Séminaire Lotharingien de Combinatoire **85B** (2021)
Article #54, 12 pp.

Proceedings of the 33rd Conference on Formal Power Series and Algebraic Combinatorics (Ramat Gan)

Positroids, knots, and q,t-Catalan numbers

Pavel Galashin*[1] and Thomas Lam†[2]

[1] *Department of Mathematics, University of California, Los Angeles*
[2] *Department of Mathematics, University of Michigan, Ann Arbor*

Abstract. We relate the mixed Hodge structure on the cohomology of open positroid varieties (in particular, their Betti numbers over $\mathbb{C}$ and point counts over $\mathbb{F}_q$) to Khovanov–Rozansky homology of the associated links. We deduce that the mixed Hodge polynomials of top-dimensional open positroid varieties are given by rational q,t-Catalan numbers. Via the curious Lefschetz property, this implies the q,t-symmetry and unimodality properties of rational q,t-Catalan numbers. We show that the q,t-symmetry phenomenon is a manifestation of Koszul duality for category $\mathcal{O}$, and discuss relations with equivariant derived categories of flag varieties, and open Richardson varieties.

Keywords: Positroid varieties, q,t-Catalan numbers, HOMFLY polynomial, Khovanov–Rozansky homology, mixed Hodge structure, equivariant cohomology, Koszul duality.

1 Introduction

The Poincaré polynomial of the complex Grassmannian $\mathrm{Gr}(k,n)$ is well known to be given by the Gaussian polynomial $\begin{bmatrix} n \\ k \end{bmatrix}_q$. The number of points of $\mathrm{Gr}(k,n)$ over a finite field $\mathbb{F}_q$ is given by the same polynomial. The reason these two polynomials coincide is that the mixed Hodge structure on the cohomology of $\mathrm{Gr}(k,n)$ is pure. The situation is different when one considers the top-dimensional positroid variety $\Pi^\circ_{k,n} \subset \mathrm{Gr}(k,n)$, introduced in [18] building on the results of [25]. The space $\Pi^\circ_{k,n}$ consists of row spans of full rank $k \times n$ matrices whose cyclically consecutive maximal minors are all nonzero. It turns out that the Poincaré polynomial and the point count of $\Pi^\circ_{k,n}$ are given by two different q-analogs of (rational) Catalan numbers: one of our main results is that when $\gcd(k,n) = 1$, the Poincaré polynomial of $\Pi^\circ_{k,n}$ is given by $\sum_{P \in \mathrm{Dyck}_{k,n-k}} q^{\mathrm{area}(P)}$ while the number of points of $\Pi^\circ_{k,n}$ over $\mathbb{F}_q$ equals $\frac{1}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q$, up to a simple factor. The mixed Hodge structure on $H^\bullet(\Pi^\circ_{k,n})$ is non-pure, and we show that its bigraded Poincaré polynomial $\mathcal{P}(\Pi^\circ_{k,n}; q, t)$ coincides with the rational q,t-Catalan number $C_{k,n-k}(q,t)$ introduced

*galashin@math.ucla.edu. P.G. was supported by an Alfred P. Sloan Research Fellowship and by the National Science Foundation under Grants No. DMS-1954121 and No. DMS-2046915.

†tfylam@umich.edu. T.L. was supported by a von Neumann Fellowship from the Institute for Advanced Study and by grant DMS-1464693 from the National Science Foundation.

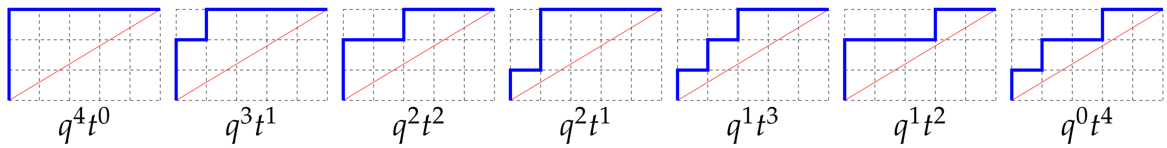


Figure 1: Computing the rational q,t-Catalan number $C_{3,5}(q,t)$.

in [9, 21]. Our proof proceeds via relating both sides to Khovanov–Rozansky knot homology [17]. Our results apply more generally to arbitrary positroid and Richardson varieties.

2 Positroid varieties and Catalan numbers

Rational q,t-Catalan numbers. Let a and b be coprime positive integers. The *rational q,t-Catalan number* $C_{a,b}(q,t) \in \mathbb{N}[q,t]$ was introduced by Loehr–Warrington [21], generalizing the work of Garsia–Haiman [9]. It is defined as follows:

$$C_{a,b}(q,t) := \sum_{P \in \mathrm{Dyck}_{a,b}} q^{\mathrm{area}(P)} t^{\mathrm{dinv}(P)}, \tag{2.1}$$

where $\mathrm{Dyck}_{a,b}$ is the set of lattice paths P inside a rectangle of height a and width b that stay above the diagonal, $\mathrm{area}(P)$ is the number of unit squares fully contained between P and the diagonal, and $\mathrm{dinv}(P)$ is the number of pairs (h,v) satisfying the following conditions: h is a horizontal step of P, v is a vertical step of P that appears to the right of h, and there exists a line of slope a/b (parallel to the diagonal) intersecting both h and v. For example, Figure 1 shows that

