Séminaire Lotharingien de Combinatoire **85B** (2021)
Article #57, 12 pp.

Proceedings of the 33rd Conference on Formal Power Series and Algebraic Combinatorics (Ramat Gan)

q-Whittaker functions, finite fields, and Jordan forms

Steven N. Karp*[1] and Hugh Thomas†[1]

[1] *Laboratoire de combinatoire et d'informatique mathématique, Université du Québec à Montréal, CP 8888, Succ. Centre-ville, Montréal, QC H3C 3P8, Canada*

Abstract. The q-Whittaker function $W_\lambda(\mathbf{x};q)$ associated to a partition λ is a q-analogue of the Schur function $s_\lambda(\mathbf{x})$, and the $t=0$ specialization of the Macdonald polynomial $P_\lambda(\mathbf{x};q,t)$. We give a new formula for $W_\lambda(\mathbf{x};q)$ in terms of partial flags compatible with a nilpotent endomorphism over the finite field of size $1/q$, analogous to a well-known formula for the Hall–Littlewood functions. We show that considering pairs of partial flags and taking Jordan forms leads to a probabilistic bijection between nonnegative-integer matrices and pairs of semistandard tableaux of the same shape, which we call the *q-Burge correspondence*. In the $q \to 0$ limit, we recover a description of the classical Burge correspondence (also known as column RSK) due independently to Gansner (1981), Spaltenstein (1982), and Steinberg (1988) for permutation matrices, and to Rosso (2012) in general. Finally, we apply the q-Burge correspondence to prove enumerative formulas for certain modules over the preprojective algebra of a path quiver.

Keywords: q-Whittaker function, finite field, Jordan form, partial flag variety, Burge correspondence, RSK correspondence, preprojective algebra, socle filtration.

1 Introduction

The *Cauchy identity* for the Schur functions $s_\lambda(\mathbf{x})$

$$\prod_{i,j\geq 1} \frac{1}{1-x_iy_j} = \sum_{\lambda} s_\lambda(\mathbf{x})s_\lambda(\mathbf{y}) \tag{1.1}$$

plays an important role in the theories of symmetric functions and Schur processes. A bijective proof of (1.1) is provided by the celebrated *Robinson–Schensted–Knuth correspondence*, or its column-insertion variant, the *Burge correspondence* [4] (which will be more relevant for us). Namely, we expand the left-hand side of (1.1) as a weighted sum over nonnegative-integer matrices, and expand the right-hand side as a weighted sum over pairs of semistandard tableaux of the same shape, and these correspondences give a

*S.N.K. was supported by an NSERC postdoctoral fellowship.
†H.T. was partially supported by an NSERC Discovery Grant and the Canada Research Chairs program.

weight-preserving bijection between such matrices and pairs of tableaux. For example, the coefficient of $x_1x_2y_1y_2$ on each side of (1.1) is 2, and the Burge correspondence acts by

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\mapsto\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array},\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right),\qquad\begin{bmatrix}0&1\\1&0\end{bmatrix}\mapsto\left(\begin{array}{|c|c|}\hline 1&2\\\hline\end{array},\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\right). \tag{1.2}$$

The goal of this paper is to give a combinatorial and algebraic understanding of the following q-analogue of (1.1) [10, VI.(4.13)]:

$$\prod_{\substack{i,j\geq 1,\\ d\geq 0}}\frac{1}{1-x_iy_jq^d}=\sum_{\lambda}\frac{(1-q)^{-\lambda_1}}{\prod_{i\geq 1}[\lambda_i-\lambda_{i+1}]_q!}W_\lambda(\mathbf{x};q)W_\lambda(\mathbf{y};q). \tag{1.3}$$

Here the Schur function $s_\lambda(\mathbf{x})$ has been replaced by the *q-Whittaker function* $W_\lambda(\mathbf{x};q)$, which is the $t=0$ specialization of the Macdonald polynomial $P_\lambda(\mathbf{x};q,t)$; see Section 2 for a precise definition and other notation. We can expand the left-hand side of (1.3) as a q-weighted sum over nonnegative-integer matrices, and expand the right-hand side as a q-weighted sum over pairs of semistandard tableaux of the same shape. For example, extracting the coefficients of $x_1x_2y_1y_2$ in (1.3) and rescaling by $(1-q)^2$, we obtain

$$\begin{array}{ccccccc} 1 & + & 1 & = & (1-q) & + & (1+q). \\ \begin{bmatrix}1&0\\0&1\end{bmatrix} & & \begin{bmatrix}0&1\\1&0\end{bmatrix} & & \left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array},\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right) & & \left(\begin{array}{|c|c|}\hline 1&2\\\hline\end{array},\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\right)\end{array}$$

