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# Modules of the 0-Hecke algebra arising from standard permuted composition tableaux

Seung-Il Choi*[1], Young-Hun Kim†[2,3], Sun-Young Nam‡[2], and Young-Tak Oh§[2]

[1]*Center for Quantum Structures in Modules and Spaces, Seoul National University,*
[2]*Department of Mathematics, Sogang University*
[3]*Research Institute for Basic Science, Sogang University*

**Abstract.** We study structures of the $H_n(0)$-module $\mathbf{S}^\sigma_\alpha$ arising from standard permuted composition tableaux of shape $\alpha$ and type $\sigma$. Precisely, we investigate the indecomposability of $\mathbf{S}^\sigma_\alpha$, the characteristic relations among $\mathbf{S}^\sigma_\alpha$'s, the quasisymmetric Schur-expansion of the image of $\mathbf{S}^\sigma_\alpha$ under the quasisymmetric characteristic, and the projective cover of every indecomposable direct summand $\mathbf{S}^\sigma_{\alpha,E}$ of $\mathbf{S}^\sigma_\alpha$. And then, we reveal hidden connections to other indecomposable $H_n(0)$-modules $\mathcal{V}_\alpha$ and $X_\alpha$ arising from standard immaculate tableaux and standard extended tableaux, respectively.

**Keywords:** 0-Hecke algebra, projective cover, quasisymmetric Schur function

## 1 Introduction

The 0-Hecke algebra $H_n(0)$ is a $\mathbb{C}$-algebra generated by $\pi_1, \ldots, \pi_{n-1}$ subject to the following relations:

$$\begin{aligned} \pi_i^2 &= \pi_i && \text{for } 1 \le i \le n-1, \\ \pi_i\pi_{i+1}\pi_i &= \pi_{i+1}\pi_i\pi_{i+1} && \text{for } 1 \le i \le n-2, \\ \pi_i\pi_j &= \pi_j\pi_i && \text{if } |i-j| > 1. \end{aligned}$$

In 1979, Norton [10] classified all projective indecomposable $H_n(0)$-modules $\mathcal{P}_I$ ($I \subseteq \{1, \ldots, n-1\}$) and all simple $H_n(0)$-modules $\mathbf{F}_I$ ($I \subseteq \{1, \ldots, n-1\}$) up to equivalence, which are defined by

$$\mathcal{P}_I := H_n(0) \cdot \overline{\pi}_{w_0(I)}\pi_{w_0(I^c)} \quad \text{and} \quad \mathbf{F}_I := \operatorname{top}(\mathcal{P}_I).$$

*ignatioschoi@snu.ac.kr. The first author was supported by the National Research Foundation of Korea(NRF) grant funded by the Korean government(MSIT) (No. NRF-2019R1C1C1010668).

†yhkim14@sogang.ac.kr. The second author was supported by NRF grant funded by the Korean Government (No. NRF-2020R1F1A1A01071055).

‡synam.math@gmail.com. The third author was supported by Basic Science Research Program through NRF funded by the Ministry of Education (No. NRF-2019R1I1A1A01062658).

§ytoh@sogang.ac.kr. The fourth author was supported by NRF grant funded by the Korean Government (No. NRF-2020R1F1A1A01071055).

Here $\overline{\pi}_i := \pi_i - 1$, $w_0(I)$ is the longest element of the parabolic subgroup $\mathfrak{S}_{n,I}$, $I^{\mathrm{c}} := \{1, \dots, n-1\} \setminus I$, and $\mathrm{top}(\mathcal{P}_I)$ is the *top* of $\mathcal{P}_I$.

Let us denote by $\alpha \models n$ a composition $\alpha$ of $n$. Krob and Thibon [9] introduced an $H_n(0)$-action on the $\mathbb{C}$-span of $\mathrm{SRT}(\alpha)$, the set of standard ribbon tableaux of shape $\alpha$ using generators $\overline{\pi}_i$'s and denoted the resulting $H_n(0)$-module by $\mathbf{P}_\alpha$. It turns out that $\mathbf{P}_\alpha$ is isomorphic to $\mathcal{P}_{\mathrm{set}(\alpha)}$, where $\mathrm{set}(\alpha)$ is the set corresponding to $\alpha$. Later, Huang [7] introduced their induced modules in terms of standard ribbon tableaux of shape $\boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ ranges over the set of generalized compositions.

