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Specializations of colored quasisymmetric functions

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Abstract. We propose a unified approach to prove general formulas for the joint distribution of an Eulerian statistic, a Mahonian statistic and the color sum statistic over a set of colored permutations by specializing Poirier's colored quasisymmetric functions. We apply this method to derive formulas for joint Euler–Mahonian and color sum distributions on colored permutations, derangements and involutions. A number of known formulas are recovered as special cases of our results, including formulas of Biagioli–Zeng, Assaf, Haglund–Loehr–Remmel, Biagioli–Caselli, Faliharimalala–Zeng and Désarménien–Foata. Several new results are also obtained.

Keywords: Quasisymmetric function, generating function, specialization, descent set, major index, Euler–Mahonian distribution, Eulerian polynomial

1 Introduction

For a positive integer n, let $\mathfrak{S}_n$ be the set of permutations of $[n] := \{1, 2, \ldots, n\}$. For $w \in \mathfrak{S}_n$, an index $i \in [n-1]$ is called a *descent* of w, if $w(i) > w(i+1)$. The set of all descents of w, written $\mathrm{Des}(w)$, is called the *descent set* of w. The cardinality and the sum of all elements of $\mathrm{Des}(w)$ are written as $\mathrm{des}(w)$ and $\mathrm{maj}(w)$, respectively, and called the *descent number* and *major index* of w. A statistic on $\mathfrak{S}_n$ which is equidistributed with des (resp. maj) is called *Eulerian* (resp. *Mahonian*). Let

$$A_n(x, q) := \sum_{w \in \mathfrak{S}_n} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)}$$

be the generating polynomial for the joint distribution $(\mathrm{des}, \mathrm{maj})$ on $\mathfrak{S}_n$. The polynomial $A_n(x) := A_n(x, 1)$ is called the n-th *Eulerian polynomial* and constitutes one of the most important polynomials in combinatorics (see, for example, [3]).

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It follows from the work of MacMahon [15] that

$$\sum_{m\geq 0}[m+1]_q^n\, x^m \;=\; \frac{A_n(x,q)}{(1-x)(1-xq)\cdots(1-xq^n)} \tag{1.1}$$

$$A_n(1,q) \;=\; [1]_q[2]_q\cdots[n]_q, \tag{1.2}$$

where $[n]_q := 1+q+\cdots+q^{n-1}$ is the q-analogue of n. These formulas serve as the basis for many generalizations to Coxeter groups and r-colored permutation groups, that is wreath products $\mathbb{Z}_r \wr \mathfrak{S}_n$. Identities such as Equation (1.1), involving Euler–Mahonian distributions will be called *Euler–Mahonian identities*. As mentioned in [5, Section 1], a general approach to prove Euler–Mahonian identities, among others, is via the theory of symmetric/quasisymmetric functions. We illustrate this approach by proving Equations (1.1) and (1.2) in a unified way, by specializing Gessel's fundamental quasisymmetric functions [19, Section 7.19]. This proof serves as a prototype for the proofs of all applications presented here (see Section 4).

Let $\mathbf{x} = (x_1, x_2, \dots)$ be a sequence of commuting indeterminates. The *fundamental quasisymmetric function* associated to $S \subseteq [n-1]$ is defined by

$$F_{n,S}(\mathbf{x}) \;:=\; \sum_{\substack{i_1\geq i_2\geq\cdots\geq i_n\geq 1\\ j\in S\Rightarrow i_j>i_{j+1}}} x_{i_1}x_{i_2}\cdots x_{i_n}. \tag{1.3}$$

The original definition actually defines $F_{n,S}(\mathbf{x})$ as in Equation (1.3) with $1\leq i_1\leq i_2\leq \cdots \leq i_n$ instead. Our choice of definition will become clear in Section 3. The *stable principal specialization*, written ps, and *principal specialization* of order m, written ps_m, of a formal power series in $\mathbf{x}$ are defined [19, Section 7.8] by the substitutions $x_i = q^{i-1}$, for all $i\geq 1$ and

$$x_i \;=\; \begin{cases} q^{i-1}, & \text{if } 1\leq i\leq m\\ 0, & \text{if } i>m,\end{cases}$$

respectively.