We see that for $q\neq 0$, there is no q-weight preserving bijection between the two matrices labeling the terms of the left-hand side and the two pairs of tableaux labeling the terms of the right-hand side. Nevertheless, we can define a *probabilistic bijection*, for example one which acts by

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\mapsto\begin{cases}\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array},\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right), & \text{with probability } 1-q;\\ \left(\begin{array}{|c|c|}\hline 1&2\\\hline\end{array},\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\right), & \text{with probability } q,\end{cases}\qquad\begin{bmatrix}0&1\\1&0\end{bmatrix}\mapsto\left(\begin{array}{|c|c|}\hline 1&2\\\hline\end{array},\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\right). \tag{1.4}$$

Such a probabilistic bijection proving (1.3) was given by Matveev and Petrov [11] via a q-row insertion algorithm, building on work of Borodin and Petrov [3]; we refer to [1, Section 1] for further discussion and references. We prove (1.3) by introducing a new probabilistic bijection which we call the *q-Burge correspondence*, which is defined when $1/q$ is a prime power via the geometry of nilpotent endomorphisms over the finite field $\mathbb{F}_{1/q}$. In the limit $q\to 0$, when we replace $\mathbb{F}_{1/q}$ by an infinite field, we recover a description of the classical Burge correspondence [4], due independently to Gansner [7], Spaltenstein [15], and Steinberg [17, 18] in the case of permutation matrices, and due to Rosso [14] in general.

In order to state our result, we introduce some terminology. If N is a nilpotent endomorphism of $\Bbbk^n$, we say that a partial flag of subspaces

$$F : 0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k = \Bbbk^n$$

in $\Bbbk^n$ is *strictly compatible* with N if

$$N(F_i) \subseteq F_{i-1} \quad \text{for } 1 \leq i \leq k. \tag{1.5}$$

Restricting N to $F_0, \ldots, F_k$ gives a sequence of nilpotent endomorphisms whose (conjugated) Jordan form types define a semistandard tableau $\mathsf{JF}^\top(N; F)$ of size n on the alphabet $\{1, \ldots, k\}$. If $F = (F_0, \ldots, F_k)$ and $F' = (F'_0, \ldots, F'_l)$ are two partial flags in $\Bbbk^n$, we define the *relative position of* (F, F') as the $k \times l$ nonnegative-integer matrix M satisfying

$$\dim(F_i \cap F'_j) = \sum_{1 \leq i' \leq i,\, 1 \leq j' \leq j} M_{i',j'} \quad \text{for all } 0 \leq i \leq k,\ 0 \leq j \leq l.$$

Theorem 1.6. *Given a $k \times l$ nonnegative-integer matrix M whose entries sum to n, let (F, F') be a pair of partial flags in $\mathbb{F}_{1/q}^n$ with relative position M. If T and T' are semistandard tableaux of size n of the same shape, we define $\mathsf{p}_M(T, T')$ as the probability that a uniformly random nilpotent endomorphism N of $\Bbbk^n$ which is strictly compatible with both F and F' satisfies*

$$\mathsf{JF}^\top(N; F) = T \quad \text{and} \quad \mathsf{JF}^\top(N; F') = T'.$$

(This definition does not depend on the choice of (F, F').)

(i) The map p defines a probabilistic bijection proving (1.3) whenever $1/q$ is a prime power.[1]

(ii) Transposing the matrix corresponds to swapping the tableaux: $\mathsf{p}_M(T, T') = \mathsf{p}_{M^\top}(T', T)$.

(iii) As $q \to 0$, the probability $\mathsf{p}_M(T, T')$ converges to 1 if the Burge correspondence sends M to (T, T'), and converges to 0 otherwise.

Example 1.7. Let us show that p as defined in Theorem 1.6 leads to (1.4). For $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we can take $F = F'$ to be the partial flag

$$0 = F_0 \subseteq F_1 = \langle e_1 \rangle \subseteq F_2 = \langle e_1, e_2 \rangle.$$

Then the nilpotent endomorphisms of $\mathbb{F}_{1/q}^2$ which are strictly compatible with $F = F'$ are precisely of the form

$$N = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \quad (a \in \mathbb{F}_{1/q}).$$

[1]This establishes (1.3) for all q, since it is sufficient to verify equality at infinitely many values of q.

If $a \neq 0$ (which occurs with probability $1-q$), then we obtain the pair of tableaux $\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right)$, while if $a = 0$ (which occurs with probability q), we obtain the pair of tableaux $\left(\begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}, \begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}\right)$.