The representation theory of the 0-Hecke algebra has strong connections to quasisymmetric functions. The most striking feature might be that the Grothendieck ring $\mathcal{G}_0(H_n(0))$ is isomorphic to QSym, the ring of quasisymmetric functions, which is a quasi-analogue of the Frobenius correspondence between characters of symmetric groups and symmetric functions. In [5], the $\mathbb{C}$-linear map

$$\mathrm{ch} : \bigoplus_{n \geq 0} \mathcal{G}_0(H_n(0)) \to \mathrm{QSym}, \quad [\mathbf{F}_I] \mapsto F_{\mathrm{comp}(I)}, \tag{1.1}$$

called the *quasisymmetric characteristic*, was introduced as a ring isomorphism. Here $\mathrm{comp}(I)$ the composition corresponding to $I$ and $F_{\mathrm{comp}(I)}$ the *fundamental quasisymmetric function* of $\mathrm{comp}(I)$.

From the viewpoint of this correspondence, when we have a family of notable quasisymmetric functions, it would be of great importance to investigate if these elements appear as the image of the isomorphism classes of certain modules with nice properties under the quasisymmetric characteristic. The related studies have been done for the quasisymmetric Schur functions and their permuted version [12, 13], the dual immaculate functions [2], and the extended Schur functions [11], all of which form a basis of QSym.

Let us explain these results in more detail. From now on, we fix $\alpha \models n$ and $\sigma \in \mathfrak{S}_{\ell(\alpha)}$, where $\ell(\alpha)$ is the length of $\alpha$.

Tewari and van Willigenburg [12] defined an $H_n(0)$-action on $\mathrm{SRCT}(\alpha)$, the set of *standard reverse composition tableaux of shape* $\alpha$, and showed that the resulting $H_n(0)$-module $\mathbf{S}_\alpha$ has the *quasisymmetric Schur function* $\mathcal{S}_\alpha$ as the image under (1.1).

As a far-reaching generalization, they [13] introduced new combinatorial objects, called *standard permuted composition tableaux of shape* $\alpha$ *and type* $\sigma$, by weakening the condition on the first column in standard reverse composition tableaux. It was shown in [13] that $\mathrm{SPCT}^\sigma(\alpha)$, the set of standard permuted composition tableaux of shape $\alpha$ and type $\sigma$, has an $H_n(0)$-action and the resulting $H_n(0)$-module $\mathbf{S}^\sigma_\alpha$ shares many properties with $\mathbf{S}_\alpha$. For instance, one can give a colored graph structure on $\mathrm{SPCT}^\sigma(\alpha)$ as in [12].

Let $\mathcal{E}^\sigma(\alpha)$ be the set of connected components and $\mathbf{S}^\sigma_{\alpha,E}$ the $H_n(0)$-submodule of $\mathbf{S}^\sigma_\alpha$ whose underlying space is the $\mathbb{C}$-span of $E$ for $E \in \mathcal{E}^\sigma(\alpha)$. Then, as $H_n(0)$-modules,

$$\mathbf{S}^\sigma_\alpha \cong \bigoplus_{E \in \mathcal{E}^\sigma(\alpha)} \mathbf{S}^\sigma_{\alpha,E}.$$

The first concern of this extended abstract is to demonstrate noteworthy properties of $\mathbf{S}^{\sigma}_{\alpha}$ compared to $\mathbf{S}_{\alpha}$ as follows:

- the indecomposability of $\mathbf{S}^{\sigma}_{\alpha,E}$ and of $\mathbf{S}^{\sigma}_{\alpha}$,
- characteristic relations among $\mathbf{S}^{\sigma}_{\alpha}$'s and the quasisymmetric Schur expansion of $\mathrm{ch}([\mathbf{S}^{\sigma}_{\alpha}])$,
- the projective cover of $\mathbf{S}^{\sigma}_{\alpha,E}$.

Next, let us review other $H_n(0)$-modules arising from two other tableaux. The first one, called a *standard immaculate tableau*, is introduced by Berg, Bergeron, Saliola, Serrano, and Zabrocki [2]. They defined an $H_n(0)$-action on the set of standard immaculate tableaux of shape $\alpha$. And then, they showed that the resulting module $\mathcal{V}_\alpha$ is an indecomposable $H_n(0)$-module generated by a single tableau whose quasisymmetric characteristic is the *dual immaculate function* $\mathfrak{S}^*_\alpha$. The second one, called a *standard extended tableau*, is introduced by Assaf and Searles. Later, on the set of standard extended tableaux, Searles [11] defined an $H_n(0)$-action and constructed indecomposable $H_n(0)$-modules $X_\alpha$ generated by a single tableau whose quasisymmetric characteristics are the extended Schur functions.

