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Hurwitz numbers for reflection groups

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Abstract. We give formulas for the number of transitive reflection factorizations of a parabolic quasi-Coxeter element in a Weyl group or complex reflection group, generalizing the Hurwitz formulas for the symmetric group.

Keywords: transitive factorizations, quasi-Coxeter elements, Weyl groups

1 Introduction

In the late 19th century, Hurwitz (an early disciple of Riemann surface theory) was the first to recognize that the structure of Riemann surfaces with finitely many branch points is intrinsically combinatorial. In particular, he showed in [12] that they are determined via (classes of) factorizations of the identity in the symmetric group $\mathfrak{S}_n$. In the same work [12, § 7], he gave a complete analysis for the case of genus-0 surfaces with all but one branch points being simple. In combinatorial terms, these correspond to minimum-length *transitive* factorizations $\tau_1 \cdots \tau_k = \sigma$ of a given element σ in $\mathfrak{S}_n$ in transpositions τ_i, with transitivity referring to the natural action of the group $\langle \tau_i \rangle_{i=1}^k$ on the set $[n] := \{1, \ldots, n\}$. If $\lambda = (\lambda_1, \cdots, \lambda_c)$ is the cycle structure of σ, in which case it is easy to see that $k = n + c - 2$, Hurwitz's celebrated formula for the number $H_0(\lambda)$ of such factorizations has a beautiful product structure

$$H_0(\lambda) = (n + c - 2)! \cdot n^{c-3} \cdot \prod_{i=1}^{c} \frac{\lambda_i^{\lambda_i}}{(\lambda_i - 1)!}. \tag{1.1}$$

In particular, for factorizations of the identity, this gives $H_0(1^n) = (2n-2)! \cdot n^{n-3}$. These *(single) Hurwitz numbers of genus* 0 also count certain connected planar graphs embedded in the sphere (*planar maps*; see, e.g., [16]).

In the 1980s, work of Stanley [19] and Jackson [13] rekindled the interest in the enumeration of factorizations in $\mathfrak{S}_n$ even if, at the time, they were unaware of the topological

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context. In a different vein, the next few decades saw the emergence of Coxeter combinatorics; one of its main breakthroughs was the realization that theorems about $\mathfrak{S}_n$ are often shadows of more general results that hold for all reflection groups W. In our context of factorizations, this means replacing transpositions with reflections $\tau_i \in W$.

The intersection of these two areas has witnessed a lot of research activity recently [4, 5, 17], especially for factorizations of Coxeter elements $c \in W$, generalizing the long cycle case $\lambda = (n)$. In general, however, analogs of (1.1) have been hard to find, not least because it is unclear how to define transitivity in reflection groups. In $\mathfrak{S}_n$, transitivity corresponds to the connectedness of the map or Riemann surface, both of which have no analogs for W.

An equivalent way to interpret this notion of transitivity is to require that the factorization cannot be realized in any *proper* Young subgroup of $\mathfrak{S}_n$. This approach is valid for arbitrary reflection groups W, where we will thus say that $\tau_1 \cdots \tau_k = g$ is a *transitive reflection factorization* of an element $g \in W$, if the τ_i are reflections and they generate the whole group W. We are particularly interested in the "genus-0" case, where the number k of terms is minimum for the element g. We call this number k the *transitive reflection length* of g and denote it by $\ell_R^{\text{tr}}(g)$ (leaving the symbol $\ell_R(g)$ to stand for the usual reflection length of g, which does not require transitivity of the factorization). We write $F_W^{\text{tr}}(g)$ for the number of such *minimum length* transitive reflection factorizations of g.

In this work we establish (essentially) uniform formulas for these counts $F_W^{\text{tr}}(g)$, which then might be called *W-Hurwitz numbers*. They generalize the previously mentioned Hurwitz formulas to any Weyl group W (see also § 4 for the complex case) and any parabolic quasi-Coxeter element $g \in W$ (this is a wide class of elements which for example contains all $g \in \mathfrak{S}_n$). Partial results for the "higher genus" case appear in § 5.

There are many combinatorial approaches to the proof and interpretation of the Hurwitz formulas (e.g., [3, 11]). In particular, Duchi–Poulalhon–Schaeffer [7] give a bijective proof where the term n^{c-3} in (1.1) roughly counts certain trees on the c-many cycles of $g \in \mathfrak{S}_n$. Our main Theorem 1.1 indicates that something similar happens for all reflection groups W. Those trees are replaced by the collection $RGS(W)$ of (length-n) *reflection generating sets* of W, and a relative version $RGS(W, g)$ (see § 2). For the special case that $g = \text{id} \in W$, Thm. 1.1 for instance implies the formula

$$F_W^{\text{tr}}(id) = (2n)! \cdot |RGS(W)| \cdot \frac{1}{I(W)}, \tag{1.2}$$

where $n = \text{rank}(W)$ and $I(W)$ is the connection index of the Weyl group W. Indeed, for $\mathfrak{S}_n$, the collection $RGS(\mathfrak{S}_n)$ corresponds precisely to the n^{n-2} trees on $[n]$ and since $I(\mathfrak{S}_n) = n$, this gives a direct generalization of the Hurwitz number $H_0(1^n)$ of (1.1).

