Séminaire Lotharingien de Combinatoire **85B** (2021)
Article #50, 12 pp.

Proceedings of the 33rd Conference on Formal Power Series and Algebraic Combinatorics (Ramat Gan)

Perfect models and Gelfand W-graphs

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Abstract. A Gelfand model for an algebra is a module isomorphic to a direct sum of irreducible modules, with every isomorphism class of irreducible modules represented exactly once. We introduce and study the notion of a perfect model for a finite Coxeter group; such a model is a certain set of discrete data parametrizing a Gelfand model for the associated Iwahori–Hecke algebra. We classify which Coxeter groups have perfect models, and then describe explicit Gelfand models for the classical finite Coxeter groups. This generalizes separate constructions of Adin, Postnikov, and Roichman and of Araujo and Bratten. Our Gelfand models have interesting "canonical bases" that give rise to associated W-graphs. We classify the molecules in these W-graphs when W is a symmetric group, and conjecture that in type A every molecule is a cell.

Keywords: Coxeter systems, Iwahori–Hecke algebras, W-graphs, Gelfand models, perfect involutions, quasiparabolic sets

1 Introduction

In this article we study certain uniform ways of constructing multiplicity-free representations of Iwahori–Hecke algebras that are also instances of *W-graphs*. We begin by explaining the definition of the models that are our primary subject.

Let (W, S) be a Coxeter system with length function $\ell : W \to \mathbb{N} := \{0, 1, 2, \dots\}$. Define $\mathrm{Aut}(W, S)$ to be the group of *Coxeter automorphisms* of W, that is, group automorphisms $\varphi \in \mathrm{Aut}(W)$ with $\varphi(S) = S$. Let $W^+ = W \rtimes \mathrm{Aut}(W, S)$ be the semidirect product whose elements are the pairs (w, φ) with $w \in W$ and $\varphi \in \mathrm{Aut}(W, S)$, with multiplication

$$(v, \alpha)(w, \beta) := (v \cdot \alpha(w), \alpha\beta).$$

Extend the length function of W to W^+ by setting $\ell((w, \varphi)) = \ell(w)$. We view W as a subgroup of W^+ by identifying $w \in W$ with $(w, \mathrm{id}_W) \in W^+$.

Let $z = (w, \varphi) \in W^+$. Following [21], we define z to be a *perfect involution* if $z^2 = (zt)^4 = 1$ for all reflections $t \in \{vsv^{-1} : v \in W, s \in S\}$. Let $\mathscr{I} = \mathscr{I}(W, S)$ be the set of perfect involutions in W^+. The group W acts on $\mathscr{I}$ by conjugation

$$v : (w, \varphi) \mapsto v(w, \varphi)v^{-1} = (v \cdot w \cdot \varphi(v)^{-1}, \varphi).$$

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As in [21], we say an element $z \in \mathscr{I}$ is *W-minimal* if $\ell(szs) \geq \ell(z)$ for all $s \in S$, and *W-maximal* if $\ell(szs) \leq \ell(z)$ for all $s \in S$. Each W-conjugacy class in $\mathscr{I}$ contains a unique W-minimal element (and a unique W-maximal element when $|W| < \infty$) [21, Cor. 2.10].

If $C_W(\theta) := \{w \in W : wz = zw\}$ where $z \in \mathscr{I}$ is W-minimal, then $C_W(z)$ is a *quasiparabolic subgroup* of W in the sense of Rains and Vazirani [21, Thm. 4.6]. Among other consequences, this implies that the permutation representation of W acting on $W/C_W(z)$ naturally deforms to a representation of the Iwahori–Hecke algebra of (W, S).

Example 1.1. The sets $\mathscr{I}(W, S)$ for classical Coxeter systems are described in [21, §9]:

(A) The symmetric group S_n of permutations of $[n] := \{1, 2, \dots, n\}$ is a Coxeter group relative to the generating set $\{s_1, s_2, \dots, s_{n-1}\}$ where $s_i := (i, i+1)$. The only non-identity element of $\mathrm{Aut}(S_n, \{s_1, s_2, \dots, s_{n-1}\})$ is the inner automorphism $\mathrm{Ad}(w_0)$ induced by $w_0 = n \cdots 321$. When n is odd every perfect involution in S_n^+ is central. When n is even there are three S_n-minimal elements in S_n^+: the identity element $1 \in S_n$, the fixed-point-free involution $s_1 s_3 s_5 \cdots s_{n-1} \in S_n$, and $(1, \mathrm{Ad}(w_0)) \in S_n^+$.

(BC) The group W_n^{BC} of permutations of $\{-n, \dots, -1, 0, 1, \dots, n\}$ that commute with the negation map is a Coxeter group relative to the generators $\{s_0, s_1, \dots, s_{n-1}\}$ with $s_0 := (-1, 1)$ and $s_i := (-i-1, -i)(i, i+1)$ for $i > 0$. The one-line representation of $w \in W_n^{\mathsf{BC}}$ is the word $w(1)w(2)\cdots w(n)$ where we write $\bar{i}$ in place of $-i$. When $n > 2$ there are only trivial Coxeter automorphisms and the perfect involutions in W_n^{BC} are the elements $w = w^{-1}$ such that $|w| : i \mapsto |w(i)|$ is the identity map or a fixed-point-free permutation of $[n]$. The W_n^{BC}-minimal perfect involutions consist of the $n+1$ elements $\theta_i := \bar{1}\bar{2}\cdots\bar{i}(i+1)\cdots n$ plus $s_1 s_3 s_5 \cdots s_{n-1}$ if n is even.

(D) The subgroup W_n^{D} of $w \in W_n^{\mathsf{BC}}$ such that $|\{i \in [n] : w(i) < 0\}|$ is even is a Coxeter group relative to the generators $\{s_{-1}, s_1, s_2, \dots, s_{n-1}\}$ where $s_{-1} := (-2, 1)(-1, 2)$ and s_i for $i \geq 0$ is as in (BC). For $n > 4$ there is just one nontrivial Coxeter automorphism $w \mapsto w^* := s_0 w s_0$. The perfect involutions in $(W_n^{\mathsf{D}})^+$ are the pairs (w, φ) with $\varphi \in \{1, *\}$ and $w^{-1} = \varphi(w)$ such that $|w| \in S_n$ is 1 or fixed-point-free. The W_n^{D}-minimal perfect involutions are θ_i for i even, $\tilde{\theta}_i := (1\bar{2}\bar{3}\cdots\bar{i}(i+1)\cdots n, *)$ for i odd, plus both $s_1 s_3 s_5 \cdots s_{n-1}$ and $s_{-1} s_3 s_5 \cdots s_{n-1}$ if n is even.