On the other hand, for $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we can take F and F' to be

$$0 = F_0 \subseteq F_1 = \langle e_1 \rangle \subseteq F_2 = \langle e_1, e_2 \rangle \quad \text{and} \quad 0 = F_0' \subseteq F_1' = \langle e_2 \rangle \subseteq F_2' = \langle e_1, e_2 \rangle.$$

Then the only nilpotent endomorphism of $\mathbb{F}_{1/q}^2$ which is strictly compatible with both F and F' is the zero endomorphism, and so (with probability 1) we obtain the pair of tableaux $\left(\begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}, \begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}\right)$. In summary, we have

$$\mathsf{p}_{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right) = 1-q, \quad \mathsf{p}_{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}\left(\begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}, \begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}\right) = q, \quad \mathsf{p}_{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}\left(\begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}, \begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}\right) = 1. \quad \diamondsuit$$

In general, as in Example 1.7, given M we can always choose the pair of partial flags (F, F') in Theorem 1.6 so that the nilpotent endomorphisms N compatible with both F and F' have the following form: each entry is either zero or an arbitrary element of $\mathbb{F}_{1/q}$.

An important open problem is to determine whether $\mathsf{p}_M(T, T')$ defines a polynomial in q, and if so, to find a combinatorial interpretation of it. We remark that the combinatorics of p is different from the q-column insertion of O'Connell and Pei [13] and the q-row insertion of Borodin and Petrov [3] and Matveev and Petrov [11].

Along the way to proving Theorem 1.6, we find a new formula for $W_\lambda(\mathbf{x}; q)$:

Theorem 1.8. *Let λ be a partition of n, and fix a nilpotent endomorphism N of $\mathbb{F}_{1/q}^n$ whose (conjugated) Jordan form type is λ. Then for any weak composition $(\alpha_1, \dots, \alpha_k)$ of n, the coefficient of $x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ in $W_\lambda(\mathbf{x}; q)$ equals $q^{\sum_{i \ge 1}\binom{\lambda_i}{2} - \sum_{i=1}^k \binom{\alpha_i}{2}}$ times the number of partial flags*

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k = \mathbb{F}_{1/q}^n, \quad \dim(F_i) = \alpha_1 + \cdots + \alpha_i,$$

which are strictly compatible with N.

Example 1.9. Let us use Theorem 1.8 to find the coefficient of $x_1^2 x_2^2$ in $W_{(3,1)}(\mathbf{x}; q)$, when $\lambda := (3,1)$ and $\alpha := (2,2)$. We may take N to be the nilpotent endomorphism of $\mathbb{F}_{1/q}^4$ given by

$$N := \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We must enumerate partial flags

$$0 = F_0 \subseteq F_1 \subseteq F_2 = \mathbb{F}_{1/q}^4, \quad \dim(F_1) = 2,$$

which are strictly compatible with N. The condition $N(F_2) \subseteq F_1$ means that F_1 contains e_1, and the condition $N(F_1) \subseteq F_0$ means that $F_1 \subseteq \langle e_1, e_2, e_3 \rangle$. Therefore F_1 is the direct sum of $\langle e_1 \rangle$ and an arbitrary 1-dimensional subspace of $\langle e_2, e_3 \rangle$, of which there are $\frac{1}{q} + 1$ in total. Therefore the coefficient of $x_1^2 x_2^2$ in $W_{(3,1)}(\mathbf{x}; q)$ equals

$$q^{\binom{3}{2}+\binom{1}{2}-\binom{2}{2}-\binom{2}{2}}\left(\tfrac{1}{q}+1\right) = 1 + q. \qquad \Diamond$$

We note that since $W_\lambda(\mathbf{x}; q)$ is a symmetric function, the enumeration in Theorem 1.8 does not depend on the order of the parts of the composition α, but this is far from obvious. We also remark that there is a well-known formula for the *Hall–Littlewood functions* which is similar to Theorem 1.8 (see, e.g., [10, Chapter II] or [12, Corollary 2.13]), where we instead enumerate partial flags which are *weakly* compatible with N (i.e. we only require $N(F_i) \subseteq F_i$ in (1.5)). However, we do not know how to deduce one formula from the other.