Among elements $g \in W$, the parabolic quasi-Coxeter ones have a unique decomposition $g = g_1 \cdot g_2 \cdots g_m$ that generalizes the cycle-type decomposition of permutations and where each g_i is a quasi-Coxeter element of some irreducible parabolic subgroup of

W (see § 2). We write $F^{\text{red}}(g_i)$ for the number of *reduced* (i.e., of minimum length, and thereby not transitive unless $\ell_R(g_i) = n$) reflection factorizations of g_i. These quantities are analogs of the numbers $\lambda_i^{\lambda_i - 2}$ in $\mathfrak{S}_n$ and are described in § 2; they are all given by product formulas. With these notations, our main theorem is as follows.

Theorem 1.1. *For a Weyl group W, and any parabolic quasi-Coxeter element $g \in W$ with decomposition $g = g_1 \cdot g_2 \cdots g_m$, the number $F_W^{tr}(g)$ of minimum-length transitive reflection factorizations of g is given by the formula*

$$F_W^{tr}(g) = \ell_R^{tr}(g)! \cdot |RGS(W,g)| \cdot \frac{I(W_g)}{I(W)} \cdot \prod_{i=1}^{m} \frac{F^{red}(g_i)}{\ell_R(g_i)!}, \tag{1.3}$$

where $\ell_R^{tr}(g)$ is the transitive reflection length of g, W_g is the smallest parabolic subgroup containing g, and $I(W)$ and $F^{red}(g_i)$ are as above.

The result for reducible groups follows easily from the case of irreducible groups that we prove in Section 3. In Section 4 we generalize (conjecturally unless $g = \text{id}$) Theorem 1.1 to all well-generated complex reflection groups. The appearance of the quantity $|RGS(W,g)|$ in the formula (1.3) seems very reasonable, but it cannot be explained on a purely naive basis. In particular, the number of subsets of the terms in a minimum-length transitive reflection factorization of g that belong to $RGS(W,g)$ is not constant, already in $\mathfrak{S}_n$ and for the identity $g = \text{id}$.

2 Quasi-Coxeter elements and reflection generating sets

In this abstract, we present Thm. 1.1 in the level of generality of *Weyl groups* (even though it can be further extended, see § 4). These are finite subgroups W of $\text{GL}(\mathbb{R}^n)$, generated by euclidean reflections, that further admit the W-equivariant, essential, *root lattice* $\mathcal{Q}$ and *coroot lattice* $\check{\mathcal{Q}}$ (see the classical reference [14]). They have an assortment of related objects; in particular, roots $\alpha \in \mathcal{Q}$ and coroots $\check{\alpha} \in \check{\mathcal{Q}}$, and a weight lattice $\mathcal{P}$ defined as the dual lattice to $\check{\mathcal{Q}}$. There is an inclusion $\mathcal{Q} \subset \mathcal{P}$ and the index $I(W) := [\mathcal{P} : \mathcal{Q}]$ is an important invariant of W, known as its *connection index*; it agrees with the determinant of the Cartan matrix $\big[(\alpha_i, \check{\alpha}_j)\big]_{i,j}$ for a set of simple roots $\{\alpha_i\}$. Weyl groups are classified in three infinite families $\mathfrak{S}_{n+1} = A_n, B_n, D_n$, where n is the *rank* of the group, and five exceptional cases E_6, E_7, E_8, F_4, and G_2. The corresponding connection indices are $n+1, 2, 4$ for the infinite families and $3, 2, 1, 1, 1$ for the exceptional groups.

Our main Theorem 1.1 extends the Hurwitz formulas (1.1) to the collection of parabolic quasi-Coxeter classes in Weyl groups W. To define those, recall first that a *standard* parabolic subgroup W_I of W is any subgroup generated by a subset $\{s_i\}_{i \in I}$ of the simple generators of W and that any subgroup conjugate to some W_I is simply called a *parabolic subgroup* of W. An element $g \in W$ is then called *parabolic quasi-Coxeter* if it has a minimum-length reflection factorization $g = t_1 \cdots t_{\ell_R(g)}$ whose terms $\{t_i\}$ generate a parabolic subgroup of W, and it is called *quasi-Coxeter* if they generate all of W. To justify the name, notice that Coxeter elements (products of the simple generators $\{s_i\}$ in any order, and their conjugates) are always quasi-Coxeter. Similarly, parabolic Coxeter elements are parabolic quasi-Coxeter. To see finally that Theorem 1.1 fully generalizes the Hurwitz formulas, notice that in $\mathfrak{S}_n$ all reflection subgroups are parabolic, therefore all elements are parabolic quasi-Coxeter elements.