Given $J \subset S$, let $W_J := \langle s \in J \rangle$. Then (W_J, J) is a Coxeter system whose length function is $\ell|_{W_J}$. Write $\mathscr{I}_J := \mathscr{I}(W_J, J)$ for the set of perfect involutions in $(W_J)^+$.

Definition 1.2. A *model triple* $(J, z_{\min}, \sigma)$ for a finite Coxeter group W consists of a set $J \subset S$, a W_J-minimal element $z_{\min} \in \mathscr{I}_J$, and a linear character $\sigma : W_J \to \{\pm 1\}$. A set $\mathscr{P}$ of model triples for W is a *perfect model* if $\sum_{(J, z_{\min}, \sigma) \in \mathscr{P}} \mathrm{Ind}_{C_{W_J}(z_{\min})}^{W} \mathrm{Res}_{C_{W_J}(z_{\min})}^{W_J}(\sigma) = \sum_{\chi \in \mathrm{Irr}(W)} \chi$, where $\mathrm{Irr}(W)$ is the set of complex irreducible characters of W and Ind_H^G and Res_H^G denote the usual operations of induction and restriction for groups $H \subset G$.

Our first objective is to classify which finite Coxeter groups have perfect models. We will then investigate how to use such models to construct some interesting representations of the Iwahori–Hecke algebras of finite classical Coxter systems. Since space is limited, we have omitted all proofs in this extended abstract; for these details, see [20].

2 Classification of perfect models

A Coxeter system (W, S) is *irreducible* if the only disjoint decomposition $S = S' \sqcup S''$ in which every $s' \in S'$ commutes with every $s'' \in S''$ has either $S' = \varnothing \neq S''$ or $S' \neq \varnothing = S''$. An *irreducible factor* of (W, S) is a parabolic subsystem (W_J, J) that is irreducible.

Theorem 2.1. A finite Coxeter group W has a perfect model if and only if each of its irreducible factors has a perfect model. The irreducible Coxeter systems with perfect models are those of type A_{n-1}, BC_n, D_{2n+1}, H_3, and $\mathsf{I}_2(n)$ for all $n \geq 2$. The irreducible Coxeter systems without perfect models are those of type D_{2n}, E_6, E_7, E_8, F_4, and H_4.

Perfect models for the finite classical Coxeter groups are given below. The model for type A_{n-1} is well-known; see [1, 13, 15]. What we describe for type BC_n (respectively, D_n) is a coarser version of the models in [2, 4, 18] (respectively, [8, 9, 10, 17]).

Definition 2.2. Let $W \in \{S_n, W_n^{\mathsf{BC}}, W_n^{\mathsf{D}}\}$ be a classical Weyl group. Write $\mathbb{1} = \mathbb{1}_W : w \mapsto 1$ and $\mathrm{sgn} = \mathrm{sgn}_W : w \mapsto (-1)^{\ell(w)}$ for the trivial and sign representations.

(A) Assume $W = S_n$ with $n \geq 1$. For each integer $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$, form a triple $(J, z_{\min}, \sigma)$ by taking $J = \{s_1, s_2, \dots, s_{n-1}\} \setminus \{s_{2k}\}$ so that $\langle J \rangle = S_{2k} \times S_{n-2k}$, and then setting

$$z_{\min} = s_1 s_3 s_5 \cdots s_{2k-1} \quad \text{and} \quad \sigma = \mathbb{1}_{S_{2k}} \times \mathrm{sgn}_{S_{n-2k}}.$$

(BC) Assume $W = W_n^{\mathsf{BC}}$ with $n \geq 2$. For each $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$, form a triple $(J, z_{\min}, \sigma)$ by taking $J = \{s_0, s_1, s_2, \dots, s_{n-1}\} \setminus \{s_{2k}\}$ so that $\langle J \rangle = W_{2k}^{\mathsf{BC}} \times S_{n-2k}$, and then setting

$$z_{\min} = s_1 s_3 s_5 \cdots s_{2k-1} \quad \text{and} \quad \sigma = \mathbb{1}_{W_{2k}^{\mathsf{BC}}} \times \mathrm{sgn}_{S_{n-2k}}.$$

(D) Assume $W = W_n^{\mathsf{D}}$ with $n \geq 3$. For each $0 < k \leq \lfloor \frac{n}{2} \rfloor$, form a triple $(J, z_{\min}, \sigma)$ by taking $J = \{s_{-1}, s_1, s_2, \dots, s_{n-1}\} \setminus \{s_{2k}\}$ so that $\langle J \rangle = W_{2k}^{\mathsf{D}} \times S_{n-2k}$, and then setting

$$z_{\min} = s_1 s_3 s_5 \cdots s_{2k-1} \quad \text{and} \quad \sigma = \mathbb{1}_{W_{2k}^{\mathsf{D}}} \times \mathrm{sgn}_{S_{n-2k}}.$$

Also include one additional triple $(J, z_{\min}, \sigma) = (\{s_1, s_2, \dots, s_{n-1}\}, 1, \mathrm{sgn}_{S_n})$.

Let $\mathscr{P} = \mathscr{P}(W)$ be the set of $1 + \lfloor \frac{n}{2} \rfloor$ triples $(J, z_{\min}, \sigma)$ listed in each case.

Theorem 2.3. Assume W is one of the classical Weyl groups S_{n-1}, W_n^{BC}, or W_{2n+1}^{D} for an integer $n \geq 2$. Then the set $\mathscr{P}(W)$ given in Definition 2.2 is a perfect model.