$$C_{3,5}(q,t) = q^4 + q^3t + q^2t^2 + q^2t + qt^3 + qt^2 + t^4. \tag{2.2}$$

Positroid varieties in the Grassmannian. The *Grassmannian* $\mathrm{Gr}(k,n)$ is the space of linear k-dimensional subspaces of $\mathbb{C}^n$. Alternatively, it can be identified with the space of full rank $k \times n$ matrices modulo row operations. Building on Postnikov's cell decomposition [25] of its totally nonnegative part, Knutson–Lam–Speyer [18] constructed a stratification $\mathrm{Gr}(k,n) = \bigsqcup_{f \in S_{k,n}} \Pi_f^\circ$ of the Grassmannian into *(open) positroid varieties*. Roughly speaking,[1] each positroid variety Π_f° corresponds to a permutation $f \in S_n$ such that $\#\{1 \le i \le n \mid f(i) < i\} = k$; the set of such permutations is denoted by $S_{k,n}$. For

[1] More precisely, positroid varieties are in bijection with *decorated* permutations, where a decoration of f is an arbitrary coloring of fixed points of f into black and white colors. The actual set $S_{k,n}$ consists of decorated permutations f satisfying $\#\{1 \le i \le n \mid f(i) < i\} + \#\{\text{black fixed points of } f\} = k$. The most interesting special case for us occurs when f is a single cycle, where the decoration is trivial.

each $f \in S_{k,n}$, the space Π_f° is a smooth algebraic variety. Two basic questions one can ask about such a space are: what is the number of points in $\Pi_f^\circ(\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$ with q elements, and what is the cohomology of Π_f° considered as a variety over $\mathbb{C}$?

These two questions turn out to be closely related to each other through the work of Deligne [4] on *mixed Hodge structures*, explored in the case of cluster varieties in [19]. Since the work of Scott [27], positroid varieties have been expected to admit a natural *cluster algebra* structure arising from Postnikov diagrams. We recently proved this conjecture building on the results of [20, 23, 28].

Theorem 2.1 ([7]). *The coordinate ring of each positroid variety Π_f° is isomorphic to the associated cluster algebra.*

This result allows one to study Π_f° as a *cluster variety*, in which case Deligne's mixed Hodge structure can be explored using the machinery developed by Lam–Speyer [19]. The mixed Hodge structure endows the cohomology $H^\bullet(\Pi_f^\circ)$ of Π_f° with a second grading, and the suitably renormalized Poincaré polynomial $\mathcal{P}(\Pi_f^\circ; q, t)$ of this bigraded vector space answers both of the above questions simultaneously:

Theorem 2.2 ([19, 7]). *For each $f \in S_{k,n}$, the* bigraded Poincaré polynomial $\mathcal{P}(\Pi_f^\circ; q, t) \in \mathbb{N}[q^{\frac{1}{2}}, t^{\frac{1}{2}}]$ *satisfies the following properties:*

(i) *q,t-symmetry: $\mathcal{P}(\Pi_f^\circ; q, t) = \mathcal{P}(\Pi_f^\circ; t, q)$;*
(ii) *q,t-unimodality: for each d, the coefficients of $\mathcal{P}(\Pi_f^\circ; q, t)$ at $q^d t^0, q^{d-1}t^1, \dots, q^0 t^d$ form a unimodal sequence;*
(iii) *$\mathcal{P}(\Pi_f^\circ; q^2, 1)$ equals the Poincaré polynomial of Π_f° (considered as a variety over $\mathbb{C}$);*
(iv) *$q^{\frac{1}{2}\dim \Pi_f^\circ} \cdot \mathcal{P}(\Pi_f^\circ; q, t)|_{t^{\frac{1}{2}} = -q^{-\frac{1}{2}}}$ equals the point count $\#\Pi_f^\circ(\mathbb{F}_q)$.*

The positroid stratification contains a unique open stratum, the *top-dimensional positroid variety* $\Pi_{k,n}^\circ := \Pi_{f_{k,n}}^\circ$ corresponding to the permutation $f_{k,n} \in S_{k,n}$ sending $i \mapsto i + k$ modulo n for all $1 \le i \le n$. It is given by

$$\Pi_{k,n}^\circ := \{\mathrm{RowSpan}(A) \mid A \in \mathrm{Mat}(k, n; \mathbb{C}) : \Delta_{1,2,\dots,k}(A), \Delta_{2,3,\dots,k+1}(A), \dots, \Delta_{n,1,\dots,k-1}(A) \neq 0\}.$$

Here $\Delta_I(A)$ is the maximal minor of A with column set $I \subset \{1, 2, \dots, n\}$, $|I| = k$. We are ready to state (the most important special case of) our main result.

Theorem 2.3. *Assume that* $\gcd(k, n) = 1$. *Then*

$$\mathcal{P}(\Pi_{k,n}^\circ; q, t) = \left(q^{\frac{1}{2}} + t^{\frac{1}{2}}\right)^{n-1} C_{k,n-k}(q, t). \tag{2.3}$$

The equality (2.3) arises as a conjecture from the works [30, 29] and we thank Vivek Shende for drawing our attention to the conjecture. We generalize Theorem 2.3 to all positroid varieties in Theorem 4.2 below.