The principal specialization of order m and stable principal specialization of the quasisymmetric generating function of a subset $\mathcal{A}\subseteq\mathfrak{S}_n$ are given by the following formulas [12, Theorem 5.3]

$$\sum_{m\geq 1}\mathrm{ps}_m\, F(\mathcal{A};\mathbf{x})\, x^{m-1} \;=\; \frac{\sum_{w\in\mathcal{A}} x^{\mathrm{des}(w)}\, q^{\mathrm{maj}(w)}}{(1-x)(1-xq)\cdots(1-xq^n)} \tag{1.4}$$

$$\mathrm{ps}\, F(\mathcal{A};\mathbf{x}) \;=\; \frac{\sum_{w\in\mathcal{A}} q^{\mathrm{maj}(w)}}{(1-q)(1-q^2)\cdots(1-q^n)}, \tag{1.5}$$

where $F(\mathcal{A};\mathbf{x}) := \sum_{w\in\mathcal{A}} F_{n,\mathrm{Des}(w)}(\mathbf{x})$. The quasisymmetric function $F(\mathfrak{S}_n;\mathbf{x})$ is known to have [19, Corollary 7.12.5] the following nice form

$$F(\mathfrak{S}_n;\mathbf{x}) = (x_1+x_2+\cdots)^n. \tag{1.6}$$

The proof of Equations (1.1) and (1.2) follows by taking the principal specialization of order m and the stable principal specialization of Equation (1.6) and substituting in Equations (1.4) and (1.5) for $\mathcal{A} = \mathfrak{S}_n$, respectively.

In our recent paper [17], we provide a unified approach to prove Euler–Mahonian identities on sets of colored permutations by specializing Poirier's colored analogue of quasisymmetric functions [18]. A first instance of this technique appears in the work of Athanasiadis [3, Equation (45)]. Our choice of colored quasisymmetric functions, and much of the motivation behind this paper comes from the fact that Poirier's signed analogue of the fundamental quasisymmetric functions was recently employed by Adin et al. [1] in order to define and study a signed analogue of the concept of fine sets and fine characters of Adin and Roichman (see, for example, [1, Section 1]).

This extended abstract is a summary of our results in [17] with applications involving color sum Euler–Mahonian identities. In Section 2, we review colored permutation statistics and recall the definition of fundamental colored quasisymmetric functions. In Section 3, we present the main results on specializations of colored quasisymmetric functions and in Section 4, we apply these results to prove formulas for the joint Euler–Mahonian and color sum distributions on colored permutations, derangements and absolute involutions.

2 Preliminaries

This section provides key definitions regarding colored permutation statistics and colored quasisymmetric functions. For a survey on Euler–Mahonian identities for the colored permutation groups, we refer the reader to [17, Section 2.1]. We assume familiarity with basic concepts in the theory of symmetric functions as presented in [19, Section 7]. For nonnegative integers $0 \le k \le n$, define

$$(x;q)_n := \begin{cases} 1, & \text{if } n = 0 \\ (1-x)(1-xq)\cdots(1-xq^{n-1}), & \text{if } n \ge 1 \end{cases}$$

$$\binom{n}{k}_q := \begin{cases} 1, & \text{if } n = 0 \\ \dfrac{[n]_q!}{[k]_q![n-k]_q!}, & \text{if } n \ge 1 \end{cases}$$

and set $(q)_n := (q;q)_n$.

2.1 Colored permutation statistics

Fix a positive integer r and let

$$\Omega_{n,r} := \{1^0, 2^0, \ldots, n^0, 1^1, 2^1, \ldots, n^1, \ldots, 1^{r-1}, 2^{r-1}, \ldots, n^{r-1}\}$$

be the set of *r-colored integers*. We will often identify colored integers i^0 with i. The *r-colored permutation group*, denoted by $\mathfrak{S}_{n,r}$, consists of all permutations w of $\Omega_{n,r}$ such that $w(a^0) = b^j \Rightarrow w(a^i) = b^{i+j}$, where $i + j$ is computed modulo r and the product of $\mathfrak{S}_{n,r}$ is composition of permutations. The r-colored permutation group can be realized as the wreath product group $\mathbb{Z}_r \wr \mathfrak{S}_n$, where the elements of $\mathbb{Z}_r$ are thought of as colors. The elements of $\mathfrak{S}_{n,r}$ are represented in window notation as $w = w(1)^{c_1} w(2)^{c_2} \cdots w(n)^{c_n}$, where $w(1)w(2)\cdots w(n) \in \mathfrak{S}_n$ is the *underlying permutation* and $(c_1, c_2, \ldots, c_n)$ is the *color vector* of w. We will represent both the colored permutation and the underlying permutation by the same letter.