Let G be a finite group. Choose representatives for the distinct conjugacy classes of involutions $g = g^{-1} \in G$. For each representative involution g, form the centralizer $H = C_G(g) = \{x \in G : xg = gx\}$ and choose a linear character $\lambda : H \to \mathbb{C}$. The resulting set of pairs (H, λ) is an *involution model* for G if $\sum_{(H,\lambda)} \mathrm{Ind}_H^G(\lambda) = \sum_{\chi \in \mathrm{Irr}(G)} \chi$. The existence of an involution model implies that $|\{g \in G : g = g^{-1}\}| = \sum_{\chi \in \mathrm{Irr}(G)} \deg \chi$, which holds if and only if all representations of G are realizable over $\mathbb{R}$ (see [7, §2]).

The involution models for finite Coxeter groups are classified in [5, 23]. Surprisingly, although we do not know of any general procedure for converting involution models to perfect models or vice versa, this classification is identical to the one in Theorem 2.1:

Corollary 2.4. A finite Coxeter group has a perfect model if and only if it has an involution model.

3 Gelfand models for Iwahori–Hecke algebras

Let x be an indeterminate. The *(single-parameter) Iwahori–Hecke algebra* of a Coxeter system (W, S) is the free $\mathbb{Q}[x, x^{-1}]$-module $\mathcal{H} = \mathcal{H}(W)$ with basis $\{H_w : w \in W\}$, equipped with the unique algebra structure in which $H_w H_s = H_{sw}$ if $\ell(sw) > \ell(w)$ and $H_w H_s = H_{sw} + (x - x^{-1}) H_w$ if $\ell(sw) < \ell(w)$ for all $s \in S$ and $w \in W$.

A *Gelfand model* for $\mathcal{H}$ is an $\mathcal{H}$-module isomorphic to the direct sum of all irreducible $\mathcal{H}$-modules. Any perfect model for W gives rise to a Gelfand model for $\mathcal{H}$. Our results in this section are formal consequences of this fact and Theorem 2.3.

For integers $i > 0$, define permutations $s_i := (-i-1, -i)(i, i+1)$, $s_0 := (-1, 1)$, and $s_{-i} = (-i-1, i)(-i, i+1)$. We realize the Coxeter groups of type A_{n-1}, BC_n, and D_n as $S_n := \langle s_1, s_2, \ldots, s_{n-1} \rangle \subset W_n^{\mathsf{D}} := \langle s_{-1}, s_1, s_2, \ldots, s_{n-1} \rangle \subset W_n^{\mathsf{BC}} := \langle s_0, s_1, s_2, \ldots, s_{n-1} \rangle$. Assume below that W is one of these groups with S the set of simple generators just listed. Then the length function of W is determined by the following properties:

- If $w \in W$ and $i > 0$ then $\ell(ws_i) < \ell(w)$ if and only if $w(i) > w(i+1)$.
- If $w \in W = W_n^{\mathsf{BC}}$ then $\ell(ws_0) < \ell(w)$ if and only if $w(1) < 0$.
- If $w \in W = W_n^{\mathsf{D}}$ then $\ell(ws_{-1}) < \ell(w)$ if and only if $-w(2) > w(1)$.

Let $\mathcal{F}_{2n} := \left\{ z \in W_{2n}^{\mathsf{BC}} : z = z^{-1} \text{ and } |z(i)| \neq i \text{ for all } i \in [2n] \right\} \subset W_{2n}^{\mathsf{D}}$. We say that an integer $i > 0$ is a *visible descent* of $z \in \mathcal{F}_{2n}$ if $z(i+1) < \min\{i, z(i)\}$ or $z(i) < -i$. Next:

- Define $\mathcal{K}_n^{\mathsf{BC}}$ to be the set of $z \in \mathcal{F}_{2n}$ with no visible descents greater than n.
- Define $\mathcal{K}_{n-1}^{\mathsf{A}} := \mathcal{K}_n^{\mathsf{BC}} \cap S_{2n}$.

- Define $\mathcal{K}_n^{\mathsf{D}}$ to be the subset of $z \in \mathcal{K}_n^{\mathsf{BC}}$ for which $|\{i \in [n] : z(i) < -i\}|$ is even.

Define $\mathcal{K} = \mathcal{K}(W)$ to be $\mathcal{K}_{n-1}^{\mathsf{A}}$, $\mathcal{K}_n^{\mathsf{BC}}$, or $\mathcal{K}_n^{\mathsf{D}}$ according to whether W is S_n, W_n^{BC}, or W_n^{D}. When evaluating $\ell(z)$ for $z \in \mathcal{K}$, we consider $\mathcal{K}_{n-1}^{\mathsf{A}} \subset S_{2n}$, $\mathcal{K}_n^{\mathsf{BC}} \subset W_{2n}^{\mathsf{BC}}$, and $\mathcal{K}_n^{\mathsf{D}} \subset W_{2n}^{\mathsf{D}}$.

Remark 3.1. One can characterize the elements of $\mathcal{K}$ more explicitly. Namely, $z \in \mathcal{F}_{2n}$ belongs to $\mathcal{K}_n^{\mathsf{BC}}$ if and only if there is an integer $0 \leq i \leq n$ with $i \equiv n \pmod 2$ such that

$$-n \leq z(n+1) < z(n+2) < \cdots < z(n+i) \leq n \quad \text{and} \quad z(n+i+2j) = n+i+2j-1$$

for each integer $j > 0$ with $n+i+2j \leq 2n$. This suggests the following notation. Suppose $w = w^{-1} \in W_n^{\mathsf{BC}}$. Let $a_1 > a_2 > \cdots > a_p$ (respectively, $b_1 < b_2 < \cdots < b_q$) be the numbers $a \in [n]$ with $w(a) = -a$ (respectively, $b \in [n]$ with $w(b) = b$). Define $\underline{w} \in \mathcal{K}_n^{\mathsf{BC}}$ to be the unique element mapping $a_i \mapsto -n-i$ and $b_i \mapsto n+p+i$ and $c \mapsto w(c)$ for $c \in [n] \setminus \{a_1, a_2, \dots, a_p, b_1, b_2, \dots, b_q\}$. For example, we would have

$$\underline{2134} = 21563487 \in \mathcal{K}_3^{\mathsf{A}} \quad \text{and} \quad \underline{\bar{3}2\bar{1}\bar{4}\bar{5}} = \bar{3}, 8, \bar{1}, \bar{7}, \bar{6}, \bar{5}, \bar{4}, 2, 10, 9 \in \mathcal{K}_5^{\mathsf{BC}}$$

When $W \in \{S_n, W_n^{\mathsf{BC}}\}$, the map $w \mapsto \underline{w}$ is a bijection $\{w = w^{-1} \in W\} \to \mathcal{K}$. In type D, if we set $e(w) := |\{i \in [n] : w(i) < -i\}|$ for $w \in W_n^{\mathsf{BC}}$, then the same map is a bijection $\{w = w^{-1} \in W_n^{\mathsf{D}} : e(w) \text{ is even}\} \sqcup \{w = w^{-1} \in W_n^{\mathsf{BC}} \setminus W_n^{\mathsf{D}} : e(w) \text{ is odd}\} \to \mathcal{K}_n^{\mathsf{D}}$.