Quasi-Coxeter elements were introduced by Voigt [21] and rediscovered and generalized to the parabolic case by Baumeister et al. [1]. Both works considered the Hurwitz action on reduced (i.e., minimum-length) reflection factorizations of elements $g \in W$ and showed that it is transitive if and only if g is parabolic quasi-Coxeter. The sizes of these Hurwitz orbits, equivalently the numbers $F^{\text{red}}(g)$ of reduced reflection factorizations of g, were computed by Kluitmann and Voigt [15], independently by Stump (private communication); below we give them only for the infinite families. It was observed that they all factor as products of small primes and a uniform formula for them has been proposed by the first author [6]. In the groups $\mathfrak{S}_n$ and B_n the only quasi-Coxeter classes are the Coxeter ones, but in D_n we have $\lfloor \frac{n}{2} \rfloor$ classes denoted by $D_{k,n-k}$ which contain Coxeter elements of the $B_k \times B_{n-k}$ reflection subgroups of B_n (viewing D_n as a subgroup of B_n).

g	$\mathfrak{S}_n$	B_n	$D_{k,n-k}$
$F^{\text{red}}(g)$	n^{n-2}	n^n	$2 \cdot (n-1) \cdot \binom{n-2}{k-1} \cdot k^k \cdot (n-k)^{n-k}$

Table 1: Enumeration of reduced reflection factorizations for quasi-Coxeter elements.

For any element g of W, we write W_g for its *parabolic closure*, i.e., the smallest parabolic subgroup that contains it. The group W_g can be written as a product $W_g = W_1 \times \cdots \times W_m$ of irreducible components. Gobet [9] defined a generalization of the cycle-decomposition of permutations and showed that parabolic quasi-Coxeter elements have a unique such decomposition $g = g_1 \cdot g_2 \cdots g_m$ where each g_i is quasi-Coxeter for W_i. The number $F^{\text{red}}(g)$ of reduced reflection factorizations of g will be a product of the $F^{\text{red}}(g_i)$ and a multinomial coefficient, thus also a product of small primes.

For a Weyl group W of rank n, we will call any set of n reflections that generate W, a *reflection generating set* of W and we will write $RGS(W)$ for the collection of such sets. Every reduced reflection factorization of a parabolic quasi-Coxeter element g can be minimally extended to a reflection generating set of W [1, Prop. 6.2], which then allows for a relative version of this notion. For each such g, we define the collection $RGS(W, g)$ of *relative reflection generating sets* for g as ($\mathcal{R}$ stands for the set of reflections of W)

$$RGS(W, g) := \Big\{ S \subset \mathcal{R} \colon |S| = n - \ell_R(g) \text{ such that } \langle S, T_g \rangle = W \Big\}, \tag{2.1}$$

where T_g is any set of reflections that make up a reduced reflection factorization of g.

Genus-1 Hurwitz numbers

Formula (1.1) is for the *(single) Hurwitz numbers of genus* 0. In general, the genus-g Hurwitz numbers $H_g(\lambda)$ count transitive factorizations $\tau_1 \cdots \tau_k$ of a given element σ in $\mathfrak{S}_n$ of cycle type λ into $k = n + c + 2g - 2$ transpositions τ_i. In addition to the product formula for $H_0(\lambda)$, the genus-1 numbers $H_1(\lambda)$ have a closed formula that we will use in the proof of our main theorem. Let e_i denote the ith elementary symmetric function.

Lemma 2.1 (Goulden-Jackson [10])**.** *For $\lambda = (\lambda_1, \ldots, \lambda_c)$,*

$$H_1(\lambda) = \frac{1}{24}(n+c)! \left(\prod_{i=1}^{c} \frac{\lambda_i^{\lambda_i}}{(\lambda_i - 1)!} \right) \left(n^c - n^{c-1} - \sum_{i=2}^{c} (i-2)! n^{c-i} e_i(\lambda) \right).$$

In particular $H_1(1^n) = \frac{1}{24}(2n)!(n^n - n^{n-1} - \sum_{i=2}^{n} \binom{n}{i}(i-2)! n^{n-i})$.

3 Main results

In order to enumerate minimum-length reflection factorizations, it is necessary first to understand the length at which these factorizations occur. Our first result gives a formula for the transitive reflection length of parabolic quasi-Coxeter elements.