Consider the following total order

$$1^{r-1} <_c \cdots <_c n^{r-1} <_c \cdots <_c 1^1 <_c \cdots <_c n^1 <_c 1^0 <_c \cdots <_c n^0$$

on $\Omega_{n,r}$, sometimes called the *color order*. For $w \in \mathfrak{S}_{n,r}$, define $\mathrm{Des}(w)$ to be the set of all indices $i \in [n-1]$ such that $w(i) >_c w(i+1)$ together with 0, whenever $w(1)$ has nonzero color and let $\mathrm{des}(w)$ be its cardinality. Write $\mathrm{Des}^*(w) := \mathrm{Des}(w) \setminus \{0\}$ and $\mathrm{des}^*(w)$ for its cardinality. Also, define

$$\begin{aligned} \mathrm{maj}(w) &:= \sum_{i \in \mathrm{Des}^*(w)} i \\ \mathrm{csum}(w) &:= c_1 + c_2 + \cdots + c_n \\ \mathrm{fmaj}(w) &:= r\,\mathrm{maj}(w) + \mathrm{csum}(w) \\ \mathrm{fdes}(w) &:= r\,\mathrm{des}^*(w) + c_1, \end{aligned}$$

the *major index, color sum, flag major index* and *flag descent number* of w, respectively. For a complete account of these statistics we refer to [17, Section 2.1].

For a pair of statistics $(\mathrm{eul}, \mathrm{mah})$ on $\mathfrak{S}_{n,r}$ and $\mathcal{A} \subseteq \mathfrak{S}_{n,r}$, we define the following generating polynomials

$$\begin{aligned} \mathcal{A}_{n,r}^{(\mathrm{eul},\mathrm{mah},\mathrm{csum})}(x,q,p) &:= \sum_{w \in \mathcal{A}} x^{\mathrm{eul}(w)} q^{\mathrm{mah}(w)} p^{\mathrm{csum}(w)} \\ \mathcal{A}_{n,r}^{(\mathrm{mah},\mathrm{csum})}(q,p) &:= \mathcal{A}_{n,r}^{\mathrm{eul},\mathrm{mah},\mathrm{csum}}(1,q,p) \\ \mathcal{A}_{n,r}^{(\mathrm{eul},\mathrm{csum})}(x,p) &:= \mathcal{A}_{n,r}^{\mathrm{eul},\mathrm{mah},\mathrm{csum}}(x,1,p) \\ \mathcal{A}_{n,r}^{(\mathrm{eul},\mathrm{mah})}(x,q) &:= \mathcal{A}_{n,r}^{\mathrm{eul},\mathrm{mah},\mathrm{csum}}(x,q,1). \end{aligned}$$

2.2 Colored quasisymmetric functions

Let $\mathbf{x}^{(j)} = (x_1^{(j)}, x_2^{(j)}, \ldots)$ be sequences of commuting indeterminates, for every $0 \leq j \leq r-1$. We consider formal power series in $\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(r-1)}$ with complex coefficients.

The *fundamental colored quasisymmetric function* associated to $w \in \mathfrak{S}_{n,r}$ with color vector $(c_1, c_2, \ldots, c_n)$ is defined by

$$F_w(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(r-1)}) \ := \sum_{\substack{i_1 \geq i_2 \geq \cdots \geq i_n \geq 1 \\ j \in \mathrm{Des}^*(w) \ \Rightarrow \ i_j > i_{j+1}}} x_{i_1}^{(c_1)} x_{i_2}^{(c_2)} \cdots x_{i_n}^{(c_n)}. \tag{2.1}$$

This colored analogue of Gessel's fundamental quasisymmetric functions was introduced by Poirier [18] and further studied in [14] and more recently, for $r = 2$, in [1]. One can define (colored) quasisymmetric functions indexed by *colored compositions*, that is compositions of a positive integer whose parts are colored. This is the approach of [14, Section 3]. Our definition is slightly different, but equivalent to, the one given in [14] with the inequalities under the summation on the right-hand side of Equation (2.1) being reversed.

Lastly, for a subset $\mathcal{A} \subseteq \mathfrak{S}_{n,r}$, let

$$F(\mathcal{A}; \mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(r-1)}) \ := \sum_{w \in \mathcal{A}} F_w(\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(r-1)})$$

be the colored quasisymmetric generating function associated to $\mathcal{A}$.