Fix $z \in \mathcal{K}$. Let $\mathrm{Des}(z) := \{s \in S : \ell(zs) < \ell(z)\}$ and $\mathrm{Asc}(z) := \{s \in S : \ell(zs) > \ell(z)\}$ where ℓ is the length function of S_{2n}, W_{2n}^{BC}, or W_{2n}^{D}, according to whether the Coxeter group W is S_n, W_n^{BC}, or W_n^{D}, respectively. Next let

$$\mathrm{Des}^{=}(z) := \{s \in S : sz = zs\} \quad \text{and} \quad \mathrm{Asc}^{=}(z) := \{s \in S : zsz \in \{s_i : i > n\}\},$$

and define $\mathrm{Des}^{<}(z)$ and $\mathrm{Asc}^{<}(z)$ to be the complements of $\mathrm{Des}^{=}(z) \sqcup \mathrm{Asc}^{=}(z)$ in $\mathrm{Des}(z)$ and $\mathrm{Asc}(z)$, respectively.

Finally, define $\mathcal{M} = \mathcal{M}(W)$ and $\mathcal{N} = \mathcal{N}(W)$ to be the free $\mathbb{Q}[x, x^{-1}]$-modules with respective bases $\{M_z : z \in \mathcal{K}\}$ and $\{N_z : z \in \mathcal{K}\}$. Our third main result is the following:

Theorem 3.2. Assume (W, S) is a classical finite Coxeter system of type A_{n-1}, BC_n, or D_n for some $n \geq 2$. Then there is a unique $\mathcal{H}$-module structure on $\mathcal{M}$ in which

$$H_s M_z = \begin{cases} M_{szs} & \text{if } s \in \mathrm{Asc}^{<}(z) \\ M_{szs} + (x - x^{-1}) M_z & \text{if } s \in \mathrm{Des}^{<}(z) \\ x M_z & \text{if } s \in \mathrm{Des}^{=}(z) \\ -x^{-1} M_z & \text{if } s \in \mathrm{Asc}^{=}(z) \end{cases} \quad \text{for all } s \in S \text{ and } z \in \mathcal{K},$$

and there is a unique $\mathcal{H}$-module structure on $\mathcal{N}$ in which

$$H_s N_z = \begin{cases} N_{szs} & \text{if } s \in \mathrm{Asc}^{<}(z) \\ N_{szs} + (x - x^{-1}) N_z & \text{if } s \in \mathrm{Des}^{<}(z) \\ -x^{-1} N_z & \text{if } s \in \mathrm{Des}^{=}(z) \\ x N_z & \text{if } s \in \mathrm{Asc}^{=}(z) \end{cases} \quad \text{for all } s \in S \text{ and } z \in \mathcal{K}.$$

These $\mathcal{H}$-modules are multiplicity-free direct sums of irreducible submodules. If (W,S) is not of type D_n with n even, then $\mathcal{M}$ and $\mathcal{N}$ are both Gelfand models for $\mathcal{H}$.

The module $\mathcal{M}(S_n)$ can be identified with the Gelfand model for $\mathcal{H}(S_n)$ that Adin, Postnikov, and Roichman study in [1]. On the other hand, $\mathcal{N}(W)$ for each group $W \in \{S_n, W_n^{\mathsf{BC}}, W_n^{\mathsf{D}}\}$ is a deformation of the W-representation described by Araujo and Bratten in [3]. Theorem 3.2 implies that $\mathcal{M}(S_n)$ and $\mathcal{N}(S_n)$ are both abstractly isomorphic to the type A_{n-1} version of the $\mathcal{H}$-module studied by Lusztig and Vogan in [16]; see [16, §5.3]. It is not clear how to write down these isomorphisms in a concrete way. Our construction for types BC_n and D_{2n+1} partly resolves an open problem mentioned in [2, §7].

4 Gelfand W-graphs

Let $p \mapsto \overline{p}$ denote the ring automorphism of $\mathbb{Q}[x,x^{-1}]$ sending $x \mapsto x^{-1}$. A map $\phi : \mathcal{A} \to \mathcal{B}$ between $\mathbb{Q}[x,x^{-1}]$-modules is *antilinear* if $\phi(pa) = \overline{p} \cdot \phi(a)$ for all $a \in \mathcal{A}$. For the Iwahori–Hecke algebra $\mathcal{H}$ of any Coxeter system (W,S), there is a unique antilinear map $\mathcal{H} \to \mathcal{H}$, written $H \mapsto \overline{H}$ and called the *bar operator*, such that $\overline{H_w} = (H_{w^{-1}})^{-1}$ for all $w \in W$. This map is a ring involution.

The famous *Kazhdan–Lusztig basis* of $\mathcal{H}$ [14] consists of the unique elements $\underline{H}_w$ for $w \in W$ satisfying $\overline{\underline{H}_w} = \underline{H}_w \in H_w + \sum_{\ell(y)<\ell(w)} x^{-1}\mathbb{Z}[x^{-1}]H_y$. Our third main result constructs analogous "canonical bases" for the modules $\mathcal{M}$ and $\mathcal{N}$ from Theorem 3.2. An *$\mathcal{H}$-compatible bar operator* on an $\mathcal{H}$-module $\mathcal{A}$ is an antilinear map $A \mapsto \overline{A}$ with $\overline{HA} = \overline{H} \cdot \overline{A}$ for all $H \in \mathcal{H}$ and $A \in \mathcal{A}$. For the rest of this section, assume (W,S) is a classical finite Coxeter system and define $\mathcal{K}$, $\mathcal{M}$, and $\mathcal{N}$ as in Theorem 3.2.