We outline the proofs of our results, which will appear in the full version of this paper. We approach Theorem 1.8 by proving a refined version, showing that the contribution of a semistandard tableau T to $W_\lambda(\mathbf{x}; q)$ is given by enumerating partial flags F satisfying $\mathsf{JF}^\top(N; F) = T$. We proceed by induction on k, which reduces the proof to enumerating subspaces F_{k-1} of $\mathbb{F}_{1/q}$ which contain $N(\mathbb{F}_{1/q})$ and satisfy certain rank conditions; we are able to explicitly carry out this calculation. For part (i) of Theorem 1.6, we must show that in (1.3), the contribution to the right-hand side from a fixed pair of tableaux (T, T') of the same shape λ equals the contribution to the left-hand side from all nonnegative-integer matrices M, each weighted by $\mathsf{p}_M(T, T')$. To do so, we show that each of these contributions equals, up to an explicit scalar factor, the number of triples (F, F', N), such that N is a nilpotent endomorphism of $\mathbb{F}_{1/q}^n$ which is strictly compatible with both F and F' and satisfies $\mathsf{JF}^\top(N; F) = T$ and $\mathsf{JF}^\top(N; F') = T'$. We enumerate such triples in two ways. For the left-hand side, we first enumerate all F, then all F' such that (F, F') has given relative position M, and finally all N satisfying the conditions above (which involves $\mathsf{p}_M(T, T')$). For the right-hand side, we first enumerate all N with $\mathsf{JF}^\top(N) = \lambda$ (which is well-known), and then, using the refined version of Theorem 1.8, separately enumerate all F and all F' satisfying the conditions above. Part (ii) of Theorem 1.6 follows from the definitions, and we deduce part (iii) using the work of Rosso [14].

The remainder of this abstract is organized as follows. In Section 2, we define the q-Whittaker functions $W_\lambda(\mathbf{x}; q)$ and introduce other notation. In Section 3, we review the classical Burge correspondence, which is the $q = 0$ specialization of our q-Burge correspondence. Finally, in Section 4, we apply the q-Burge correspondence to prove enumerative results about modules over the preprojective algebra of a path quiver. The idea is that each triple (F, F', N) defines such a module, where M records its isomorphism type over the path algebra and the pair (T, T') records its socle filtration.

2 q-Whittaker functions

In this section we define some notation used in Section 1. We refer to [2] for background on q-Whittaker functions, and to [10, 16] for background on symmetric functions.

Definition 2.1. Given nonnegative integers $n \geq k$, we define

$$[n]_q := 1 + q + \cdots + q^{n-1}, \quad [n]_q! := [n]_q[n-1]_q \cdots [1]_q, \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Definition 2.2 (cf. [10, VI.(7.13′)])**.** Let λ be a partition of n. A *semistandard tableau* T is a filling of the boxes of λ with positive integers which is weakly increasing along rows and strictly increasing along columns. For $i \geq 1$ and $j \geq 0$, we let $T^{(j)}$ denote the partition formed by the entries $1, \dots, j$ of T, and let $T_i^{(j)}$ denote the ith part of $T^{(j)}$. We define the *q-weight of T* as

$$\mathrm{wt_q}(T) := \prod_{i,j \geq 1} \begin{bmatrix} T_i^{(j)} - T_{i+1}^{(j)} \\ T_i^{(j)} - T_i^{(j-1)} \end{bmatrix}_q \in \mathbb{N}[q].$$

Given indeterminates $\mathbf{x} = (x_1, x_2, \dots)$, we define the *$q$-Whittaker function*

$$W_\lambda(\mathbf{x}; q) := \sum_T \mathrm{wt_q}(T)\mathbf{x}^T,$$

where the sum is over all semistandard tableaux T of shape λ, and $\mathbf{x}^T := \prod_{i \geq 1} x_i^{\#i\text{'s in } T}$.

Example 2.3. Let $\lambda := (2,2)$. Then $W_\lambda(\mathbf{x}; q)$ equals

$$\sum_{1 \leq i_1 < i_2} \underset{\begin{array}{|c|c|}\hline i_1 & i_1 \\ \hline i_2 & i_2 \\ \hline\end{array}}{x_{i_1}^2 x_{i_2}^2} + \sum_{1 \leq i_1 < i_2 < i_3} \Big(\underset{\begin{array}{|c|c|}\hline i_1 & i_1 \\ \hline i_2 & i_3 \\ \hline\end{array}}{(1+q)x_{i_1}^2 x_{i_2} x_{i_3}} + \underset{\begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline i_2 & i_3 \\ \hline\end{array}}{(1+q)x_{i_1} x_{i_2}^2 x_{i_3}} + \underset{\begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline i_3 & i_3 \\ \hline\end{array}}{(1+q)x_{i_1} x_{i_2} x_{i_3}^2} \Big)$$

$$+ \sum_{1 \leq i_1 < i_2 < i_3 < i_4} \Big(\underset{\begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline i_3 & i_4 \\ \hline\end{array}}{(1+q)^2 x_{i_1} x_{i_2} x_{i_3} x_{i_4}} + \underset{\begin{array}{|c|c|}\hline i_1 & i_3 \\ \hline i_2 & i_4 \\ \hline\end{array}}{(1+q)x_{i_1} x_{i_2} x_{i_3} x_{i_4}} \Big). \qquad \diamond$$