3 Specializations

This section provides general formulas for the generating polynomials of joint Mahonian and color sum statistics and joint Euler–Mahonian and color sum statistics on sets of colored permutations by specializing fundamental colored quasisymmetric functions. We first state the results and at the end of the section comment on their proofs, the complete versions of which can be found at [17, Section 3].

We begin by considering the specialization $\mathrm{ps}_{q,p}^{(r)}$ defined by the substitutions $x_i^{(j)} = q^{i-1}p^j$ for every $i \geq 1$ and $0 \leq j \leq r-1$, the specialization $\mathrm{ps}_{q,p,m}^{(r)}$ defined by the substitutions

$$\begin{cases} x_i^{(0)} = q^{i-1}, & 1 \leq i \leq m \\ x_i^{(j)} = q^{i-1}p^j, & 1 \leq j \leq r-1 \text{ and } 1 \leq i \leq m-1 \\ x_i^{(j)} = 0, & \text{otherwise} \end{cases}$$

and the specialization $\widetilde{\mathrm{ps}}_{q,p,m}^{(r)}$, defined by the substitutions $x_i^{(j)} = q^{i-1}p^j$ for every $1 \leq i \leq m$ and $0 \leq j \leq r-1$ and $x_i^{(j)} = 0$ otherwise.

Theorem 3.1. *For a positive integer n and every $\mathcal{A} \subseteq \mathfrak{S}_{n,r}$, we have*

$$\mathrm{ps}_{q,p}^{(r)} F(\mathcal{A}; \mathbf{x}^{(0)}, \ldots, \mathbf{x}^{(r-1)}) = \frac{\mathcal{A}_{n,r}^{(\mathrm{maj},\mathrm{csum})}(q,p)}{(q)_n} \tag{3.1}$$

$$\sum_{m \geq 1} \mathrm{ps}_{q,p,m}^{(r)} F(\mathcal{A}; \mathbf{x}^{(0)}, \ldots, \mathbf{x}^{(r-1)})\, x^{m-1} = \frac{\mathcal{A}_{n,r}^{(\mathrm{des},\mathrm{maj},\mathrm{csum})}(x,q,p)}{(x;q)_{n+1}} \tag{3.2}$$

$$\sum_{m \geq 1} \widetilde{\mathrm{ps}}_{q,p,m}^{(r)} F(\mathcal{A}; \mathbf{x}^{(0)}, \ldots, \mathbf{x}^{(r-1)})\, x^{m-1} = \frac{\mathcal{A}_{n,r}^{(\mathrm{des}^*,\mathrm{maj},\mathrm{csum})}(x,q,p)}{(x;q)_{n+1}}. \tag{3.3}$$

Next, we consider the specialization $\psi_{q,p}^{(r)}$ defined by the substitutions $x_i^{(j)} = q^{r(i-1)+j} p^j$ for every $i \geq 1$ and $0 \leq j \leq r-1$, the specialization $\psi_{q,p,m}^{(r)}$ defined by the substitutions

$$\begin{cases} x_i^{(0)} = q^{r(i-1)}, & 1 \leq i \leq m \\ x_i^{(j)} = q^{r(i-1)+j} p^j, & 1 \leq j \leq r-1 \text{ and } 1 \leq i \leq m-1 \\ x_i^{(j)} = 0, & \text{otherwise} \end{cases}$$

and the specialization $\widetilde{\psi}_{q,p,m}^{(r)}$ defined by the substitutions $x_i^{(j)} = q^{r(i-1)+j} p^j$ for every $1 \leq i \leq m$ and $0 \leq j \leq r-1$ and $x_i^{(j)} = 0$ otherwise.

Theorem 3.2. *For a positive integer n and every $\mathcal{A} \subseteq \mathfrak{S}_{n,r}$, we have*

$$\psi_{q,p}^{(r)} F(\mathcal{A}; \mathbf{x}^{(0)}, \ldots, \mathbf{x}^{(r-1)}) = \frac{\mathcal{A}^{(\mathrm{fmaj},\mathrm{csum})}(q,p)}{(q^r)_n} \tag{3.4}$$

$$\sum_{m \geq 1} \psi_{q,p,m}^{(r)} F(\mathcal{A}; \mathbf{x}^{(0)}, \ldots, \mathbf{x}^{(r-1)})\, x^{m-1} = \frac{\mathcal{A}^{(\mathrm{des},\mathrm{fmaj},\mathrm{csum})}(x,q,p)}{(x;q^r)_{n+1}} \tag{3.5}$$

$$\sum_{m \geq 1} \widetilde{\psi}_{q,p,m}^{(r)} F(\mathcal{A}; \mathbf{x}^{(0)}, \ldots, \mathbf{x}^{(r-1)})\, x^{m-1} = \frac{\mathcal{A}^{(\mathrm{des}^*,\mathrm{fmaj},\mathrm{csum})}(x,q,p)}{(x;q^r)_{n+1}}. \tag{3.6}$$