Theorem 4.1. There are unique $\mathcal{H}$-compatible bar operators on $\mathcal{M}$ and $\mathcal{N}$ with $\overline{M_z} = M_z$ and $\overline{N_z} = N_z$ for all $z \in \mathcal{K}$ with $\mathrm{Des}^{<}(z) = \varnothing$. Each of these bar operators is an involution. Additionally, the module $\mathcal{M}$ has a unique basis $\{\underline{M}_z : z \in \mathcal{K}\}$ satisfying $\overline{\underline{M}_z} = \underline{M}_z \in M_z + \sum_{\ell(y)<\ell(z)} x^{-1}\mathbb{Z}[x^{-1}]M_y$. The module $\mathcal{N}$ likewise has a unique basis $\{\underline{N}_z : z \in \mathcal{K}\}$ satisfying $\overline{\underline{N}_z} = \underline{N}_z \in N_z + \sum_{\ell(y)<\ell(z)} x^{-1}\mathbb{Z}[x^{-1}]N_y$.

The structure constants for multiplication $\mathcal{H} \times \mathcal{M} \to \mathcal{M}$ and $\mathcal{H} \times \mathcal{N} \to \mathcal{N}$ in the bases $\{\underline{M}_z\}$ and $\{\underline{N}_z\}$ may be encoded using Kazhdan and Lusztig's notion of a *W-graph* from [14]. We review the definition below, following the conventions in [22].

Definition 4.2. An *S-labeled graph* $\Gamma = (V, \omega, I)$ is a set V with maps $\omega : V \times V \to \mathbb{Z}[x,x^{-1}]$ and $I : V \to \{$ subsets of S $\}$. We often think of this structure as a weighted directed graph on V with edges $u \xrightarrow{\omega(u,v)} v$ for each $u, v \in V$ with $\omega(u,v) \neq 0$.

Definition 4.3. An S-labeled graph $\Gamma = (V, \omega, I)$ is a *W-graph* if the free $\mathbb{Q}[x,x^{-1}]$-module

$\mathcal{Y}(\Gamma)$ with basis $\{Y_v : v \in V\}$ has an $\mathcal{H}$-module structure where

$$H_s Y_u = \begin{cases} xY_u & \text{if } s \notin I(u) \\ -x^{-1}Y_u + \sum_{\substack{v\in V \\ s\notin I(v)}} \omega(u,v)Y_v & \text{if } s \in I(u) \end{cases} \quad \text{for all } s \in S \text{ and } u \in V. \tag{4.1}$$

Definition 4.4. A W-graph $\Gamma = (V, \omega, I)$ is *quasi-admissible* if (1) the corresponding directed graph is bipartite, (2) we always have $\omega(u,v) \in \mathbb{Z}$, (3) we have $\omega(u,v) = 0$ whenever $I(u) \subset I(v)$, and (4) we have $\omega(u,v) = \omega(v,u)$ whenever $I(u) \not\subset I(v) \not\subset I(u)$. A quasi-admissible W-graph is *admissible* if we always have $\omega(u,v) \in \mathbb{N}$.

Define $\mathbf{m}_{yz}, \mathbf{n}_{yz} \in \mathbb{Z}[x^{-1}]$ for $y,z \in \mathcal{K}$ to be the polynomials with $\underline{M}_z = \sum_{y\in\mathcal{K}} \mathbf{m}_{yz} M_y$ and $\underline{N}_z = \sum_{y\in\mathcal{K}} \mathbf{n}_{yz} N_y$. Write $\mu^{\mathbf{m}}_{yz} := [x^{-1}]\mathbf{m}_{yz}$ and $\mu^{\mathbf{n}}_{yz} := [x^{-1}]\mathbf{n}_{yz}$ for the coefficients of x^{-1} in these polynomials. For $z \in \mathcal{K}$, define

$$\mathrm{Asc}^{\mathbf{m}}(z) := \mathrm{Asc}^{<}(z) \sqcup \mathrm{Asc}^{=}(z) \quad \text{and} \quad \mathrm{Asc}^{\mathbf{n}}(z) := \mathrm{Asc}^{<}(z) \sqcup \mathrm{Des}^{=}(z).$$

Let $\omega^{\mathbf{m}}(y,z) := \mu^{\mathbf{m}}_{yz} + \mu^{\mathbf{m}}_{zy}$ if $\mathrm{Asc}^{\mathbf{m}}(y) \not\subset \mathrm{Asc}^{\mathbf{m}}(z)$ and $\omega^{\mathbf{m}}(y,z) := 0$ otherwise for $y,z \in \mathcal{K}$. Likewise define $\omega^{\mathbf{n}}(y,z) := \mu^{\mathbf{n}}_{yz} + \mu^{\mathbf{n}}_{zy}$ if $\mathrm{Asc}^{\mathbf{n}}(y) \not\subset \mathrm{Asc}^{\mathbf{n}}(z)$ and $\omega^{\mathbf{n}}(y,z) := 0$ otherwise. Now let $\Gamma^{\mathbf{m}} = \Gamma^{\mathbf{m}}(W) := (\mathcal{K}, \omega^{\mathbf{m}}, \mathrm{Asc}^{\mathbf{m}})$ and $\Gamma^{\mathbf{n}} = \Gamma^{\mathbf{n}}(W) := (\mathcal{K}, \omega^{\mathbf{n}}, \mathrm{Asc}^{\mathbf{n}})$.

Theorem 4.5. The S-labeled graphs $\Gamma^{\mathbf{m}}$ and $\Gamma^{\mathbf{n}}$ are quasi-admissible W-graphs. The linear maps $Y_z \mapsto \underline{M}_z$ and $Y_z \mapsto \underline{N}_z$ are isomorphisms $\mathcal{Y}(\Gamma^{\mathbf{m}}) \cong \mathcal{M}$ and $\mathcal{Y}(\Gamma^{\mathbf{n}}) \cong \mathcal{N}$.

