function elements of $\mathcal{D}_\nu$ in terms of the number of valleys and returns:

$$H_\nu(x,y) \stackrel{\mathsf{def}}{=} \sum_{\mu \in \mathcal{D}_\nu} x^{\mathsf{val}(\mu)} y^{\mathsf{ret}(\mu)}. \tag{3.2}$$

In Figure 2, we have labeled in each of the paths the valleys by a blue dot, and we have circled the returns in red. Additionally, we have noted the term each path contributes to $H_{EENEN}(x,y)$ in blue (bottom expression). Adding up, we obtain

$$H_{EENEN}(x,y) = x^2y^2 + x^2y + x^2 + 2xy + 3x + 1.$$

4 Proof of the $F = H$ correspondence

In this section, we prove Theorem 1.1. To illustrate this result, we reconsider our running example for $\nu = EENEN$. We have

$$\begin{aligned}
x^2 H_{EENEN}\left(\frac{x+1}{x}, \frac{y+1}{x+1}\right) &= (y+1)^2 + (x+1)(y+1) + (x+1)^2 + 2x(y+1) \\
&\qquad + 3x(x+1) + x^2 \\
&= y^2 + 2y + 1 + xy + x + y + 1 + x^2 + 2x + 1 + 2xy \\
&\qquad + 2x + 3x^2 + 3x + x^2 \\
&= 5x^2 + 3xy + y^2 + 8x + 3y + 3 \\
&= F_{EENEN}(x,y).
\end{aligned}$$

Now, in general, if we plug in the definition of H_ν in (1.2), we obtain:

$$\begin{aligned}
x^{\mathsf{deg}(\nu)} H_\nu\left(\frac{x+1}{x}, \frac{y+1}{x+1}\right) &= x^{\mathsf{deg}(\nu)} \sum_{\mu \in \mathcal{D}_\nu} \left(\frac{x+1}{x}\right)^{\mathsf{val}(\mu)} \left(\frac{y+1}{x+1}\right)^{\mathsf{ret}(\mu)} \\
&= \sum_{\mu \in \mathcal{D}_\nu} x^{\mathsf{deg}(\nu) - \mathsf{val}(\mu)} (x+1)^{\mathsf{val}(\mu) - \mathsf{ret}(\mu)} (y+1)^{\mathsf{ret}(\mu)}.
\end{aligned}$$

This is certainly a polynomial in x and y with nonnegative integer coefficients, because $\mathsf{val}(\mu) \geq \mathsf{ret}(\mu)$ and $\mathsf{deg}(\nu) = \max\{\mathsf{val}(\mu) \mid \mu \in \mathcal{D}_\nu\}$. Theorem 1.1 is then equivalent to the following proposition.

Proposition 4.1. *For every northeast path ν, the following holds:*

$$F_\nu(x,y) = \sum_{\mu \in \mathcal{D}_\nu} x^{\mathsf{deg}(\nu) - \mathsf{val}(\mu)} (x+1)^{\mathsf{val}(\mu) - \mathsf{ret}(\mu)} (y+1)^{\mathsf{ret}(\mu)}. \tag{4.1}$$

In order to prove this proposition we will transform this expression to another expression in terms of ν-trees, using the bijection from Section 2.4 (see Proposition 4.6). The second ingredient in our proof will be to show that the term associated with a ν-tree in this new expression is equal to the sum of terms contributed by a specific group of faces in the definition of the F-triangle (see Proposition 4.5). In order to shape our intuition, these groups can be visualized (as shadowed groups); as in Figure 5 for our running example.

We start by explaining that the bijection Φ from ν-paths to ν-trees sends the valleys and returns to ascents and relevant nodes. We omit the proof for space constraints.

Lemma 4.2. *Let μ be a ν-path and $T = \Phi(\mu)$ be its corresponding ν-tree. Then,*

(i) $\mathsf{val}(\mu) = \mathsf{asc}(T)$, *and*

(ii) $\mathsf{ret}(\mu) = \mathsf{rel}(T)$.