We remark that Theorem 3.2 follows from Theorem 3.1 by setting $q \to q^r$ and $p \to qp$. Lastly, we consider a more complicated specialization $\phi_{q,p,m}^{(r)}$ defined as follows: If $m = rs + t$, for some $1 \leq t \leq r$ and $s \geq 0$, then let $x_i^{(j)} = 0$ if the pair (i,j) is lexicographically greater than the pair $(m-t+1, t-1)$ and otherwise

$$x_i^{(j)} = \begin{cases} q^{i+j-1} p^j, & \text{if } i \equiv 1 \pmod{r} \\ 0, & \text{if } i \not\equiv 1 \pmod{r}. \end{cases}$$

We illustrate the definition of $\phi^{(r)}_{q,p,m}$ by considering a specific example for $r = 3$ and $m \in \{7, 8, 9\}$. The substitutions become

$$(x_i^{(j)})_{\substack{0\leq j\leq 2\\ i\geq 1}} = \begin{cases} \begin{pmatrix} 1 & 0 & 0 & q^3 & 0 & 0 & q^6 & 0 & \cdots \\ qp & 0 & 0 & q^4p & 0 & 0 & 0 & 0 & \cdots \\ q^2p^2 & 0 & 0 & q^5p^2 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix}, & \text{if } m = 7 \\ \begin{pmatrix} 1 & 0 & 0 & q^3 & 0 & 0 & q^6 & 0 & \cdots \\ qp & 0 & 0 & q^4p & 0 & 0 & q^7p & 0 & \cdots \\ q^2p^2 & 0 & 0 & q^5p^2 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix}, & \text{if } m = 8 \\ \begin{pmatrix} 1 & 0 & 0 & q^3 & 0 & 0 & q^6 & 0 & \cdots \\ qp & 0 & 0 & q^4p & 0 & 0 & q^7p & 0 & \cdots \\ q^2p^2 & 0 & 0 & q^5p^2 & 0 & 0 & q^8p^2 & 0 & \cdots \end{pmatrix}, & \text{if } m = 9. \end{cases}$$

Theorem 3.3. *For a positive integer n and every $\mathcal{A} \subseteq \mathfrak{S}_{n,r}$, we have*

$$\sum_{m\geq 1} \phi^{(r)}_{q,p,m} F(\mathcal{A}; \mathbf{x}^{(0)}, \ldots, \mathbf{x}^{(r-1)})\, x^{m-1} = \frac{\mathcal{A}^{(\text{fdes},\text{fmaj},\text{csum})}(x, q, p)}{(1-x)(1-x^rq^r)(1-x^rq^{2r})\cdots(1-x^rq^{nr})}. \tag{3.7}$$

The proofs of Theorems 3.1 and 3.3 are similar to each other and similar to those of Equations (1.4) and (1.5) and can be found in [17, Section 3]. The key observation in the proof of Equation (3.2), for example, is the following identity

$$\mathrm{ps}^{(r)}_{q,p,m}\, F_w(\mathbf{x}^{(0)}, \ldots, \mathbf{x}^{(r-1)}) = \sum_{\substack{m:=i_0\geq i_1\geq\cdots\geq i_n\geq 1\\ j\in\mathrm{Des}(w)\ \Rightarrow\ i_j>i_{j+1}}} q^{i_1+\cdots+i_n-n+\mathrm{csum}(w)} p^{\mathrm{csum}(w)}. \tag{3.8}$$

This is the case, because under the specialization $\mathrm{ps}^{(r)}_{q,p,m}$ substitutions $x_m^{(1)}, x_m^{(2)}, \ldots, x_m^{(r-1)}$ occur only if the color of the first entry of w is nonzero, which is exactly when 0 is considered a descent of w, explaining the first inequality under the sum on the right-hand side of Equation (3.8). Lastly, notice that our choice of the direction of inequalities in the definition of F_w (see Equation (2.1)) allows us to deal with the major index instead of the comajor index, as done in [12, Lemma 5.2] for example. This explains the motivation behind our choice.