These W-graphs are not usually admissible. Define a *Gelfand W-graph* to be a W-graph Γ such that $\mathcal{Y}(\Gamma)$ is a Gelfand model for $\mathcal{H}(W)$. Theorem 4.5 constructs a pair of Gelfand W-graphs $\Gamma^{\mathbf{m}}$ and $\Gamma^{\mathbf{n}}$ for the groups $W \in \{S_n, W^{\mathsf{BC}}_n, W^{\mathsf{D}}_{2n+1}\}$. In the next section, we explain a precise sense in which these Gelfand W-graphs should be considered as the canonical ones for classical types.

5 Model equivalence

Let $\mathbb{T} = (J, z_{min}, \sigma)$ be a model triple for a finite Coxeter system (W,S). Each map $\alpha \in \mathrm{Aut}(W,S)$ extends to an automorphism of W^+ that preserves $\mathscr{I}(W,S)$ by the formula $\alpha : (w,\varphi) \mapsto (\alpha(w), \alpha\varphi\alpha^{-1})$. Using this convention, we define

$$\mathbb{T}^{\alpha} := (\alpha(J), \alpha(z_{min}), \sigma\alpha^{-1}) \text{ for } \alpha \in \mathrm{Aut}(W,S) \quad \text{and} \quad \overline{\mathbb{T}} := (J, z_{min}, \sigma\cdot\mathrm{sgn}).$$

Both $\mathbb{T}^{\alpha}$ and $\overline{\mathbb{T}}$ are again model triples for (W,S). Write w_J for the longest element in W_J and set $w_0 := w_S$. If $z = (y,\theta) \in \mathscr{I}_J$ for $y \in W_J$ and $\theta \in \mathrm{Aut}(W_J, J)$ then we set

$$z^{\vee} := (y^{\vee}, \theta^{\vee}) \text{ where } \begin{cases} y^{\vee} := w_0 \cdot w_J \cdot y \cdot w_0, \\ \theta^{\vee} := \mathrm{Ad}(w_0) \circ \mathrm{Ad}(w_J) \circ \theta \circ \mathrm{Ad}(w_0), \end{cases} \tag{5.1}$$

writing $\mathrm{Ad}(g) : w \mapsto gwg^{-1}$ for the inner automorphism induced by $g \in W$. One can check that $z^\vee \in \mathscr{I}_{w_0Jw_0}$ and that if z is W_J-maximal then $z^\vee$ is $W_{w_0Jw_0}$-minimal. Thus, if $z_{\max}$ is the unique W_J-maximal element in the W_J-orbit of $z_{\min}$, then

$$\mathbb{T}^\vee := (w_0Jw_0, z^\vee_{\max}, \sigma \circ \mathrm{Ad}(w_0)) \tag{5.2}$$

is another model triple. Finally, if $\mathbb{T}' = (J', z'_{\min}, \sigma')$ is a model triple such that $J = J'$, $C_{W_J}(z_{\min}) = C_{W_J}(z'_{\min})$, and $\mathrm{Res}^{W_J}_{C_{W_J}(z_{\min})}(\sigma) = \mathrm{Res}^{W_J}_{C_{W_J}(z_{\min})}(\sigma')$, then we write $\mathbb{T} \equiv \mathbb{T}'$.

Definition 5.1. Let $\sim$ be the transitive closure of the relation on model triples that has $\mathbb{T} \sim \overline{\mathbb{T}} \sim \mathbb{T}^\vee$, $\mathbb{T} \sim \mathbb{T}^\alpha$ for all $\alpha \in \mathrm{Aut}(W, S)$, and $\mathbb{T} \sim \mathbb{T}'$ whenever $\mathbb{T} \equiv \mathbb{T}'$. Two perfect models $\mathscr{P}$ and $\mathscr{P}'$ for the same finite Coxeter group W are *equivalent* if there is a bijection $\mathscr{P} \to \mathscr{P}'$ such that if $(J, z_{min}, \sigma) \mapsto (J', z'_{min}, \sigma')$ then $(J, z_{min}, \sigma) \sim (J', z'_{min}, \sigma')$.

Let us explain what makes this is a sensible definition. Results in [19, 21] associate to any model triple $\mathbb{T} = (J, z_{min}, \sigma)$ a pair of W_J-graphs with vertex set $W_J/C_{W_J}(z_{min})$. Howlett and Yin's method of W-graph induction [11, 12] transforms these structures into certain W-graphs, which we call $\Gamma^{\mathbf{m}}(\mathbb{T})$ and $\Gamma^{\mathbf{n}}(\mathbb{T})$, with vertex set $W^J \times W_J/C_{W_J}(z_{min})$. Here W^J is the set of minimal length left coset representatives of W_J in W.

We do not have room here to fully explain these constructions (see [20]). We comment, however, that the W-graphs in Theorem 4.5 can be identified with the disjoint unions of $\Gamma^{\mathbf{m}}(\mathbb{T})$ and $\Gamma^{\mathbf{n}}(\mathbb{T})$ as $\mathbb{T}$ ranges over the model triples in Theorem 2.3, and our notion of equivalence for model triples interacts nicely with these smaller W-graphs:

- Replacing $\mathbb{T}$ by $\mathbb{T}^\alpha$ corresponds to applying α to the vertices of $\Gamma^{\mathbf{m}}(\mathbb{T})$ and $\Gamma^{\mathbf{n}}(\mathbb{T})$.
- It holds that $\Gamma^{\mathbf{m}}(\overline{\mathbb{T}}) = \Gamma^{\mathbf{n}}(\mathbb{T})$ and $\Gamma^{\mathbf{n}}(\overline{\mathbb{T}}) = \Gamma^{\mathbf{m}}(\mathbb{T})$.
- If $\mathbb{T} \equiv \mathbb{T}'$ then $\Gamma^{\mathbf{m}}(\mathbb{T}) \cong \Gamma^{\mathbf{m}}(\mathbb{T}')$ and $\Gamma^{\mathbf{n}}(\mathbb{T}) \cong \Gamma^{\mathbf{n}}(\mathbb{T}')$.
- The W-graph $\Gamma^{\mathbf{m}}(\mathbb{T}^\vee)$ is dual to $\Gamma^{\mathbf{n}}(\mathbb{T})$, and $\Gamma^{\mathbf{n}}(\mathbb{T}^\vee)$ is dual to $\Gamma^{\mathbf{m}}(\mathbb{T})$.

There is a precise notion of *duality* for W-graphs [22, Prop. 1.2], but here it suffices to define this as meaning that the underlying directed graphs are anti-isomorphic, that is, they are isomorphic after all of the directed edges in one graph are reversed.