Proposition 3.1. *Suppose that w is a parabolic quasi-Coxeter element in a Weyl group W of rank n. Then the transitive reflection length of w is given by the formula*

$$\ell_R^{tr}(w) = 2n - \ell_R(w) = n + \dim \mathrm{fix}(w).$$

Proof. Fix any element w of a Weyl group W and any transitive reflection factorization of w of minimum length, and let $\ell = \ell_R(w)$ be the reflection length of w. We have by [18, Cor. 1.4] that there is another transitive factorization

$$w = r_1 \cdots r_\ell \cdot r_{\ell+1} \cdot r_{\ell+1}^{-1} \cdots r_{\ell+k} \cdot r_{\ell+k}^{-1} \tag{3.1}$$

of the same length $\ell_R^{\mathrm{tr}}(w) = \ell + 2k$. The factors $r_1, \cdots, r_{\ell+k}$ in this second factorization must also generate W, so $\ell + k \geq n$. Therefore

$$\ell_R^{\mathrm{tr}}(w) = \ell + 2k = 2(\ell + k) - \ell \geq 2n - \ell_R(w).$$

On the other hand, since w is parabolic quasi-Coxeter, we have by [1, Prop. 6.2] that there is some length-ℓ reflection factorization $w = r_1 \cdots r_\ell$ of w that can be extended to a length-n factorization $w' = r_1 \cdots r_\ell \cdot r_{\ell+1} \cdots r_n$ of some element w' in which the factors generate the group W. Then $w = r_1 \cdots r_\ell \cdot r_{\ell+1} \cdot r_{\ell+1}^{-1} \cdots r_n \cdot r_n^{-1}$ is a reflection factorization of w of length $\ell + 2(n - \ell) = 2n - \ell_R(w)$ whose factors generate W, and consequently $\ell_R^{\mathrm{tr}}(w) \leq 2n - \ell_R(w)$. □

We prove now our Theorem 1.1, enumerating transitive reflection factorizations of parabolic quasi-Coxeter elements in Weyl groups. We split the proof into two parts: we first handle the classical types A_n, B_n, and D_n, and then separately the exceptional types. We sketch the proof in types A_n and D_n; the proof of type B_n is similar.

Sketch of proof of Theorem 1.1 in classical types

Type A: As we have mentioned before, in type A all elements are parabolic quasi-Coxeter. We start with some $g \in \mathfrak{S}_n$ of cycle type $\lambda = (\lambda_1, \cdots, \lambda_c)$ and we first look at the right side of (1.3). By Prop. 3.1, we have that $\ell_R^{\mathrm{tr}}(g) = n + c - 2$. The set $RGS(\mathfrak{S}_n, g)$ can be described via trees on the c-many cycles of g: after adding a reduced factorization of g to a relative generating set, the total set of transpositions will give a tree on $[n]$. Since there are $\lambda_i \cdot \lambda_j$ possible edges between the ith and jth cycles, each tree on $[c]$ appears with total multiplicity $\prod_{i=1}^{c} \lambda_i^{\deg(T_i)}$. Now the weighted Cayley theorem implies that

$$|RGS(\mathfrak{S}_n, g)| = \sum_{T \text{ a tree on } [c]} \prod_{i=1}^{c} \lambda_i^{\deg(T_i)} = \lambda_1 \cdots \lambda_c \cdot (\lambda_1 + \cdots + \lambda_c)^{c-2} = \left(\prod \lambda_i\right) \cdot n^{c-2}.$$

Moreover, the parabolic subgroup corresponding to g is just a Young subgroup of type $\mathfrak{S}_\lambda := \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_c}$, which means that its connection index is $I(\mathfrak{S}_\lambda) = \lambda_1 \cdots \lambda_c$. Finally, we have $\prod F^{\mathrm{red}}(g_i) = \prod_{i=1}^{c} \lambda_i^{\lambda_i - 2}$ corresponding to minimum-length reflection factorizations of the cycles g_i of g. Multiplying all these quantities together gives exactly the expression of formula (1.1) for $H_0(\lambda)$, which is by definition the left side of (1.3).

Type D: We represent the elements in D_n as signed permutations acting on $\pm i$, $i = 1, \ldots, n$. Any element g in D_n is a product of disjoint cycles. Such cycles are positive or negative depending on if the number of sign changes in the cycle is even or odd. Given an element g in D_n, the projection $\pi(g)$ in $\mathfrak{S}_n$ is the permutation obtained by removing the signs from the entries. Following Zaslavsky [22], one may think of sets of reflections in type D as the edges of a signed graph.