Our proof of Theorem 2.3 involves a number of ingredients, including Khovanov–Rozansky knot homology and equivariant perverse sheaves. The point count specialization ($t^{\frac{1}{2}} = -q^{-\frac{1}{2}}$) turns out to require less advanced machinery. Namely, let us denote $[n]_q := 1 + q + \cdots + q^{n-1}$, $[n]_q! := [1]_q[2]_q \cdots [n]_q$, and $\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}$. We give an elementary proof of the following special case of Theorem 2.3.

Proposition 2.4. *Assume that* $\gcd(k,n) = 1$*. Then* $\#\Pi^{\circ}_{k,n}(\mathbb{F}_q) = (q-1)^{n-1} \cdot \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q$*. In other words, the probability that a uniformly random k-dimensional subspace of* $(\mathbb{F}_q)^n$ *belongs to* $\Pi^{\circ}_{k,n}(\mathbb{F}_q)$ *equals* $\frac{(q-1)^n}{q^n-1}$*.*

Remark 2.5. Surprisingly, this probability $\frac{(q-1)^n}{q^n-1}$ does not depend on k. We do not have a combinatorial explanation for this phenomenon.

Our proof proceeds by associating a *link* $\hat{\beta}_f$ to each positroid variety Π°_f (Section 3) and then comparing the point count $\#\Pi^{\circ}_f(\mathbb{F}_q)$ to the HOMFLY polynomial of $\hat{\beta}_f$. The HOMFLY polynomial is categorified by Khovanov–Rozansky knot homology, and our proof of Theorem 2.3 may be considered a "categorification" of Proposition 2.4.

Remark 2.6. After discovering the proof of Proposition 2.4 via the HOMFLY polynomial, we found that it can also be deduced from the results of [30, 29]. Our proof is new and yields a generalization (Theorem 3.4) of Proposition 2.4 to arbitrary open positroid varieties.

Torus action. The appearance of the extra factor $\left(q^{\frac{1}{2}} + t^{\frac{1}{2}}\right)^{n-1}$ in (2.3), as well as the condition $\gcd(k,n) = 1$ are neatly explained by the *torus action* on $\mathrm{Gr}(k,n)$. Let $T \cong (\mathbb{C}^*)^{n-1}$ be the quotient of the group of diagonal $n \times n$ matrices by the group of scalar matrices. The group T acts on $\mathrm{Gr}(k,n)$ by rescaling the columns of $k \times n$ matrices. This action leaves each positroid variety Π°_f invariant. We say that T acts *freely* on Π°_f if all non-identity elements of T act on Π°_f without fixed points. It is straightforward to check that the action of T on Π°_f is free if and only if the permutation f is a single cycle. Note that $f_{k,n}$ is a single cycle if and only if $\gcd(k,n) = 1$. We will see later in Definition 3.2 that f is a single cycle precisely when the associated link $\hat{\beta}_f$ has a single component, i.e., is a knot.

Let $\mathrm{ncyc}(f)$ denote the number of cycles of $f \in S_n$, and let $S^{\mathrm{ncyc}=1}_{k,n} := \{f \in S_{k,n} \mid \mathrm{ncyc}(f) = 1\}$. For $f \in S^{\mathrm{ncyc}=1}_{k,n}$, the quotient Π°_f/T is again a smooth cluster variety, and Theorem 2.2 applies to it. The associated bigraded Poincaré polynomials are related as

$\mathcal{P}(\Pi_f^\circ; q,t) = \left(q^{\frac{1}{2}} + t^{\frac{1}{2}}\right)^{n-1} \cdot \mathcal{P}(\Pi_f^\circ/T; q,t)$. In particular, in the setting of Theorem 2.3, we find

$$\mathcal{P}(\Pi_{k,n}^\circ/T; q,t) = C_{k,n-k}(q,t). \tag{2.4}$$

Combining this with Theorem 2.2 has consequences for q,t-Catalan numbers and positroid varieties which can be stated in an elementary way. Let us denote $d_{k,n} := (k-1)(n-k-1) = \dim(\Pi_{k,n}^\circ/T)$.

Corollary 2.7. *Assume that* $\gcd(k,n) = 1$. *We have:*

(i) *q,t-symmetry: $C_{k,n-k}(q,t) = C_{k,n-k}(t,q)$;*
(ii) *q,t-unimodality: for each d, the coefficients of $C_{k,n-k}(q,t)$ at $q^d t^0, q^{d-1}t^1, \ldots, q^0 t^d$ form a unimodal sequence;*
(iii) *the Poincaré polynomial of $\Pi_{k,n}^\circ/T$ is given by*

$$\sum_d q^{\frac{d}{2}} \dim H^{d_{k,n}-d}(\Pi_{k,n}^\circ/T) = C_{k,n-k}(q,1) = \sum_{P \in \mathrm{Dyck}_{k,n-k}} q^{\mathrm{area}(P)}; \tag{2.5}$$

(iv) *the number of $\mathbb{F}_q$-points of $\Pi_{k,n}^\circ/T$ is given by*

$$\#(\Pi_{k,n}^\circ/T)(\mathbb{F}_q) = \frac{1}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q = q^{\frac{1}{2}d_{k,n}} \cdot C_{k,n-k}(q, 1/q). \tag{2.6}$$

Remark 2.8. When $a = n$ and $b = n+1$, $C_{a,b}(q,t)$ recovers the famous *q,t-Catalan numbers $C_n(q,t)$* of Garsia and Haiman [9]. The fact that $C_n(q,t)$ is q,t-symmetric and q,t-unimodal follows from the results of Haiman [14, 13]. For arbitrary a,b, the q,t-symmetry property follows from the celebrated recent proof of the rational shuffle conjecture [22]. To our knowledge, q,t-unimodality of $C_{k,n-k}(q,t)$ is a new result.