4 Applications; Color sum Euler–Mahonian identities on colored permutations, derangements and absolute involutions

This section applies the theorems of Section 3 to prove color sum Euler–Mahonian identities on colored permutations, derangements and absolute involutions .

For a positive integer n, we have

$$F(\mathfrak{S}_{n,r};\mathbf{x}^{(0)},\ldots,\mathbf{x}^{(r-1)}) \;=\; (e_1(\mathbf{x}^{(0)})+\cdots+e_1(\mathbf{x}^{(r-1)}))^n, \tag{4.1}$$

where $e_1(\mathbf{x}^{(j)}) := \sum_{i\geq 1} x_i^{(j)}$, for every $0 \leq j \leq r-1$. This formula reduces to Equation (1.6) for $r = 1$. It appears in [18, Proposition 1.13] in a more general setting, but can also be proved using a colored version of the Robinson–Schensted correspondence and the Frobenius formula for $\mathfrak{S}_{n,r}$ (see [17, Lemma 2.1]). The following corollary follows from Theorem 3.1 for $\mathcal{A} = \mathfrak{S}_{n,r}$ by specializing Equation (4.1).

Corollary 4.1. *For a positive integer n, we have*

$$\mathfrak{S}_{n,r}^{(\mathrm{maj},\mathrm{csum})}(q,p) \;=\; [r]_p^n[n]_q! \tag{4.2}$$

and

$$\sum_{m\geq 0}\left([m+1]_q + p[r-1]_p[m]_q\right)^n x^m \;=\; \frac{\mathfrak{S}_{n,r}^{(\mathrm{des},\mathrm{maj},\mathrm{csum})}(x,q,p)}{(x;q)_{n+1}} \tag{4.3}$$

$$\sum_{m\geq 0}\left([r]_p[m+1]_q\right)^n x^m \;=\; \frac{\mathfrak{S}_{n,r}^{(\mathrm{des}^*,\mathrm{maj},\mathrm{csum})}(x,q,p)}{(x;q)_{n+1}}. \tag{4.4}$$

Equations (4.2) and (4.3) are due to Assaf [2, Equation (13)] and Biagioli and Zeng [7, Equation (8.1)], respectively. The following corollary follows from Theorem 3.2 for $\mathcal{A} = \mathfrak{S}_{n,r}$ by specializing Equation (4.1).

Corollary 4.2. *For a positive integer n, we have*

$$\mathfrak{S}_{n,r}^{(\mathrm{fmaj},\mathrm{csum})}(q,p) \;=\; [r]_{pq}^n[n]_{q^r}! \tag{4.5}$$

and

$$\sum_{m\geq 0}\left([m+1]_{q^r} + pq[m]_{q^r}[r-1]_{pq}\right)^n x^m \;=\; \frac{\mathfrak{S}_{n,r}^{(\mathrm{des},\mathrm{fmaj},\mathrm{csum})}(x,q,p)}{(x;q^r)_{n+1}} \tag{4.6}$$

$$\sum_{m\geq 0}\left([r]_{pq}[m+1]_{q^r}\right)^n x^m \;=\; \frac{\mathfrak{S}_{n,r}^{(\mathrm{des}^*,\mathrm{fmaj},\mathrm{csum})}(x,q,p)}{(x;q^r)_{n+1}}. \tag{4.7}$$

Equation (4.5) refines Haglund, Loehr and Remmel's formula [13, Equation (34)] for the distribution of the flag major index over colored permutations and Equation (4.6) appears in the work of Biagioli and Caselli [6, Theorem 5.2].

Remark 4.3. (a) Equations (4.3) and (4.6) for $q = 1$ become

$$\sum_{m\geq 0}(m[r]_p+1)^n x^m \;=\; \frac{\mathfrak{S}_{n,r}^{(\mathrm{des},\mathrm{csum})}(x,p)}{(1-x)^{n+1}}, \tag{4.8}$$

which reduces to an identity of Brenti [8, Equation (12)] for $r = 2$. In particular, the polynomial $\mathfrak{S}_{n,r}^{(\mathrm{des},\mathrm{csum})}(x,p) \;:=\; \sum_{i=0}^{n} a_{n,r,i}(p)x^i$ satisfies the formula

$$(m[r]_p+1)^n \;=\; \sum_{i=0}^{n} a_{n,r,i}(p)\binom{m+n-i}{n}$$

and therefore has only real roots for every positive integer n and every $p \geq 1$ (cf. [8, Corollary 3.7]). Although this result may not be new, it served as a motivation to introduce the parameter p which keeps track of the color sum.