The group D_n has two types of parabolic subgroups: those of the form $\mathfrak{S}_\lambda$ for $\lambda \vdash n$, and those of the form $D_k \times \mathfrak{S}_\lambda$ for $\lambda \vdash n - k$. The parabolic quasi-Coxeter elements for the first type have all cycles positive, while those of the second type have exactly two negative cycles. We consider the second case first.

Suppose that g has two negative cycles and $c - 2$ positive cycles, and the two negative cycles have sizes k_1, k_2 (with $k_1 + k_2 = k$). We have in this case that $\ell_R(g) = n - c + 2$ and therefore $\ell_R^{\mathrm{tr}}(g) = n + c - 2$, which is also the length of a shortest transitive $\mathfrak{S}_n$-factorization of the projection $\pi(g)$. It follows that the projection into $\mathfrak{S}_n$ of a shortest transitive factorization of g produces a shortest transitive factorization of $\pi(g)$. Fix a subset of factors in such a factorization that form a spanning tree on $[n]$. It is not difficult to see that we may choose the signs on the other $c - 1$ factors independently, and the signs on the $n-1$ fixed edges will be uniquely determined by the choices and the requirement to be a factorization of g. It follows immediately in this case that the left side of (1.3) is equal to $2^{c-1} \cdot H_0(\lambda + k_1 + k_2) = 2^{c-1} \cdot (n+c-2)! \cdot n^{c-3} \cdot \frac{k_1^{k_1} k_2^{k_2}}{(k_1-1)!(k_2-1)!} \prod \frac{\lambda_i^{\lambda_i}}{(\lambda_i-1)!}$.

Still considering the same element g with two negative cycles, we move to the right side of (1.3). A relative generating set of reflections for such an element g must induce a connected graph when taken together with a shortest reflection factorization of g. Taking the contraction of such a graph with respect to the factors in the shortest factorization, we see that the factors in the relative generating set form a tree whose $c-1$ vertices are the $c-2$ positive cycles of g and a single vertex for the two negative cycles, where an edge connected to a vertex C has $|C|$ choices of endpoint in that cycle. Thus, taking account of sign, we have $|RGS(D_n, g)| = 2^{c-2} \sum_{T \text{ a tree on } [c-1]} \prod_{i=1}^{c-2} \lambda_i^{\deg(i)} \cdot k^{\deg(c-1)}$. By the weighted Cayley theorem, this simplifies to

$$2^{c-2} \cdot (\lambda_1 + \ldots + \lambda_{c-2} + k)^{c-3} \cdot k \cdot \prod_i \lambda_i = 2^{c-2} \cdot n^{c-3} \cdot k \cdot \prod_i \lambda_i.$$

For this element g we have $\frac{I(W_g)}{I(W)} = \frac{4\prod_i \lambda_i}{4}$. Finally, since there are $2\frac{(k-1)!}{(k_1-1)!(k_2-1)!} k_1^{k_1} k_2^{k_2}$ shortest factorizations of a quasi-Coxeter element in D_k whose cycles have lengths k_1 and k_2 (see Table 1), we have $\prod F^{\text{red}}(g_i) = 2\frac{(k-1)!}{(k_1-1)!(k_2-1)!} k_1^{k_1} k_2^{k_2} \cdot \prod_i^{c-2} \lambda_i^{\lambda_i - 2}$. Plugging these factors in to the right side of (1.3) gives exactly the value $F_W^{\text{tr}}(w)$ computed in the previous paragraph.

Finally, suppose that g has c cycles, all positive. Thus $\ell_R(g) = n - c$. Since the minimum-length reflection factorizations of g are precisely the same as those restricted to its parabolic subgroup, we have $\prod F_{D_n}^{\text{red}}(g_i) = \prod F_{\mathfrak{S}_n}^{\text{red}}(g_i) = \prod_i^c \lambda_i^{\lambda_i - 2}$. A relative generating set must consist of c edges and induce a connected signed graph when taken together with the a shortest reflection factorization of g. Taking the contraction of such a graph with respect to the factors in the shortest factorization, the transposition-like factors in the relative generating set form a signed "unicycle": a connected graph with c vertices (the cycles of g) and c edges, where again the edges are weighted so that an edge connected to cycle C_i has $|C_i| = \lambda_i$ choices of endpoint in that cycle. Moreover, in the unique cycle in this graph, the number of negatively signed transpositions must be odd (otherwise, all reflections belong to a subgroup conjugate to $\mathfrak{S}_n$). We may count such graphs as follows: if the cycle is a loop, with both endpoints in cycle C_i, then it may be chosen in $\binom{\lambda_i}{2}$ ways, while the rest of the set forms a signed weighted tree on c vertices. Thus, the contribution from this case is

$$2^{c-1} n^{c-2} \prod_i \lambda_i \cdot \sum_i \binom{\lambda_i}{2} = 2^{c-2} n^{c-2} (-n + \sum_i \lambda_i^2) \prod_i \lambda_i.$$