Example 2.9. For $k=3$, $n=8$, the coordinate ring of $\Pi_{k,n}^\circ/T$ is a cluster algebra of type E_8 (with no frozen variables). The associated *mixed Hodge table* recording the dimensions of $H^{k,(p,p)}(\Pi_{3,8}^\circ/T)$ has the following form.

H^k	H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7	H^8
$k-p=0$	1	0	1	0	1	0	1	0	1
$k-p=1$					1	0	1		

See [19, Table 5]. The grading conventions are chosen so that the first row contributes $q^4 + q^3t + q^2t^2 + qt^3 + t^4$ while the second row contributes $q^2t + qt^2$ to $\mathcal{P}(\Pi_{k,n}^\circ; q,t)$. Comparing the result with (2.2), we find $\mathcal{P}(\Pi_{k,n}^\circ/T; q,t) = C_{k,n-k}(q,t)$.

The polynomial $C_{3,5}(q,t)$ given in (2.2) is indeed q,t-symmetric and q,t-unimodal: fixing the total degree of q and t, it splits into polynomials $q^4 + q^3t + q^2t^2 + qt^3 + t^4$ and $q^2t + qt^2$. We also have $C_{3,5}(q,1) = q^4 + q^3 + 2q^2 + 2q + 1$; the coefficient of $q^{d/2}$ is equal to $\dim H^{d_{k,n}-d}(\Pi_{k,n}^\circ/T)$ for each d.

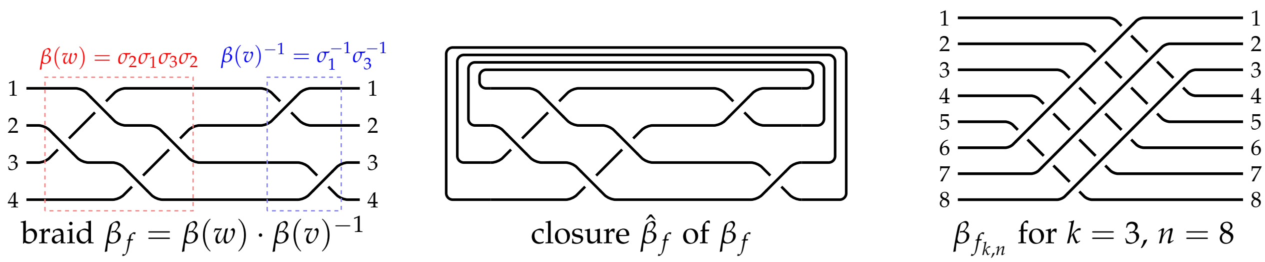


Figure 2: Braids and links associated to positroid varieties.

3 Links associated to positroid varieties

In order to explain how knot theory comes into play, we need a way to represent $f \in S_{k,n}$ in a slightly different form. Let us say that a permutation $w \in S_n$ is *k-Grassmannian* if $w^{-1}(1) < w^{-1}(2) < \cdots < w^{-1}(k)$ and $w^{-1}(k+1) < \cdots < w^{-1}(n)$. We denote by $\leq$ the (strong) Bruhat order on S_n. The following result is well known.

Proposition 3.1 ([18]). *For every $f \in S_{k,n}$, there exists a unique*[2] *pair of permutations $v, w \in S_n$ such that $v \leq w$, w is k-Grassmannian, and $f = wv^{-1}$.*

For example, when $f = f_{k,n}$, we have $w = f$ and $v = \mathrm{id}$. The dimension of Π_f° equals $\ell_{v,w} := \ell(w) - \ell(v)$, where $\ell(w)$ is the number of inversions of w.

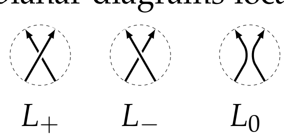
The group S_n is generated by simple transpositions $s_i = (i, i+1)$ for $1 \leq i \leq n-1$. Similarly, let $\mathcal{B}_n$ be the *braid group* on n strands, generated by $\sigma_1, \ldots, \sigma_{n-1}$ with relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$. Connecting the corresponding endpoints of a braid β gives rise to a *link* called the *closure* $\hat{\beta}$ of β; see Figure 2.

For each element $u \in S_n$, let $\beta(u)$ denote the corresponding braid, obtained by choosing a reduced word $u = s_{i_1}s_{i_2}\cdots s_{i_{\ell(u)}}$ for u and then replacing each s_i with σ_i.

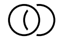
Definition 3.2. For $f \in S_{k,n}$ and $v \leq w \in S_n$ as in Proposition 3.1, let $\beta_f := \beta(w) \cdot \beta(v)^{-1}$. We refer to the closure $\hat{\beta}_f$ as *the link associated to* f. See Figure 2 for an example.

Observe that $\hat{\beta}_f$ is a *knot* (i.e., has one connected component) if and only if $f \in S_{k,n}^{\mathrm{ncyc}=1}$. We note that two other (more complicated) ways of assigning a Legendrian or a transverse link to a positroid variety have appeared recently in [29, 6].