(b) Setting $p = x$ in Equations (4.4) and (4.7) yields Euler–Mahonian idendities for the pairs $(\mathrm{ldes},\mathrm{maj})$ and $(\mathrm{ldes},\mathrm{fmaj})$

$$\sum_{m\geq 0}[m+1]_q^n\, x^m \;=\; \frac{\mathfrak{S}_{n,r}^{(\mathrm{ldes},\mathrm{maj})}(x,q)}{(x;q)_{n+1}[r]_x^n}$$
$$\sum_{m\geq 0}[m+1]_{q^r}^n\, x^m \;=\; \frac{\mathfrak{S}_{n,r}^{(\mathrm{ldes},\mathrm{fmaj})}(x,q)}{(x;q^r)_{n+1}[r]_{qx}^n},$$

where the *length descent number* of a colored permutation $w \in \mathfrak{S}_{n,r}$ is defined by [4, Definition 5.1]

$$\mathrm{ldes}(w) := \mathrm{des}^*(w) + \mathrm{csum}(w).$$

The following corollary appears in the work of Biagioli and Caselli [6, Theorem 5.4] and can be proved by taking the $\phi_{q,p,m}^{(r)}$ specialization of Equation (4.1) and substituting in Theorem 3.3 for $\mathcal{A} = \mathfrak{S}_{n,r}$.

Corollary 4.4. *For a nonnegative integer m, we write $m = r\mathrm{Q}(m) + \mathrm{R}(m)$ for some nonnegative integer $\mathrm{Q}(m)$ and $0 \leq \mathrm{R}(m) < r$. Then, we have*

$$\sum_{m\geq 0}\left([\mathrm{Q}(m)+1]_{q^r} + pq[r-1]_{pq}[\mathrm{Q}(m)]_{q^r} + pq^{r\mathrm{Q}(m)+1}[\mathrm{R}(m)]_{pq}\right)^n x^m$$
$$= \frac{\mathfrak{S}_{n,r}^{(\mathrm{fdes},\mathrm{fmaj},\mathrm{csum})}(x,q,p)}{(1-x)(1-x^rq^r)(1-x^rq^{2r})\cdots(1-x^rq^{nr})}. \tag{4.9}$$

An element of $\mathfrak{S}_{n,r}$ without fixed points of zero color is called a *colored derangement*. Let $\mathcal{D}_{n,r}$ be the set of all colored derangements in $\mathfrak{S}_{n,r}$. For a positive integer n, we have

$$F(\mathcal{D}_{n,r};\mathbf{x}^{(0)},\ldots,\mathbf{x}^{(r-1)}) \;=\; \sum_{k=0}^{n}(-1)^k e_k(\mathbf{x}^{(0)})(e_1(\mathbf{x}^{(0)})+\cdots+e_1(\mathbf{x}^{(r-1)}))^{n-k}, \tag{4.10}$$

where $e_k(\mathbf{x}^{(j)})$ is the k-th elementary symmetric function in $\mathbf{x}^{(j)}$. This formula reduces to [12, Theorem 8.1] and [1, Equation (7.8)] for $r = 1$ and $r = 2$, respectively and can be proved by trivially generalizing Adin et al.'s argument in the proof of [1, Theorem 7.3] for general r. The following corollary is due to Assaf [2, Theorem 3.2] and can be proved by taking the $\mathrm{ps}_{q,p}^{(r)}$ specialization of Equation (4.1) and substituting in Equation (3.1) for $\mathcal{A} = \mathfrak{S}_{n,r}$.

Corollary 4.5. *For a positive integer n, we have*

$$\mathcal{D}_{n,r}^{(\mathrm{maj},\mathrm{csum})}(q,p) \;=\; [r]_p^n [n]_q! \sum_{k=0}^{n} (-1)^k \frac{q^{\binom{k}{2}}}{[r]_p^k [k]_q!}. \tag{4.11}$$

The following corollary follows from Theorem 3.2 for $\mathcal{A} = \mathcal{D}_{n,r}$ by specializing Equation (4.10).