Remark 2.4. Given a partition λ, let $P_\lambda(\mathbf{x}; q, t)$ and $\widetilde{H}_\lambda(\mathbf{x}; q, t)$ denote the *Macdonald polynomial* and *modified Macdonald polynomial*, respectively, and let ω denote the standard involution on symmetric functions. Then we have [2, Section 3]

$$W_\lambda(\mathbf{x}; q) = P_\lambda(\mathbf{x}; q, 0) = q^{\deg(\widetilde{H}_\lambda)} \omega(\widetilde{H}_\lambda(\mathbf{x}; 1/q, 0)).$$

In particular, we have the following specializations of $W_\lambda(\mathbf{x}; q)$ [10, Section VI.4]:

$$W_\lambda(\mathbf{x}; 0) = s_\lambda(\mathbf{x}), \quad W_\lambda(\mathbf{x}; 1) = e_{\lambda^\top}(\mathbf{x}).$$

Definition 2.5. Let N be an endomorphism of the n-dimensional vector space V over $\Bbbk$. We say that N is *nilpotent* if some power of N is zero. If so, we can choose a basis of V so that N is represented by the $n \times n$ block-diagonal matrix

$$\begin{bmatrix} J_{\mu_1} & 0 & \cdots & 0 \\ 0 & J_{\mu_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{\mu_k} \end{bmatrix}, \text{ where } \mu_1 \geq \cdots \geq \mu_k \text{ and } J_l \text{ is the } l \times l \text{ matrix } \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Let $\lambda := \mu^\top$ denote the conjugate partition of μ; we call λ the *(conjugated) Jordan form partition* of N, denoted $\mathsf{JF}^\top(N)$. Equivalently, we have

$$\dim(\ker(N^i)) = \lambda_1 + \cdots + \lambda_i \quad \text{for all } i \geq 0.$$

For example, if N is the zero endomorphism of $\Bbbk^n$, then $\mathsf{JF}^\top(N) = (n)$.

Lemma 2.6. *Let N be a nilpotent endomorphism of $\Bbbk^n$ which is strictly compatible with the partial flag $F = (F_0, \ldots, F_k)$ (see (1.5)). Let T be the tableau of shape $\mathsf{JF}^\top(N)$ such that $T^{(j)} = \mathsf{JF}^\top(N|_{F_j})$ for $0 \leq j \leq k$, where $N|_{F_j}$ denotes the endomorphism N restricted to F_j. Then T is a semistandard tableau, which we denote by $\mathsf{JF}^\top(N; F)$.*

3 Burge correspondence

In this section we review the Burge correspondence [4], following [6, Appendix A].

Definition 3.1 ([6, Appendix A])**.** Let T be a semistandard tableau, and let $i \in \mathbb{Z}_{>0}$. Then the *column insertion of i into T* is the semistandard tableau obtained from T by performing the following steps. If i is greater than all the entries of T in column 1, we place i at the end of column 1. Otherwise, we replace the smallest entry $j \geq i$ in column 1 with i, and insert j into column 2 by the same procedure, and so on.

Given a $k \times l$ matrix M of nonnegative integers, we define semistandard tableaux $\mathsf{P}(M)$ and $\mathsf{Q}(M)$ of the same shape by the following procedure, starting with $\mathsf{P}(M)$ and $\mathsf{Q}(M)$ as the empty tableau. For $j = 1, \ldots, l$ and $i = k, \ldots, 1$ (i.e. we read M column by column, left to right, and within each column from bottom to top), we column-insert i into $\mathsf{P}(M)$ and record j in the new box of $\mathsf{Q}(M)$ a total of $M_{i,j}$ times, so that at each step $\mathsf{P}(M)$ and $\mathsf{Q}(M)$ have the same shape. The map $M \mapsto (\mathsf{P}(M), \mathsf{Q}(M))$ is called the *Burge correspondence*.

Example 3.2. Let $M := \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Then $\mathsf{P}(M)$ is obtained by column-inserting 2, 2, 1, 3, 2, 3, 2, and 1 into the empty tableau, and $\mathsf{Q}(M)$ is obtained by recording 1, 1, 1, 2, 2, 3, 3, and 3. We get

$$\mathsf{P}(M) = \begin{ytableau} 1 & 1 & 2 & 2 & 2 \\ 2 & 3 \\ 3 \end{ytableau} \quad \text{and} \quad \mathsf{Q}(M) = \begin{ytableau} 1 & 1 & 1 & 3 & 3 \\ 2 & 2 \\ 3 \end{ytableau}. \qquad \diamondsuit$$

Example 3.3. In the case of 2×2 permutation matrices, the Burge correspondence acts by (1.2). This is consistent with Theorem 1.6(iii) and the calculation in Example 1.7. $\diamondsuit$