The *HOMFLY polynomial* $P(L) = P(L; a, z)$ of an (oriented) link L is defined by a skein relation $aP(L_+) - a^{-1}P(L_-) = zP(L_0)$ and $P(\bigcirc) = 1$, where $\bigcirc$ denotes the unknot and L_+, L_-, L_0 are three links whose planar diagrams locally differ as follows:



[2]If f has fixed points, the pair (v, w) must be compatible with the decoration of f.

Example 3.3. For $n = 2$, we may take L_+ to be the closure of σ_1, in which case L_- is the closure of σ_1^{-1} and $L_0 =$  is the 2-component unlink. Applying the skein relation, we find $P(L_0) = \frac{a-a^{-1}}{z}$.

Surprisingly, the HOMFLY polynomial computes the number of $\mathbb{F}_q$-points of *any* positroid variety.

Theorem 3.4. *For all $f \in S_{k,n}$, let $P_f^{\mathrm{top}}(q)$ be obtained from the top a-degree term of $P(\hat{\beta}_f; a, z)$ by substituting $a := q^{-\frac{1}{2}}$ and $z := q^{\frac{1}{2}} - q^{-\frac{1}{2}}$. Then $\#\Pi_f^\circ(\mathbb{F}_q) = (q-1)^{n-1} \cdot P_f^{\mathrm{top}}(q)$.*

Remark 3.5. When $\gcd(k, n) = 1$, we have $f_{k,n} \in S_{k,n}^{\mathrm{ncyc}=1}$, and the associated knot $\hat{\beta}_{f_{k,n}}$ is the $(k, n-k)$-*torus knot*; see Figure 2(right). The value of $P(\hat{\beta}_{f_{k,n}}; a, z)$ was computed in [15], and its relationship with Catalan numbers was clarified in [10]. Thus Proposition 2.4 follows from Theorem 3.4 as a direct corollary.

Example 3.6. For $k = 3$, $n = 8$, one calculates (for instance, using Sage[3]) that the top a-degree term of $P(\hat{\beta}_{f_{k,n}}; a, z)$ equals $\frac{z^8+8z^6+21z^4+21z^2+7}{a^8}$. Substituting $a := q^{-\frac{1}{2}}$ and $z := q^{\frac{1}{2}} - q^{-\frac{1}{2}}$, we get

$$P_f^{\mathrm{top}}(q) = q^8 + q^6 + q^5 + q^4 + q^3 + q^2 + 1 = q^4 \cdot C_{3,5}(q, 1/q).$$

This agrees with (2.6) and Theorem 3.4.

Links associated to Richardson varieties. By Proposition 3.1, positroid varieties correspond to pairs $v \leq w$ of permutations such that w is k-Grassmannian. There is a more general class of *(open) Richardson varieties* $R_{v,w}^\circ$, indexed by all pairs $v \leq w \in S_n$, and the majority of the above story generalizes to this setting. The varieties $R_{v,w}^\circ$ form a stratification of the *complete flag variety* $\mathrm{Fl}_n(\mathbb{C})$. For a permutation $f \in S_{k,n}$ corresponding to a pair $v \leq w \in S_n$ via Proposition 3.1, the projection map $\mathrm{Fl}_n(\mathbb{C}) \to \mathrm{Gr}(k, n)$ restricts to an isomorphism $\Pi_f^\circ \cong R_{v,w}^\circ$. Thus positroid varieties are special cases of Richardson varieties.

Now, let G be a complex semisimple algebraic group of adjoint type, and choose a pair $B, B_- \subset G$ of opposite Borel subgroups. Let $T := B \cap B_-$ be the maximal torus and $W := N_G(T)/T$ the associated Weyl group. For the case $G = \mathrm{SL}_n(\mathbb{C})$, we have $W = S_n$, the subgroups $B, B_- \subset G$ consist of upper and lower triangular matrices, and $T \cong (\mathbb{C}^*)^{n-1}$ is the group of diagonal matrices modulo scalar matrices.[4] We have Bruhat decompositions $G = \bigsqcup_{w \in W} BwB = \bigsqcup_{v \in W} B_- vB$, and the intersection $BwB \cap B_- vB$ is

[3] https://doc.sagemath.org/html/en/reference/knots/sage/knots/link.html

[4] Note that G is assumed to be of *adjoint type*, thus in type A we should have $G = \mathrm{PGL}_n(\mathbb{C})$. However, we choose to work with $G = \mathrm{SL}_n(\mathbb{C})$ for simplicity.

nonempty if and only if $v \le w$ in the Bruhat order on W. For $v \le w$, we denote by $R^\circ_{v,w} := (BwB \cap B_-vB)/B$ an *open Richardson variety* inside the *complete flag variety* G/B. For $G = \mathrm{SL}_n(\mathbb{C})$, $R^\circ_{v,w}$ is the subset of $G/B = \mathrm{Fl}_n(\mathbb{C})$ obtained by specifying the dimensions of the intersections of a given flag with a given coordinate flag and its opposite coordinate flag.

In the case $G = \mathrm{SL}_n(\mathbb{C})$, one can similarly associate a braid $\beta_{v,w} := \beta(w) \cdot \beta(v)^{-1}$ to any pair $v \le w$ and consider its closure $\hat\beta_{v,w}$. We refer to the links of the form $\hat\beta_{v,w}$ as *Richardson links*. The point count $\#R^\circ_{v,w}(\mathbb{F}_q)$ is given by the *Kazhdan–Lusztig R-polynomial* [16], and both the statement and the proof of Theorem 3.4 generalize to this setting.