The *cells* of a W-graph $\Gamma = (V, \omega, I)$ are the subsets of V that make up the strongly connected components in the associated directed graph. If $C \subset V$ is a cell then $\Gamma|_C := (C, \omega|_{C\times C}, I|_C)$ is itself a W-graph, and so defines a *cell representation* $\mathcal{Y}(\Gamma|_C)$ of the Iwahori–Hecke algebra $\mathcal{H}$. Identifying the cells in any particular W-graph is a natural problem of interest. The remarks above imply that if we can describe the cells in $\Gamma^{\mathbf{m}}(\mathbb{T})$ and $\Gamma^{\mathbf{n}}(\mathbb{T})$ then we can also describe the cells in $\Gamma^{\mathbf{m}}(\mathbb{T}')$ and $\Gamma^{\mathbf{n}}(\mathbb{T}')$ when $\mathbb{T} \sim \mathbb{T}'$.

Say that a model triple (J, z_{min}, σ) is *strict* if $\sigma(s) = \sigma(t)$ whenever $s, t \in J$ have $st \neq ts$. A perfect model is *strict* if all of its model triples are strict. This always holds if there are only odd edges in the Coxeter graph of (W, S), as happens in types A_n and D_n.

Theorem 5.2. Suppose (W,S) is an irreducible finite Coxeter system. If $|S| \neq 3$ then (W,S) has at most one equivalence class of strict perfect models. Moreover, if (W,S) has a perfect model and is not of type $\mathsf{I}_2(2n)$, then (W,S) also has a strict perfect model.

Remark. In type $\mathsf{I}_2(2n)$ there are no strict perfect models. In type BC_n for $n \neq 3$, there are only 4 perfect models. Two of these are strict and the others are uninteresting "refinements" obtained by replacing a single strict model triple by a pair of non-strict ones.

To sum up, each perfect model gives rise to two Gelfand W-graphs, and for equivalent models the cells in these W-graphs are essentially the same. The Gelfand W-graphs for the classical groups in Theorem 4.5 are derived from representatives of a unique equivalence classes of (strict) perfect models. In types BC_n and D_n, there is one more property that underscores how the Gelfand W-graphs from Theorem 4.5 are canonical:

Theorem 5.3. If $W \in \{W_n^{\mathsf{BC}}, W_n^{\mathsf{D}}\}$, then $\Gamma^{\mathbf{m}}(W)$ and $\Gamma^{\mathbf{n}}(W)$ are dual in the sense of being anti-isomorphic as directed graphs (via an explicit bijection given in [20]).

The analogous property for $\Gamma^{\mathbf{m}}(S_n)$ and $\Gamma^{\mathbf{n}}(S_n)$ seems to hold if and only if $n \leq 4$.

6 Molecules in type A

It is an interesting open problem to identify the cells in the W-graphs $\Gamma^{\mathbf{m}}(W)$ and $\Gamma^{\mathbf{n}}(W)$ for each classical Weyl group W. For $W \in \{W_n^{\mathsf{BC}}, W_n^{\mathsf{D}}\}$, it follows by Theorem 5.3 that the cells in $\Gamma^{\mathbf{m}}(W)$ determine the cells in $\Gamma^{\mathbf{n}}(W$ and vice-versa. The cell classification problems for $\Gamma^{\mathbf{m}}(S_n)$ and $\Gamma^{\mathbf{n}}(S_n)$ are genuinely different.

The *molecules* in a W-graph $\Gamma = (V, \omega, I)$ are the equivalence classes in V under the transitive closure of the relation with $u \sim v$ if $\omega(u,v) \neq 0 \neq \omega(v,u)$. These subset do not inherit a W-graph structure, but every cell is a disjoint union of molecules.

Computations suggest that there are fewer cells than molecules in $\Gamma^{\mathbf{m}}(W_n^{\mathsf{BC}})$ for all $n \geq 6$ and in $\Gamma^{\mathbf{m}}(W_n^{\mathsf{D}})$ for all $n \geq 4$. Moreover, the cell representations for both of these W-graphs are often reducible. In type A, however, the following holds for at least $n \leq 10$:

Conjecture 6.1. Each molecule in $\Gamma^{\mathbf{m}}(S_n)$ and $\Gamma^{\mathbf{n}}(S_n)$ is a cell, and the cell representations associated to these S_n-graphs are all irreducible.

As a first step toward proving this conjecture, we describe the molecules in $\Gamma^{\mathbf{m}}(S_n)$ and $\Gamma^{\mathbf{n}}(S_n)$. The *(Young) diagram* of a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0)$ is the set $\mathsf{D}_\lambda = \{(i,j) \in [k] \times \mathbb{Z} : 1 \leq j \leq \lambda_i\}$. For us, a *tableau* of shape λ is just a map $T : \mathsf{D}_\lambda \to \mathbb{Z}$, which we envision as an assignment of numbers to some set of positions in a matrix. A tableau is *standard* if its rows and columns are strictly increasing and its entries are the numbers $1, 2, 3, \ldots, m$ for some $m \geq 0$ without any repetitions.

Suppose T is a tableau and $a \in \mathbb{Z}$. Define $T \xleftarrow{\mathsf{RSK}} a$ to be the tableau formed by the familiar *Schensted insertion* process. In detail, start by inserting a into the first row of T. At each stage, a number x is inserted into a row. Let y be the first entry in the row with $x < y$. If no such y exists then x is added to the end of the row; otherwise, x replaces y and y is inserted into the next row. Once this process terminates, the result is $T \xleftarrow{\mathsf{RSK}} a$.

Definition 6.2. Suppose T is a tableau and $a \leq b$ are integers. Let $T \xleftarrow{\mathfrak{m}} (a, b)$ and $T \xleftarrow{\mathfrak{n}} (a, b)$ be the tableaux given as follows.

(a) If $a = b$ then $T \xleftarrow{\mathfrak{m}} (a, b) := T \xleftarrow{\mathsf{RSK}} a$. If $a < b$ and (i, j) is the box of $T \xleftarrow{\mathsf{RSK}} a$ not in T, then form $T \xleftarrow{\mathfrak{m}} (a, b)$ from $T \xleftarrow{\mathsf{RSK}} a$ by adding b to the end of row $i + 1$.