Remark 3.4. The *Robinson–Schensted–Knuth (RSK) correspondence* is a map defined similarly to the Burge correspondence, where we use *row insertion* instead of column insertion, and read the columns of the given matrix M from top to bottom rather than bottom to top. The relationship between the RSK and Burge correspondences is explained in [6, Section A.4.2]. Namely, let (T, T') be the pair of tableaux corresponding to M under the RSK correspondence. If M is a permutation matrix, then

$$T = \mathsf{P}(M)^\top \quad \text{and} \quad T' = \mathsf{Q}(M)^\top.$$

In general, let M_{rev} (respectively, M^{rev}) denote the matrix obtained from M by reversing the order of its rows (respectively, columns). Then

$$T = \mathsf{P}(M^{\text{rev}}) = \mathsf{evac}(\mathsf{P}(M_{\text{rev}})) \quad \text{and} \quad T' = \mathsf{Q}(M_{\text{rev}}) = \mathsf{evac}(\mathsf{Q}(M^{\text{rev}})),$$

where evac is the *evacuation* map (also known as the *Schützenberger involution*).

4 The preprojective algebra and socle filtrations

In this section, we apply the q-Burge correspondence to the enumeration of quiver representations. In order to keep the exposition brief, we will not define all the necessary terms. We refer to [5, 9] for background.

Let $Q(k, l)$ denote the path quiver on $k + l - 1$ vertices with a unique sink, with $k - 1$ vertices to the right and $l - 1$ vertices to the left:

$$\begin{tikzcd} \underset{-l+1}{\bullet} \arrow[r] & \underset{-l+2}{\bullet} \arrow[r] & \underset{-l+3}{\bullet} & \cdots & \underset{-1}{\bullet} \arrow[r] & \underset{0}{\bullet} & \underset{1}{\bullet} \arrow[l] & \cdots & \underset{k-3}{\bullet} & \underset{k-2}{\bullet} \arrow[l] & \underset{k-1}{\bullet} \arrow[l] \end{tikzcd}.$$

Every finite-dimensional representation V of $Q(k, l)$ can be expressed (up to isomorphism) as a direct sum of indecomposable representations, which are indexed by the

$\binom{k+l}{2}$ subpaths of $Q(k,l)$. We will restrict our attention to representations V in which the subpath of every indecomposable summand is supported at the sink, or equivalently, representations in which all maps involved are injective. Isomorphism classes of such V are labeled by $k \times l$ matrices M of nonnegative integers, where the index (i,j) corresponds to the indecomposable representation of the path between vertices $k-i$ and $j-l$. Alternatively, we may assume that the vector space over vertex 0 is $\Bbbk^n$, and that all maps involved are identity maps; then the vector spaces at each vertex form two partial flags F and F' inside $\Bbbk^n$, and M records the relative position of (F, F'). We note that the dimension vector of V can be read off from the row and column sums of M.

We now consider ways of extending V to a module $V^\sharp$ over the *preprojective algebra* $\Pi(Q(k,l))$ of $Q(k,l)$. This amounts to additionally associating a map to the reverse of every arrow of V so that at every vertex i of $Q(k,l)$, the compositions of the maps associated to the two paths of length two from i to itself are equal up to a fixed sign. It turns out that $V^\sharp$ is given up to isomorphism by a triple (F, F', N) (in general not uniquely), where (F, F') is a pair of partial flags with relative position M, and N is a nilpotent endomorphism of $\Bbbk^n$ which is strictly compatible with both F and F':

$$\begin{tikzcd}
F'_1 \arrow[r, bend left, "\mathrm{id}"] & F'_2 \arrow[l, bend left, "-N"] \arrow[r, bend left, "\mathrm{id}"] & F'_3 \arrow[l, bend left, "-N"] & \cdots & F'_{l-1} \arrow[r, bend left, "\mathrm{id}"] & \Bbbk^n \arrow[l, bend left, "-N"] \arrow[r, bend left, "N"] & F_{k-1} \arrow[l, bend left, "\mathrm{id}"] & \cdots & F_3 \arrow[r, bend left, "N"] & F_2 \arrow[l, bend left, "\mathrm{id}"] \arrow[r, bend left, "N"] & F_1 \arrow[l, bend left, "\mathrm{id}"]
\end{tikzcd}.$$