Corollary 4.6. *For a positive integer n, we have*

$$\mathcal{D}_{n,r}^{(\mathrm{fmaj},\mathrm{csum})}(q,p) \;=\; [r]_{pq}^n [n]_{q^r}! \sum_{k=0}^{n} (-1)^k \frac{q^{r\binom{k}{2}}}{[r]_{pq}^k [k]_{q^r}!} \tag{4.12}$$

and

$$\begin{aligned} \sum_{m\geq 0} \sum_{k=0}^{n} (-1)^k q^{r\binom{k}{2}} \binom{m+1}{k}_{q^r} \left([m+1]_{q^r} + pq[m]_{q^r}[r-1]_{pq}\right)^{n-k} x^m \\ = \frac{\mathcal{D}_{n,r}^{(\mathrm{des},\mathrm{fmaj},\mathrm{csum})}(x,q,p)}{(x;q^r)_{n+1}}. \end{aligned} \tag{4.13}$$

Equation (4.12) refines Faliharimalala and Zeng's formula [10, Equation (2.5)] for the distribution of the flag major index over colored derangements (see also [11, Equation (6.8)]). Equation (4.13) appears to be new even in the case $r = 1$, where in this case refines a celebrated result of Wachs [20, Theorem 4].

For $w \in \mathfrak{S}_{n,r}$ with color vector $(c_1, \ldots, c_n)$, let $\overline{w}$ be the r-colored permutation with underlying permutation w and color vector $(-c_1, \cdots, -c_n)$. A colored permutation $w \in \mathfrak{S}_{n,r}$ such that $\overline{w}^{-1} = w$ is called an *absolute involution*. Let $\mathcal{I}_{n,r}$ be the set of all absolute involutions in $\mathfrak{S}_{n,r}$. For a discussion on colored involutions (elements $w \in \mathfrak{S}_{n,r}$ such that $w^{-1} = w$) and absolute involutions we refer to [17, Section 4.3]. We have

$$\sum_{n\geq 0} F(\mathcal{I}_{n,r}; \mathbf{x}^{(0)}, \ldots, \mathbf{x}^{(r-1)})\, z^n \;=\; \prod_{c=0}^{r-1} \prod_{i\geq 1} (1 - z x_i^{(c)})^{-1} \prod_{1\leq i<j} (1 - z^2 x_i^{(c)} x_j^{(c)})^{-1}, \tag{4.14}$$

where $F(\mathcal{I}_{0,r}; \mathbf{x}^{(0)}, \ldots, \mathbf{x}^{(r-1)}) := 1$. This formula specializes to [12, Equation (7.1)] and can be proved using a colored version of the Robinson–Schensted correspondence and [19, Corollary 7.13.8] (see [17, Theorem 4.8]).

Corollary 4.7 follows from either Equation (3.2) or Equation (3.5) for $q = 1$ and $\mathcal{A} = \mathcal{I}_{n,r}$ by specializing Equation (4.14).

Corollary 4.7. *If*

$$\mathcal{I}_{n,r}^{(\mathrm{des},\mathrm{fix},\mathrm{csum})}(x,y,p) := \begin{cases} \sum_{w\in\mathcal{I}_{n,r}} x^{\mathrm{des}(w)} y^{\mathrm{fix}(w)} p^{\mathrm{csum}(w)}, & \text{if } n \geq 1 \\ 1, & \text{if } n = 0, \end{cases}$$

where $\mathrm{fix}(w)$ *is the number of fixed points of zero color of* w*, then we have*

$$\sum_{n\geq 0} \frac{\mathcal{I}_{n,r}^{(\mathrm{des},\mathrm{fix},\mathrm{csum})}(x,y,p)}{(1-x)^{n+1}} z^n = \sum_{m\geq 0} \frac{x^m}{(1+z)^m (1-zy)^{m+1} (z;p)_r^m (z^2;p^2)_r^{\binom{m}{2}}}. \tag{4.15}$$

Equation (4.15) reduces to an identity of Désarménien and Foata [9, Equation (6.2)] for $r = 1$ and to [16, Equation (2)] for $r = 2$ and $y = p = 1$. We remark that one can prove similar formulas for the joint Euler–Mahonian and color sum distributions on absolute involutions.

We close by mentioning that formulas for color sum bimahonian and bieulerian–bimahonian distributions can be proved using the method of this paper. We refer to [17, Section 4.3] for more details.

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