Lemma 4.3. *For a ν-tree T, the following holds:*

$$(y+1)^{\mathsf{rel}(T)} = \sum_{A' \subseteq \mathsf{Rel}(T)} y^{\mathsf{rel}(T) - |A'|}, \tag{4.2}$$

$$x^{\deg(\nu) - \mathsf{asc}(T)}(x+1)^{\mathsf{asc}(T) - \mathsf{rel}(T)} = \sum_{A'' \subseteq \mathsf{Asc}(T) \setminus \mathsf{Rel}(T)} x^{\deg(\nu) - \mathsf{rel}(T) - |A''|}. \tag{4.3}$$

Proof. Since $\big|\mathsf{Rel}(T)\big| = \mathsf{rel}(T)$, Equation (4.2) follows from the Binomial Theorem:

$$(y+1)^{\mathsf{rel}(T)} = \sum_{k=0}^{\mathsf{rel}(T)} \binom{\mathsf{rel}(T)}{k} y^{\mathsf{rel}(T)-k} = \sum_{A' \subseteq \mathsf{Rel}(T)} y^{\mathsf{rel}(T) - |A'|}$$

Since $|\mathsf{Asc}(T) \setminus \mathsf{Rel}(T)| = \mathsf{asc}(T) - \mathsf{rel}(T)$, Equation (4.3) can be shown similarly:

$$\begin{aligned} x^{\deg(\nu) - \mathsf{asc}(T)}(x+1)^{\mathsf{asc}(T) - \mathsf{rel}(T)} &= x^{\deg(\nu) - \mathsf{asc}(T)} \sum_{A'' \subseteq \mathsf{Asc}(T) \setminus \mathsf{Rel}(T)} x^{\mathsf{asc}(T) - \mathsf{rel}(T) - |A''|} \\ &= \sum_{A'' \subseteq \mathsf{Asc}(T) \setminus \mathsf{Rel}(T)} x^{\deg(\nu) - \mathsf{rel}(T) - |A''|}. \end{aligned}$$ □

Lemma 4.4 ([5, Lemma 5.4])**.** *The sets $\mathsf{Asso}(\nu)$ and $\big\{(T, A) \mid T \in \mathcal{T}_\nu, A \subseteq \mathsf{Asc}(T)\big\}$ are in bijection via the map $(T, A) \mapsto T \setminus A$.*

If $C \in \mathsf{Asso}(\nu)$ is of the form $C = T \setminus A$, then we say that T is the *bottom ν-tree* of C. This terminology is motivated as follows. Recall that $\mathsf{Asso}(\nu)$ is a polyhedral complex, so any face $C \in \mathsf{Asso}(\nu)$ is itself a polytope. The edge graph of C corresponds to an interval of $\mathsf{Tam}(\nu)$ and as such inherits the orientation given by the partial order $\leq_\nu$. Then, T is the minimal element of this interval. Moreover, every ascent $p \in \mathsf{Asc}(T)$ uniquely determines a ν-tree T_p with $T \lessdot_\nu T_p$; therefore the maximal ν-tree in this interval is $T \vee \bigvee_{p \in \mathsf{Asc}(T)} T_p$ (considered as a join in the lattice $\mathsf{Tam}(\nu)$).

We denote by $\mathsf{Asso}_T(\nu)$ the set of covering ν-faces whose bottom ν-tree is T. We define

$$F_\nu^T(x,y) \stackrel{\text{def}}{=} \sum_{C \in \mathsf{Asso}_T(\nu)} x^{\mathsf{deg}(\nu)-\mathsf{dim}(C)-\mathsf{rel}(C)} y^{\mathsf{rel}(C)}. \tag{4.4}$$

In our example in Figure 5, the sets $\mathsf{Asso}_T(\nu)$ are represented by the shadowed groups. More precisely, the set $\mathsf{Asso}_T(\nu)$ consists of the faces belonging to the shadowed group containing T. The polynomial $F_\nu^T(x,y)$ is then the sum of the monomials in that shadowed group. For instance, if T_0 is the bottom tree in Figure 5, then

$$F_\nu^{T_0}(x,y) = y^2 + y + y + 1 = (y+1)^2.$$

Compare the terms in Figure 5 with the red ones (top expression per path) in Figure 2.

Proposition 4.5. *For every northeast path ν, the following holds:*

$$F_\nu^T(x,y) = x^{\mathsf{deg}(\nu)-\mathsf{asc}(T)}(x+1)^{\mathsf{asc}(T)-\mathsf{rel}(T)}(y+1)^{\mathsf{rel}(T)}. \tag{4.5}$$