Provided $k + l \geq 7$, it is an intractable (technically, *wild*) problem to classify such $V^\sharp$ up to isomorphism, but we can associate an interesting combinatorial invariant by taking the *socle filtration*. This records, for every vertex and height $j \geq 1$, the dimension of the subspace at the vertex which is annihilated after applying any j arrows of $V^\sharp$, modulo the subspace annihilated after applying any $j-1$ arrows. After deleting trivial zero entries, we obtain a *reverse plane partition* R on a $k \times l$ rectangle. By splitting apart the rectangle along its diagonal we obtain two *Gelfand–Tsetlin patterns* (as in [8, Section 1]), which we can translate into a pair (T, T') of semistandard tableaux of the same shape. It turns out that

$$\mathsf{JF}^\top(N; F) = T \quad \text{and} \quad \mathsf{JF}^\top(N; F') = T'.$$

Then we have the following analogue of Theorem 1.8 in this setting (where, as in the preceding discussion, we assume all maps of $V^\sharp$ directed towards the sink are injective):

Theorem 4.1. *Set $\Bbbk := \mathbb{F}_{1/q}$. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_l)$ be weak compositions of n, and set $c := n + \sum_{i=1}^k \binom{\alpha_i}{2} + \sum_{j=1}^l \binom{\beta_j}{2}$. Let M be a $k \times l$ matrix of nonnegative integers with row sums α and column sums β, and let R be a reverse plane partition corresponding to the pair of semistandard tableaux (T, T').*

(i) We have

$$\sum_{V^\sharp} \frac{1}{|\mathrm{Aut}(V^\sharp)|} = \frac{q^c (1-q)^{-n}}{\prod\limits_{1 \leq i \leq k,\, 1 \leq j \leq l} [M_{i,j}]_q!} \mathsf{p}_M(T, T'),$$

where the sum is over all $\Pi(Q(k,l))$-modules $V^\sharp$ up to isomorphism which are indexed by M as a representation of $Q(k,l)$ and have socle filtration R.

(ii) *We have*

$$\sum_{V^\sharp} \frac{1}{|\mathrm{Aut}(V^\sharp)|} = \frac{q^c(1-q)^{-\lambda_1}}{\prod_{i\geq 1}[\lambda_i - \lambda_{i+1}]_q!}\,\mathrm{wt}_{\mathrm{q}}(T)\,\mathrm{wt}_{\mathrm{q}}(T'),$$

where the sum is over all $\Pi(Q(k,l))$-modules $V^\sharp$ up to isomorphism which have fixed dimension vector corresponding to (α,β) and have socle filtration R.

(iii) *We have*

$$\sum_{V^\sharp} \frac{1}{|\mathrm{Aut}(V^\sharp)|} = q^c \cdot (\textit{coefficient of } x_1^{\alpha_1}\cdots x_k^{\alpha_k} y_1^{\beta_1}\cdots y_l^{\beta_l} \textit{ in } (1.3)),$$

where the sum is over all $\Pi(Q(k,l))$-modules $V^\sharp$ up to isomorphism which have fixed dimension vector corresponding to (α,β).

Example 4.2. We illustrate Theorem 4.1(i) for $M := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which has row sums $\alpha := (1,1)$ and column sums $\beta := (1,1)$. We have $c = 2$. Every $\Pi(Q(k,l))$-module $V^\sharp$ which is indexed by M as a representation of $Q(k,l)$ is, up to isomorphism, of the form

$$V^\sharp = \begin{tikzcd} \langle e_1\rangle \arrow[r, bend left, "\mathrm{id}"] & \langle e_1, e_2\rangle \arrow[l, bend left, "-N"] \arrow[r, bend right, "N"'] & \langle e_1\rangle \arrow[l, bend right, "\mathrm{id}"'] \end{tikzcd}, \quad \text{where} \quad N = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \quad (a \in \mathbb{F}_{1/q}).$$

It turns out that there are two isomorphism classes, corresponding to $a \neq 0$ and $a = 0$.

First we consider the case $a \neq 0$; we may assume that $a = 1$. The automorphism group of $V^\sharp$ (which we may regard as the set of $g \in \mathrm{GL}_2$ which fix the vector spaces at each vertex and satisfy $gNg^{-1} = N$) is

$$\left\{ \begin{bmatrix} g_{1,1} & g_{1,2} \\ 0 & g_{2,2} \end{bmatrix} : g_{1,1} = g_{2,2} \neq 0 \right\}, \quad \text{with size } \frac{1-q}{q^2}.$$

The socle filtration of $V^\sharp$ is $\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}$; splitting it apart gives the pair of Gelfand–Tsetlin patterns

$$\left(\begin{array}{cc} 1 & \\ & 1 \\ 1 & \end{array}, \begin{array}{cc} & 1 \\ 1 & \\ & 1 \end{array} \right), \text{ corresponding to the pair of tableaux } \left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \right).$$

Therefore Theorem 4.1(i) states that

$$\frac{q^2}{1-q} = q^2(1-q)^{-2}\mathsf{p}_M\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}, \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \right),$$

which is consistent with the calculation $\mathsf{p}_M\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}, \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right) = 1 - q$ in Example 1.7.