(b) If $a = b$ then $T \xleftarrow{\mathfrak{n}} (a, b) := T \xleftarrow{\mathsf{RSK}} a$. If $a < b$ and (i, j) is the box of $T \xleftarrow{\mathsf{RSK}} a$ not in T, then form $T \xleftarrow{\mathfrak{n}} (a, b)$ from $T \xleftarrow{\mathsf{RSK}} a$ by adding b to the end of column $j + 1$.

We refer to the algorithms constructing $T \xleftarrow{\mathfrak{m}} a$ and $T \xleftarrow{\mathfrak{n}} a$ as $\mathfrak{m}$-*insertion* and $\mathfrak{n}$-*insertion*. Our notion of $\mathfrak{m}$-insertion is closely related to what Beissinger describes as [6, Algorithm 3.1], while $\mathfrak{n}$-insertion appears to be new. For example, we have

$$\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \xleftarrow{\mathsf{RSK}} 2 \;=\; \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \xleftarrow{\mathfrak{m}} (2,2) \;=\; \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \xleftarrow{\mathfrak{n}} (2,2) \;=\; \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$$

while

$$\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \xleftarrow{\mathfrak{m}} (2,5) \;=\; \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \xleftarrow{\mathfrak{n}} (2,5) \;=\; \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}.$$

For $w = w_1 w_2 \cdots w_n \in S_n$ let $P_{\mathsf{RSK}}(w) = \varnothing \xleftarrow{\mathsf{RSK}} w_1 \xleftarrow{\mathsf{RSK}} w_2 \xleftarrow{\mathsf{RSK}} \cdots \xleftarrow{\mathsf{RSK}} w_n$. The molecules in the S_n-graph associated to the type A_{n-1} Kazhdan–Lusztig basis (see [14]) are precisely the sets $\{w \in S_n : P_{\mathsf{RSK}}(w) = T\}$ as T ranges over all standard tableaux with n boxes. Each of these molecules turns out to be a cell with irreducible cell representation [14].

We classify the molecules in $\Gamma^{\mathbf{m}}(S_n)$ and $\Gamma^{\mathbf{n}}(S_n)$ in a related way. Fix $z \in \mathcal{K}^{\mathsf{A}}_{n-1}$. Let $b_1 < b_2 < \cdots < b_p$ (respectively, $c_1 < c_2 < \cdots < c_q$) be the numbers $b \in [n]$ with $z(b) < b$ (respectively, the numbers $c \in [n]$ with $n < z(c)$) and set $a_i = z(b_i)$. Then define

$$\begin{aligned} P_{\mathfrak{m}}(z) &= \varnothing \xleftarrow{\mathfrak{m}} (a_1, b_1) \xleftarrow{\mathfrak{m}} \cdots \xleftarrow{\mathfrak{m}} (a_p, b_p) \xleftarrow{\mathfrak{m}} (c_1, c_1) \xleftarrow{\mathfrak{m}} \cdots \xleftarrow{\mathfrak{m}} (c_q, c_q), \\ P_{\mathfrak{n}}(z) &= \varnothing \xleftarrow{\mathfrak{n}} (a_1, b_1) \xleftarrow{\mathfrak{n}} \cdots \xleftarrow{\mathfrak{n}} (a_p, b_p) \xleftarrow{\mathfrak{n}} (c_q, c_q) \xleftarrow{\mathfrak{n}} \cdots \xleftarrow{\mathfrak{n}} (c_1, c_1). \end{aligned}$$

Let $\lambda_{\mathfrak{m}}(z)$ and $\lambda_{\mathfrak{n}}(z)$ be the shapes of $P_{\mathfrak{m}}(z)$ and $P_{\mathfrak{n}}(z)$ for $z \in \mathcal{K}^{\mathsf{A}}_{n-1}$. Given a partition λ of n, define $C^{\mathbf{m}}_{\lambda} = \{z \in \mathcal{K}^{\mathsf{A}}_{n-1} : \lambda_{\mathfrak{m}}(z) = \lambda\}$ and $C^{\mathbf{n}}_{\lambda} = \{z \in \mathcal{K}^{\mathsf{A}}_{n-1} : \lambda_{\mathfrak{n}}(z) = \lambda\}$.

Example 6.3. Suppose $\lambda = (3,1)$. Then, using the notation $\underline{w}$ from Remark 3.1, we have $C^{\mathbf{m}}_\lambda = \{\underline{2134}, \underline{3214}, \underline{4231}\}$ and $C^{\mathbf{n}}_\lambda = \{\underline{2134}, \underline{3214}, \underline{1324}\}$ since

$$P_{\mathfrak{m}}(\underline{2134}) = P_{\mathfrak{n}}(\underline{1324}) = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}, \quad P_{\mathfrak{m}}(\underline{3214}) = P_{\mathfrak{n}}(\underline{3124}) = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad P_{\mathfrak{m}}(\underline{4231}) = P_{\mathfrak{n}}(\underline{2134}) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}.$$

A theorem of Beissinger implies the following interesting property of $P_{\mathfrak{m}}(z)$:

Proposition 6.4 ([6, Thm. 3.1])**.** If $z = z^{-1} \in S_n$ is fixed-point-free, then $P_{\mathfrak{m}}(\underline{z}) = P_{\mathsf{RSK}}(z)$.

Here is our last main theorem, along with a more precise conjecture:

Theorem 6.5. As λ ranges over all partitions of n, the sets $C^{\mathbf{m}}_\lambda$ and $C^{\mathbf{n}}_\lambda$ are all nonempty and give the distinct molecules in $\Gamma^{\mathbf{m}}(S_n)$ and $\Gamma^{\mathbf{n}}(S_n)$, respectively.

Conjecture 6.6. If λ is any partition of n then the molecules $C^{\mathbf{m}}_\lambda$ and $C^{\mathbf{n}}_\lambda$ in the respective S_n-graphs $\Gamma^{\mathbf{m}}(S_n)$ and $\Gamma^{\mathbf{n}}(S_n)$ are cells, and the representations associated to these cells are both isomorphic to the usual (irreducible) Specht module $\mathbb{S}_\lambda$ of $\mathcal{H}(S_n)$.

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