4 Main results

All of the above results are actually special cases of a single statement which applies to arbitrary Richardson varieties. This includes all positroid varieties Π°_f for $f \in S_{k,n}$, where $\mathrm{ncyc}(f)$ can be arbitrary. As a warm up, we start with the non-equivariant version.

Ordinary cohomology. Let $\mathfrak{h} := \mathrm{Lie}(T)$ be the Cartan subalgebra of $\mathrm{Lie}(G)$ corresponding to T, and denote $R := \mathbb{C}[\mathfrak{h}^*]$. For $G = \mathrm{SL}_n(\mathbb{C})$, $R = \mathbb{C}[y_1, \dots, y_{n-1}]$ is the polynomial ring. Since W is a Coxeter group, we can consider the category $\mathbb{S}\mathrm{Bim}$ of *Soergel bimodules*. Each element $B \in \mathbb{S}\mathrm{Bim}$ is a graded R-bimodule, and we will be interested in its R-invariants, which by definition form the *zeroth Hochschild cohomology $HH^0(B)$ of B*. Denote $HH^0_{\mathbb{C}}(B) := HH^0(B) \otimes_R \mathbb{C}$, where $\mathbb{C} = R/(\mathfrak{h}^*)$ is an R-module on which $\mathfrak{h}^*$ acts by 0. While the functor HH^0 involves Soergel bimodules, the functor $HH^0_{\mathbb{C}}$ involves *Soergel modules* instead.

To any element $w \in W$, Rouquier [26] associates a cochain complex $F^\bullet(w)$ of Soergel bimodules. He also associates another complex $F^\bullet(w)^{-1}$ such that their tensor product $F^\bullet(w) \otimes_R F^\bullet(w)^{-1}$ is homotopic to the identity. For a braid $\beta_{v,w} = \beta(w) \cdot \beta(v)^{-1}$, we set $F^\bullet_{v,w} := F^\bullet(w) \otimes_R F^\bullet(v)^{-1}$. Applying the functor $HH^0_{\mathbb{C}}$ to each term of this complex yields a complex $HH^0_{\mathbb{C}}(F^\bullet_{v,w})$ of graded R-modules. Taking its cohomology $HHH^0_{\mathbb{C}}(F^\bullet_{v,w}) := H^\bullet(HH^0_{\mathbb{C}}(F^\bullet_{v,w}))$, we get a bigraded R-module. We denote by $H^{k,(p)}(HH^0_{\mathbb{C}}(F^\bullet_{v,w}))$ the polynomial degree $2p$ part of $H^k(HH^0_{\mathbb{C}}(F^\bullet_{v,w}))$. By convention, the elements of $\mathfrak{h}^* \subset R$ are assumed to have polynomial degree 2. On the other hand, let us denote by $H^{k,(p,p)}(R^\circ_{v,w})$ the (p,p) part of the mixed Hodge structure on $H^k(R^\circ_{v,w})$. See Example 2.9.

Theorem 4.1. *For all $v \le w \in W$ and $k, p \in \mathbb{Z}$, we have*

$$\dim H^{k,(p,p)}(R^\circ_{v,w}) = \dim H^{-k,(p)}(HH^0_{\mathbb{C}}(F^\bullet_{v,w})). \tag{4.1}$$

Equivariant cohomology. The spaces $HHH^0(F^\bullet_{v,w})$ and $HHH^0_{\mathbb{C}}(F^\bullet_{v,w})$ are closely related. By Theorem 4.1, $HHH^0_{\mathbb{C}}(F^\bullet_{v,w})$ yields the cohomology of $R^\circ_{v,w}$. It turns out that $HHH^0(F^\bullet_{v,w})$ yields the *torus-equivariant* cohomology of $R^\circ_{v,w}$.

The algebraic torus T acts on each Richardson variety $R^\circ_{v,w}$, and thus we can consider its *T-equivariant cohomology with compact support*, denoted $H^\bullet_{T,c}(R^\circ_{v,w})$. It is equipped with an action of the ring $H^\bullet_{T,c}(\mathrm{pt}) \cong R$. Similarly to the positroid case, $H^\bullet_{T,c}(R^\circ_{v,w})$ admits a second grading via the mixed Hodge structure and is therefore a bigraded R-module.

Theorem 4.2. *For all $v \le w \in W$, we have an isomorphism of bigraded R-modules*

$$H^\bullet_{T,c}(R^\circ_{v,w}) \cong HHH^0(F^\bullet_{v,w}). \tag{4.2}$$

It restricts to a vector space isomorphism $H^{\ell_{v,w}+2p+k,(p,p)}_{T,c}(R^\circ_{v,w}) \cong H^{k,(p)}(HH^0(F^\bullet_{v,w}))$ for each $k, p \in \mathbb{Z}$, where $\ell_{v,w} = \ell(w) - \ell(v) = \dim R^\circ_{v,w}$.

Koszul duality and q,t-symmetry. One can encode the dimensions of bigraded components of $HHH^0(F^\bullet(\beta))$, resp., $HHH^0_{\mathbb{C}}(F^\bullet(\beta))$ in a two-variable polynomial $\mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\beta; q, t)$, resp., $\mathcal{P}^{\mathrm{top}}_{\mathrm{KR};\mathbb{C}}(\beta; q, t)$.[5] For any $f \in S^{\mathrm{ncyc}=1}_{k,n}$, the positroid variety Π°_f/T is a cluster variety [7], so the polynomial $\mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\hat{\beta}_f; q, t)$ satisfies the properties (i)–(iv) listed in Theorem 2.2 by the results of [19]. In particular, it is q,t-symmetric and q,t-unimodal.