Now we consider the case $a = 0$. The automorphism group of $V^\sharp$ is

$$\left\{ \begin{bmatrix} g_{1,1} & g_{1,2} \\ 0 & g_{2,2} \end{bmatrix} : g_{1,1}, g_{2,2} \neq 0 \right\}, \qquad \text{with size } \frac{(1-q)^2}{q^3}.$$

The socle filtration of $V^\sharp$ is $\begin{smallmatrix} & 0 & \\ 1 & & 1 \\ & 2 & \end{smallmatrix}$; splitting it apart gives the pair of Gelfand–Tsetlin patterns

$$\left(\begin{smallmatrix} 0 & \\ & 1 \\ 2 & \end{smallmatrix}, \begin{smallmatrix} & 0 \\ 1 & \\ & 2 \end{smallmatrix} \right), \text{ corresponding to the pair of tableaux } \left(\begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}, \begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}\right).$$

Therefore Theorem 4.1(i) states that

$$\frac{q^3}{(1-q)^2} = q^2(1-q)^{-2}\mathsf{p}_M\left(\begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}, \begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}\right),$$

which is consistent with the calculation $\mathsf{p}_M\left(\begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}, \begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array}\right) = q$ in Example 1.7. $\diamondsuit$

Acknowledgments

This work arose as part of a working group at LaCIM during the academic year 2019–2020. We thank the other participants of the working group, namely Florian Aigner, François Bergeron, Benjamin Dequêne, Gabriel Frieden, Christophe Reutenauer, and Franco Saliola, for many enlightening discussions.

References

[1] F. Aigner and G. Frieden. "*q*RS*t*: A probabilistic Robinson–Schensted correspondence for Macdonald polynomials". arXiv:2009.03526.

[2] F. Bergeron. "A Survey of *q*-Whittaker polynomials". arXiv:2006.12591.

[3] A. Borodin and L. Petrov. "Nearest neighbor Markov dynamics on Macdonald processes". *Adv. Math.* **300** (2016), pp. 71–155. DOI.

[4] W. H. Burge. "Four correspondences between graphs and generalized Young tableaux". *J. Combinatorial Theory Ser. A* **17** (1974), pp. 12–30. DOI.

[5] H. Derksen and J. Weyman. *An introduction to quiver representations*. Vol. 184. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2017, pp. x+334.

[6] W. Fulton. *Young tableaux*. Vol. 35. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1997, pp. x+260.

[7] E. R. Gansner. "Acyclic digraphs, Young tableaux and nilpotent matrices". *SIAM J. Algebraic Discrete Methods* **2**.4 (1981), pp. 429–440. DOI.

[8] S. Hopkins. "RSK via local transformations" (2014). Link.

[9] A. Kirillov Jr. *Quiver representations and quiver varieties*. Vol. 174. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2016, pp. xii+295.

[10] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Second edition. Oxford Mathematical Monographs. With contributions by A. Zelevinsky. The Clarendon Press, Oxford University Press, New York, 1995, pp. x+475.

[11] K. Matveev and L. Petrov. "q-randomized Robinson–Schensted–Knuth correspondences and random polymers". *Ann. Inst. Henri Poincaré D* **4**.1 (2017), pp. 1–123. DOI.

[12] A. Mellit. "Poincaré polynomials of character varieties, Macdonald polynomials and affine Springer fibers". *Ann. of Math. (2)* **192**.1 (2020), pp. 165–228. DOI.

[13] N. O'Connell and Y. Pei. "A q-weighted version of the Robinson-Schensted algorithm". *Electron. J. Probab.* **18** (2013), no. 95, 25. DOI.

[14] D. Rosso. "Classic and mirabolic Robinson–Schensted–Knuth correspondence for partial flags". *Canad. J. Math.* **64**.5 (2012), pp. 1090–1121. DOI.

[15] N. Spaltenstein. *Classes unipotentes et sous-groupes de Borel*. Vol. 946. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1982, pp. ix+259.

[16] R. P. Stanley. *Enumerative combinatorics. Vol. 2*. Vol. 62. Cambridge Studies in Advanced Mathematics. With an appendix by Sergey Fomin. Cambridge University Press, Cambridge, 1999, pp. xii+581. DOI.

[17] R. Steinberg. "On the desingularization of the unipotent variety". *Invent. Math.* **36** (1976), pp. 209–224. DOI.

[18] R. Steinberg. "An occurrence of the Robinson–Schensted correspondence". *J. Algebra* **113**.2 (1988), pp. 523–528. DOI.