Proof. Let $C \in \mathsf{Asso}_T(\nu)$. Then $C = T \setminus A$ for some subset A of ascents of T. This subset can be written uniquely as a disjoint union $A = A' \uplus A''$, where $A' \subseteq \mathsf{Rel}(T)$ and $A'' \subseteq \mathsf{Asc}(T) \setminus \mathsf{Rel}(T)$. Then $\mathsf{rel}(C) = \mathsf{rel}(T) - |A'|$. Furthermore $\mathsf{dim}(C) = |A'| + |A''|$, and so $\mathsf{deg}(\nu) - \mathsf{dim}(C) - \mathsf{rel}(C) = \mathsf{deg}(\nu) - \mathsf{rel}(T) - |A''|$. Therefore,

$$\begin{aligned} F_\nu^T(x,y) &= \sum_{C \in \mathsf{Asso}_T(\nu)} x^{\mathsf{deg}(\nu)-\mathsf{dim}(C)-\mathsf{rel}(C)} y^{\mathsf{rel}(C)} \\ &= \sum_{A = A' \uplus A''} x^{\mathsf{deg}(\nu)-\mathsf{rel}(T)-|A''|} y^{\mathsf{rel}(T)-|A'|}. \end{aligned}$$

This is exactly the product of Equations (4.2) and (4.3), and the result follows. □

Proposition 4.6. *For every northeast path ν, the following holds:*

$$F_\nu(x,y) = \sum_{T \in \mathcal{T}_\nu} x^{\mathsf{deg}(\nu)-\mathsf{asc}(T)}(x+1)^{\mathsf{asc}(T)-\mathsf{rel}(T)}(y+1)^{\mathsf{rel}(T)}. \tag{4.6}$$

Proof. Since $\mathsf{Asso}(\nu) = \biguplus_{T \in \mathcal{T}_\nu} \mathsf{Asso}_T(\nu)$, it follows that

$$F_\nu(x,y) = \sum_{T \in \mathcal{T}_\nu} F_\nu^T(x,y).$$

The result then follows from Proposition 4.5 □

Proof of Proposition 4.1. Proposition 4.1 follows from Proposition 4.6 and Lemma 4.2, by transforming the statistics under the right flushing bijection Φ. □

Proof of Theorem 1.1. As we have already mentioned, Theorem 1.1 is equivalent to Proposition 4.1, which we have just proven. □

References

[1] D. Armstrong. "Generalized noncrossing partitions and combinatorics of Coxeter groups". *Memoirs of the American Mathematical Society* **202** (2009).

[2] C. Ceballos, W. Fang, and H. Mühle. "The Steep-Bounce zeta map in Parabolic Cataland". *Journal of Combinatorial Theory (Series A)* **172** (2020), Article 105210, 59 pages.

[3] C. Ceballos, A. Padrol, and C. Sarmiento. "Geometry of ν-Tamari lattices in types A and B". *Transactions of the American Mathematical Society* **371** (2019), pp. 2575–2622.

[4] C. Ceballos, A. Padrol, and C. Sarmiento. "The ν-Tamari lattice via ν-trees, ν-bracket vectors, and subword complexes". *The Electronic Journal of Combinatorics* **27** (2020), Research paper P1.14, 31 pages.

[5] C. Ceballos and V. Pons. "The s-Weak Order and s-Permutahedra". *Séminaire Lotharingien de Combinatoire* **82B** (2019). Proceedings of the 31st Conference on Formal Power Series and Algebraic Combinatorics, Conference paper 76, 12 pages.

[6] F. Chapoton. "Sur le nombre de réflexions pleines dans les groupes de Coxeter finis". *Bulletin of the Belgian Mathematical Society* **13** (2006), pp. 585–596.

[7] A. Garver and T. McConville. "Enumerative properties of grid-associahedra" (2017). arXiv:1705.04901.

[8] C. Krattenthaler and H. Mühle. "The rank enumeration of certain parabolic non-crossing partitions" (2019). arXiv:1910.13244.

[9] H. Mühle. "Ballot-noncrossing partitions". *Séminaire Lotharingien de Combinatoire* **82B** (2019). Proceedings of the 31st Conference on Formal Power Series and Algebraic Combinatorics, Conference paper 7, 12 pages.

[10] H. Mühle. "Hochschild lattices and shuffle lattices" (2020). arXiv:2008.13247.

[11] H. Mühle. "Noncrossing arc diagrams, Tamari lattices, and parabolic quotients of the symmetric group". *Annals of Combinatorics* (2021). To appear.

[12] L.-F. Préville-Ratelle and X. Viennot. "The enumeration of generalized Tamari intervals". *Transactions of the American Mathematical Society* **369** (2017), pp. 5219–5239.

[13] M. Thiel. "On the H-triangle of generalised nonnesting partitions". *European Journal of Combinatorics* **39** (2014), pp. 244–255.