Richardson varieties are not yet known to admit cluster structures (see [20]), in particular, it does not follow from Theorem 2.2 that $\mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\hat{\beta}_{v,w}; q, t)$ is q,t-symmetric for arbitrary $v \le w \in S_n$. We show that the q,t-symmetry phenomenon for such links is a manifestation of *Koszul duality* for mixed perverse sheaves [1, 2].

Theorem 4.3 (Koszul duality)**.** *For any $v \le w \in S_n$, we have*

$$\mathcal{P}^{\mathrm{top}}_{\mathrm{KR};\mathbb{C}}(\beta_{v,w}; q, t) = \mathcal{P}^{\mathrm{top}}_{\mathrm{KR};\mathbb{C}}(\beta_{v,w}; t, q).$$

If $\hat{\beta}_{v,w}$ is a knot then it follows that $\mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\hat{\beta}_{v,w}; q, t) = \mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\hat{\beta}_{v,w}; t, q)$. This gives a new proof of the q,t-symmetry of $C_{k,n-k}(q,t)$ for $\gcd(k,n) = 1$.

5 Catalan numbers associated to positroid varieties

An important combinatorial consequence of our results is an embedding of rational q,t-Catalan numbers $C_{k,n-k}(q,t)$ into a family of q,t-polynomials $\mathcal{P}(\Pi^\circ_f/T; q, t) \in \mathbb{N}[q^{\frac{1}{2}}, t^{\frac{1}{2}}]$ indexed by permutations $f \in S^{\mathrm{ncyc}=1}_{k,n}$ (all of which are q,t-symmetric and q,t-unimodal).

[5]The polynomial $\mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\beta; q, t)$ the top a-degree coefficient of *Khovanov–Rozansky homology* [17] of $\hat{\beta}$.

Definition 5.1. For $f \in S_{k,n}^{\text{ncyc}=1}$, define the *$f$-Catalan number* $C_f \in \mathbb{Z}$ as the $q = 1$ specialization of the point count polynomial $\#(\Pi_f^\circ/T)(\mathbb{F}_q)$.

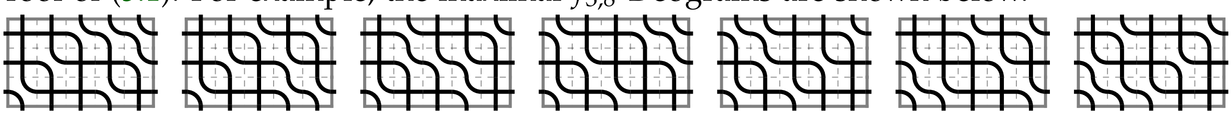
Let us focus on the case $f = f_{k,n}$ with $\gcd(k,n) = 1$. We will show that $C_{f_{k,n}} = C_{k,n-k}(1,1) = \#\text{Dyck}_{k,(n-k)}$ counts certain pipe dreams inside a $k \times (n-k)$ rectangle. This interpretation extends to arbitrary $f \in S_{k,n}$ in a straightforward fashion.

Definition 5.2. Let $\gcd(k,n) = 1$. A *maximal $f_{k,n}$-Deogram* (short for *Deodhar diagram*) is a way of placing $n-1$ elbows in a $k \times (n-k)$ rectangle and filling the rest with crossings so that (i) the resulting permutation obtained by following the paths is the identity, and (ii) the following *distinguished condition* [5] is satisfied: if any two paths have crossed an odd number of times, they cannot form an elbow.

Denote the set of maximal $f_{k,n}$-Deograms by $\text{Deo}_{f_{k,n}}^{\max}$. It follows by combining our results with [5] that $C_{f_{k,n}}$ equals the number of maximal $f_{k,n}$-Deograms:

$$C_{f_{k,n}} = \#\text{Deo}_{f_{k,n}}^{\max}. \tag{5.1}$$

An analogous result holds for arbitrary $f \in S_{k,n}$. It would be interesting to give a bijective proof of (5.1). For example, the maximal $f_{3,8}$-Deograms are shown below.



According to (5.1), these objects are in bijection with Dyck paths in Figure 1.

Problem 5.3. Find a bijection between $\text{Deo}_{f_{k,n}}^{\max}$ and $\text{Dyck}_{k,(n-k)}$ for the case $\gcd(k,n) = 1$.

For the case $n = 2k+1$ of the standard Catalan numbers, the maximal $f_{k,n}$-Deograms are easily seen (exercise) to be in bijection with *non-crossing alternating trees* on $n+1$ vertices. A recursive proof of (5.1) for the case $n = dk \pm 1$ ($d \geq 2$) was found by David Speyer. We were able to find a recursive proof of (5.1) for arbitrary k, n. The problem of finding a bijective proof remains open.

Remark 5.4. For a class of *repetition-free permutations*, a combinatorial interpretation of the numbers C_f in terms of Dyck paths avoiding a convex shape was recently given in [8]. We refer to [8, Section 7] for relations between their q,t-analogs and the results of [3, 11, 12, 24].