Otherwise, the cycle has some length $k \geq 2$. Let the vertices in the cycle be indexed by $S = \{s_1, \ldots, s_k\} \in \binom{[c]}{k}$. Contracting the edges of the cycle gives a signed weighted tree on $c-k+1$ vertices, of weights $\{\lambda_i \colon i \notin S\} \cup \{\lambda_{s_1} + \ldots + \lambda_{s_k}\}$. This weighting has the feature that the generating functions by weights for the relevant trees and unicycles are closely related: they differ by a factor of $2^{k-2}(k-1)! \cdot (\lambda_{s_1} \cdots \lambda_{s_k})^2$, where the first factor accounts for the number of different ways of inserting a signed cycle while the second factor accounts for the weights from adding the edges of the cycle. By the weighted Cayley theorem, the generating function for the trees is precisely

$$(\lambda_{s_1} + \ldots + \lambda_{s_k}) \cdot \prod_{i \notin S} \lambda_i \cdot (\lambda_1 + \ldots + \lambda_c)^{c-k-1} \cdot 2^{c-k} = (\lambda_{s_1} + \ldots + \lambda_{s_k}) \cdot \prod_{i \notin S} \lambda_i \cdot n^{c-k-1} \cdot 2^{c-k},$$

and consequently the generating function for the unicycles in this case is

$$(\lambda_{s_1} + \ldots + \lambda_{s_k})\lambda_{s_1} \cdots \lambda_{s_k} \cdot 2^{c-2}(k-1)! n^{c-k-1} \prod_i \lambda_i.$$

Thus

$$\begin{aligned} |RGS(D_n, g)| &= 2^{c-2} \prod_i \lambda_i \Big(-n^{c-1} + \sum_{k=1}^{c} (k-1)! n^{c-k-1} \sum_{S \in \binom{[c]}{k}} (\lambda_{s_1} + \ldots + \lambda_{s_k}) \lambda_{s_1} \cdots \lambda_{s_k} \Big) \\ &= 2^{c-2} \prod_i \lambda_i \Big(-n^{c-1} + \sum_{k=1}^{c} (k-1)! n^{c-k-1} m_{21^{k-1}}(\lambda) \Big), \end{aligned}$$

where $m_\mu(\lambda)$ is the *monomial symmetric function* in the variables λ. Now $m_{2,1^{k-1}} = e_{k,1} - (k+1)e_{k+1}$, and since $e_1(\lambda) = n$ the right side of the last expression becomes

$$2^{c-2} \prod_i \lambda_i \Big(- n^{c-1} + \sum_{k=1}^{c} (k-1)! n^{c-k} e_k(\lambda) - \sum_{k=1}^{c} (k-1)!(k+1) n^{c-k-1} e_{k+1}(\lambda) \Big).$$

The first term of the first sum in this expression is $n^{c-1}e_1(\lambda) = n^c$. Separating this term and combining the remaining terms with the second sum gives

$$|RGS(D_n, g)| = 2^{c-2} \prod_i \lambda_i \Big(n^c - n^{c-1} - \sum_{k=2}^{c} (k-2)! n^{c-k} e_k(\lambda) \Big),$$

so for this g the entire right side of (1.3) is

$$\frac{(n+c)!}{(n-c)!} \cdot \binom{n-c}{\lambda_1 - 1; \cdots; \lambda_c - 1} \cdot \prod_i \lambda_i^{\lambda_i} \cdot 2^{c-4} \Big(n^c - n^{c-1} - \sum_{k=2}^{c} (k-2)! n^{c-k} e_k(\lambda) \Big).$$

This is exactly $3 \cdot 2^{c-1} \cdot H_1(\lambda)$, where we used the formula for $H_1(\lambda)$ in Lemma 2.1.

Now we consider the left side of equation (1.3), still in the case that g is an element of D_n with all positive cycles. Projecting a transitive factorization of g into $\mathfrak{S}_n$, we produce a connected transposition factorization having $n+c$ factors of a permutation $\pi(g)$ whose transitive reflection length (in $\mathfrak{S}_n$) is $n + c - 2$. That is, it is a genus-1 transitive factorization in the symmetric group. The number of such factorizations is $H_1(\lambda)$. Moreover, each of these may be lifted by assigning appropriate signs to a transitive factorization of g in D_n in the same number of ways; accounting simultaneously for the requirements that the product is g and the reflections generate D_n, this number turns out to be $3 \cdot 2^{c-1}$. □

Sketch of computer proof of Theorem 1.1 for the exceptional types

For the exceptional Weyl groups G_2, F_4, E_6, E_7, and E_8, we prove our main theorem by direct calculation of both sides of (1.3) via SageMath and CHEVIE [8, 20]. It turns out that the most efficient way to do this will further count transitive reflection factorizations of arbitrary length. We follow the traditional approach via representation theory and the lemma of Frobenius [4, § 4] to first determine the exponential generating functions of *not necessarily transitive* reflection factorizations of an element g:

$$\mathcal{F}_W(g;t) := \sum_{N \geq 0} \#\{(\tau_1, \ldots, \tau_N) \in \mathcal{R}^N : \; \tau_1 \cdots \tau_N = g\} \cdot \frac{t^N}{N!}. \tag{3.2}$$

They can be computed as a finite sum of exponentials via the formula [4, (4.3)]

$$\mathcal{F}_W(g;t) = \frac{1}{|W|} \cdot \sum_{\chi \in \widehat{W}} \chi(1) \cdot \chi(g^{-1}) \cdot \exp\left(\frac{\chi(\mathcal{R})}{\chi(1)} \cdot t\right),$$

where $\widehat{W}$ denotes the set of irreducible (complex) characters of W (these are produced by CHEVIE, which realizes reflection groups via their permutation action on roots) and $\chi(\mathcal{R}) := \sum_{\tau \in \mathcal{R}} \chi(\tau)$. Any factorization enumerated in (3.2) will be transitive for the group $W' := \langle \tau_1, \ldots, \tau_N \rangle$ generated by its elements. Then, a simple inclusion-exclusion argument gives us the corresponding generating functions for *transitive* factorizations:

$$\mathcal{F}_W^{\text{tr}}(g;t) = \sum_{g \in W' \subset W} \mu(W, W') \cdot \mathcal{F}_{W'}(g;t), \tag{3.3}$$

where we sum over all *reflection* subgroups W' of W (that contain g) ordered by reverse inclusion. Having established the generating function $\mathcal{F}_W^{\text{tr}}(g;t)$, and since it remains a finite sum of exponentials, we can immediately calculate its leading term, which equals $F_W^{\text{tr}}(g) \cdot \left(t^{\ell_R^{\text{tr}}(g)} / \ell_R^{\text{tr}}(g)!\right)$. To compute finally the quantity $|RGS(W, g)|$ of (1.3), we consider all possible $(n - \ell_R(g))$-subsets of the set of reflections in $W \setminus W_g$ and check whether they extend an arbitrary reflection factorization of g to a reflection generating set of W. □

4 Generalization to complex reflection groups

There is significant evidence that our Theorem 1.1 extends naturally to all well-generated complex reflection groups (finite subgroups of $\mathrm{GL}(\mathbb{C}^n)$ generated by n unitary reflections). The quantity $|RGS(W,g)|$ is replaced by a statistic on the relative generating sets that in the case of Weyl groups always equals $I(W_g)/I(W)$. To introduce this statistic, we need an analog of roots, but we will not require that they form a W-equivariant collection. That is, for each reflection τ of W and its unique $\zeta \neq 1$ eigenvalue, we pick any nonzero element ρ_τ of the ζ-eigenspace of τ. We further normalize their lengths, defined via the standard Hermitian inner product $(\cdot,\cdot)$, to satisfy $(\rho,\rho) = 2$ for all roots ρ.

Definition 4.1. The *Grammian determinant* $\mathrm{GD}(\boldsymbol{\rho})$ of a set of roots $\boldsymbol{\rho} := \{\rho_i\}$ is

$$\mathrm{GD}(\boldsymbol{\rho}) := \det\left(\langle \rho_i, \rho_j\rangle\right)_{ij}.$$

In a Weyl group W, we may simultaneously consider roots ρ, coroots $\check{\rho}$, and *normalized* roots $\tilde{\rho}$, satisfying $(\tilde{\rho},\tilde{\rho}) = 2$ as above. Now, [2, Cor. 1.2] shows that roots ρ_i and coroots $\check{\rho}_i$ that correspond to a reflection generating set of W always form $\mathbb{Z}$-bases of the root and coroot lattices $\mathcal{Q}$ and $\check{\mathcal{Q}}$ respectively. This forces that $\det\left(\langle \rho_i, \check{\rho}_j\rangle\right)_{i,j} = I(W)$, which then further implies that $\mathrm{GD}(\tilde{\boldsymbol{\rho}}) = I(W)$ since $\det\left(\langle \rho_i, \check{\rho}_j\rangle\right)_{i,j} = \det\left(\langle \tilde{\rho}_i, \tilde{\rho}_j\rangle\right)_{i,j}$. The following result, which has a proof very much analogous to the work in § 3, is then a direct generalization of the case $g = \mathrm{id}$, displayed in (1.2), of Theorem 1.1.