Acknowledgements

We are grateful to David Speyer for his contributions to this project at the early stages. We also thank Pramod Achar, Sergey Fomin, Alexei Oblomkov, and Vivek Shende for stimulating discussions.

References

[1] A. Beilinson, V. Ginzburg, and W. Soergel. "Koszul duality patterns in representation theory". *J. Amer. Math. Soc.* **9**.2 (1996), pp. 473–527. DOI.

[2] R. Bezrukavnikov and Z. Yun. "On Koszul duality for Kac-Moody groups". *Represent. Theory* **17** (2013), pp. 1–98. DOI.

[3] J. Blasiak, M. Haiman, J. Morse, A. Pun, and G. H. Seelinger. "A Shuffle Theorem for Paths Under Any Line" (2021). arXiv:2102.07931.

[4] P. Deligne. "Théorie de Hodge. II". *Inst. Hautes Études Sci. Publ. Math.* 40 (1971), pp. 5–57. Link.

[5] V. V. Deodhar. "On some geometric aspects of Bruhat orderings. I. A finer decomposition of Bruhat cells". *Invent. Math.* **79**.3 (1985), pp. 499–511. DOI.

[6] S. Fomin, P. Pylyavskyy, E. Shustin, and D. Thurston. "Morsifications and mutations" (2017). arXiv:1711.10598.

[7] P. Galashin and T. Lam. "Positroid varieties and cluster algebras" (2019). arXiv:1906.03501.

[8] P. Galashin and T. Lam. "Positroid Catalan numbers" (2021). arXiv:2104.05701.

[9] A. M. Garsia and M. Haiman. "A remarkable q,t-Catalan sequence and q-Lagrange inversion". *J. Algebraic Combin.* **5**.3 (1996), pp. 191–244. DOI.

[10] E. Gorsky. "q,t-Catalan numbers and knot homology". *Zeta functions in algebra and geometry*. Vol. 566. Contemp. Math. Amer. Math. Soc., Providence, RI, 2012, pp. 213–232. DOI.

[11] E. Gorsky, G. Hawkes, A. Schilling, and J. Rainbolt. "Generalized q,t-Catalan numbers". *Algebr. Comb.* **3**.4 (2020), pp. 855–886. DOI.

[12] E. Gorsky, A. Neguţ, and J. Rasmussen. "Flag Hilbert schemes, colored projectors and Khovanov-Rozansky homology". *Adv. Math.* **378** (2021), pp. 107542, 115. DOI.

[13] M. Haiman. "Vanishing theorems and character formulas for the Hilbert scheme of points in the plane". *Invent. Math.* **149**.2 (2002), pp. 371–407. DOI.

[14] M. D. Haiman. "Conjectures on the quotient ring by diagonal invariants". *J. Algebraic Combin.* **3**.1 (1994), pp. 17–76. DOI.

[15] V. F. R. Jones. "Hecke algebra representations of braid groups and link polynomials". *Ann. of Math. (2)* **126**.2 (1987), pp. 335–388. DOI.

[16] D. Kazhdan and G. Lusztig. "Representations of Coxeter groups and Hecke algebras". *Invent. Math.* **53**.2 (1979), pp. 165–184. DOI.

[17] M. Khovanov and L. Rozansky. "Matrix factorizations and link homology". *Fund. Math.* **199**.1 (2008), pp. 1–91. DOI.

[18] A. Knutson, T. Lam, and D. E. Speyer. "Positroid varieties: juggling and geometry". *Compos. Math.* **149**.10 (2013), pp. 1710–1752. DOI.

[19] T. Lam and D. E. Speyer. "Cohomology of cluster varieties. I. Locally acyclic case" (2016). arXiv:1604.06843.

[20] B. Leclerc. "Cluster structures on strata of flag varieties". *Adv. Math.* **300** (2016), pp. 190–228. DOI.

[21] N. A. Loehr and G. S. Warrington. "A continuous family of partition statistics equidistributed with length". *J. Combin. Theory Ser. A* **116**.2 (2009), pp. 379–403. DOI.

[22] A. Mellit. "Toric braids and (m, n)-parking functions" (2016). arXiv:1604.07456.

[23] G. Muller and D. E. Speyer. "The twist for positroid varieties". *Proc. Lond. Math. Soc. (3)* **115**.5 (2017), pp. 1014–1071. DOI.

[24] A. Oblomkov and L. Rozansky. "HOMFLYPT homology of Coxeter links" (2017). arXiv:1706.00124.

[25] A. Postnikov. "Total positivity, Grassmannians, and networks" (2006). Preprint. Link.

[26] R. Rouquier. "Categorification of the braid groups" (2004). arXiv:math/0409593.

[27] J. S. Scott. "Grassmannians and cluster algebras". *Proc. Lond. Math. Soc. (3)* **92**.2 (2006), pp. 345–380. DOI.

[28] K. Serhiyenko, M. Sherman-Bennett, and L. Williams. "Cluster structures in Schubert varieties in the Grassmannian". *Proc. Lond. Math. Soc. (3)* **119**.6 (2019), pp. 1694–1744. DOI.

[29] V. Shende, D. Treumann, H. Williams, and E. Zaslow. "Cluster varieties from Legendrian knots". *Duke Math. J.* **168**.15 (2019), pp. 2801–2871. DOI.

[30] V. Shende, D. Treumann, and E. Zaslow. "Legendrian knots and constructible sheaves". *Invent. Math.* **207**.3 (2017), pp. 1031–1133. DOI.