Theorem 4.2. *For a well-generated complex reflection group W of rank n, the number of minimum-length transitive reflection factorizations of the identity* $\mathrm{id} \in W$ *is given by*

$$F_W^{tr}(\mathrm{id}) = (2n)! \cdot \sum_{\{\tau_i\} \in RGS(W)} \frac{1}{\mathrm{GD}(\boldsymbol{\rho_\tau})},$$

where the sum is over all subsets of n reflections $\{\tau_i\}$ *that generate W and* ρ_{τ_i} *is a root for* τ_i.

When W is not a Weyl group, this is a non-trivial sum and the numbers $\mathrm{GD}(\boldsymbol{\rho_\tau})$ that appear are not constant. For instance, in H_3, there are two "generalized" Coxeter classes related by a reflection automorphism; generating sets that correspond to factorizations of elements from one class have Grammian determinant $3+\sqrt{5}$, while the others have Grammian determinant $3-\sqrt{5}$. Notice here that the cummulative contribution of two such sets paired by the automorphism is a rational number $\frac{1}{3+\sqrt{5}} + \frac{1}{3-\sqrt{5}} = \frac{3}{2}$. Generating sets corresponding to the third quasi-Coxeter class satisfy $\mathrm{GD}(\boldsymbol{\rho}) = 2$.

There is in fact a similar full generalization of Theorem 1.1 for well generated groups, but we have not completed our work on it. We state it here as a conjecture.

Conjecture 4.3. *If W is a well-generated complex reflection group and g a parabolic quasi-Coxeter element in W with decomposition* $g = g_1 \cdots g_m$, *then*

$$F_W^{tr}(g) = \ell_R^{tr}(g)! \cdot \prod_{i=1}^{m} \frac{F^{red}(g_i)}{\ell_R(g_i)!} \cdot \sum_{\{\tau_i\} \in RGS(W,g)} \frac{\mathrm{GD}(\boldsymbol{\rho}_g)}{\mathrm{GD}(\boldsymbol{\rho_\tau} \cup \boldsymbol{\rho}_g)},$$

where $\boldsymbol{\rho_\tau}$ *denotes the set of roots associated with a relative generating set* $\boldsymbol{\tau} = \{\tau_i\}$ *and* $\boldsymbol{\rho}_g$ *corresponds to a fixed reduced reflection factorization of g.*

5 Higher genus counts in the infinite family

As discussed in Section 3, the proof for the exceptional types gives in (3.3) the complete enumeration for transitive factorizations of arbitrary length. In this section, we study the analogous generating functions for the infinite family $G(m,p,n)$ of complex reflection groups. The generating functions are defined as $\mathcal{F}^{\mathrm{tr}}_W(g;t) := \sum_{N\geq 0} F^{\mathrm{tr}}_W(g,N)\cdot t^N/N!$ where $F^{\mathrm{tr}}_W(g,N)$ is the number of length-N transitive reflection factorizations of $g \in W$. The following theorem expresses $\mathcal{F}^{\mathrm{tr}}_W(g;t)$ for elements g in $G(m,p,n)$ with only weight-0 cycles in terms of the generating functions of the symmetric group $\mathfrak{S}_n = G(1,1,n)$.

Theorem 5.1. *For an element $g \in G(1,1,n) \subset G(m,p,n)$, we have*

$$\mathcal{F}^{tr}_{G(m,p,n)}(g;t) = \frac{1}{(m/p)^{n-1}}\cdot \mathcal{F}^{tr}_{G(p,p,n)}(g;(m/p)\cdot t)\cdot \mathcal{F}^{tr}_{G(m/p,1,1)}(\mathrm{id};nt), \tag{5.1}$$

$$\mathcal{F}^{tr}_{G(m,m,n)}(g;t) = m^{c-1}\sum_{d|m}\mu(m/d)\cdot d^{2-c-n}\cdot \mathcal{F}^{tr}_{G(1,1,n)}(g;dt), \tag{5.2}$$

where c is the number of cycles of g and μ is the number-theoretic Möbius function.

The proof is similar to that presented for type D in Section 3: there is a natural projection map that sends any reflection factorization in $G(m,p,n)$ to a factorization in $\mathfrak{S}_n$ as a product of transpositions. Each of these factorizations may be lifted back to $G(m,p,n)$ in a predictable number of ways. In $G(m,p,n)$ with $p<m$, the lifting must account for the introduction of diagonal reflections (the analogues of sign-change reflections in type B), giving rise to the term corresponding to the cyclic group $G(m/p,1,1)$; in $G(m,m,n)$, the lift must generate the full $G(m,m,n)$ and not any subgroup $G(d,d,n)$ for $d \mid m$, leading to the Möbius inversion.

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