The second concern here is to reveal the hidden relationships among $\mathcal{V}_\alpha$, $X_\alpha$ and $\mathbf{S}^{\sigma}_{\alpha,C}$. We construct a series of surjections

$$\mathbf{P}_\alpha \twoheadrightarrow \mathcal{V}_\alpha \twoheadrightarrow X_\alpha \twoheadrightarrow \phi[\mathbf{S}^{\sigma}_{\widetilde{\alpha},C}], \tag{1.2}$$

where $\sigma$ ranges over the set of permutations in $\mathfrak{S}_{\ell(\alpha)}$ satisfying that $\widetilde{\alpha} = \alpha^{\mathrm{r}} \cdot \sigma$ and $\phi[\mathbf{S}^{\sigma}_{\widetilde{\alpha},C}]$ is the $\phi$-twist of $\mathbf{S}^{\sigma}_{\widetilde{\alpha},C}$. The series (1.2) enables us to deal with these modules in a uniform way although they appear in the different contexts. Then we give a surjection from $\mathbf{S}^{\sigma}_{\alpha,C}$ to $\mathbf{S}^{\sigma s_i}_{\alpha\cdot s_i,C}$ when $\ell(\sigma s_i) < \ell(\sigma)$. Using this result repeatedly, we finally arrive at a series of surjections starting from $\mathbf{P}_\alpha$ and ending at $\phi[\mathbf{S}^{\mathrm{id}}_{\alpha^{\mathrm{r}},C}]$. More precisely, for any reduced expression $s_{i_1} \cdots s_{i_k}$ of $\sigma$, we have

$$\mathbf{P}_\alpha \twoheadrightarrow \mathcal{V}_\alpha \twoheadrightarrow X_\alpha \twoheadrightarrow \phi[\mathbf{S}^{\sigma}_{\widetilde{\alpha},C}] \twoheadrightarrow \phi[\mathbf{S}^{\sigma s_{i_k}}_{\widetilde{\alpha}\cdot s_{i_k},C}] \twoheadrightarrow \cdots \twoheadrightarrow \phi[\mathbf{S}^{\mathrm{id}}_{\alpha^{\mathrm{r}},C}]$$

All the detailed proofs are omitted, which can be found in [4, 3].

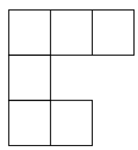
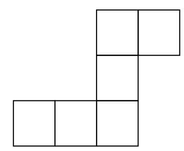
# 2 Preliminaries

## 2.1 Compositions and their diagrams

A *composition* $\alpha$ of a positive integer $n$, denoted by $\alpha \models n$, is a finite ordered list of positive integers $(\alpha_1, \ldots, \alpha_k)$ satisfying $\sum_{i=1}^{k} \alpha_i = n$. Here $k$ is called the *length* $\ell(\alpha)$ of $\alpha$ .

Throughout this subsection, let us fix $\alpha = (\alpha_1, \ldots, \alpha_k) \models n$. If $\alpha_1 \geq \cdots \geq \alpha_k$, then we say that $\alpha$ is *partition* of $n$ and denote this by $\alpha \vdash n$. The partition obtained by sorting the parts of $\alpha$ in the weakly decreasing order is denoted by $\widetilde{\alpha}$. Let $\mathrm{set}(\alpha) := \{\alpha_1, \alpha_1+\alpha_2, \ldots, \alpha_1+\alpha_2+\cdots+\alpha_{k-1}\}$. Then the set of compositions of $n$ is in bijection with the set of subsets of $[n-1]$ under the correspondence $\alpha \mapsto \mathrm{set}(\alpha)$. Let $\alpha^{\mathrm{c}}$ denote the unique composition satisfying that $\mathrm{set}(\alpha^c) = [n-1] \setminus \mathrm{set}(\alpha)$, and set $\alpha^{\mathrm{r}} := (\alpha_k, \ldots, \alpha_1)$ and $\alpha^{\mathrm{t}} := (\alpha^{\mathrm{r}})^{\mathrm{c}} = (\alpha^{\mathrm{c}})^{\mathrm{r}}$.

We define the *composition diagram* $\mathtt{cd}(\alpha)$ of $\alpha$ by a left-justified array of $n$ boxes where the $i$th row from the top has $\alpha_i$ boxes for $1 \le i \le k$. We also define the *ribbon diagram* $\mathtt{rd}(\alpha)$ of $\alpha$ by the connected skew diagram without $2 \times 2$ boxes, such that the $i$th row from the bottom has $\alpha_i$ boxes. For example, when $\alpha = (3,1,2)$, we have

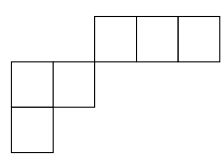
$\mathtt{cd}(\alpha) =$  and $\mathtt{rd}(\alpha) =$  .

We use $(i,j)$ to denote the box in the $i$th row and $j$th column. For a filling $\tau$ of $\mathtt{cd}(\alpha)$, we denote by $\tau_{i,j}$ the entry at $(i,j)$ in $\tau$.

Next, let us introduce a generalized notion of compositions and ribbon diagrams. A *generalized composition* $\boldsymbol{\alpha}$ of $n$, denote by $\boldsymbol{\alpha} \models n$, is a formal composition $\alpha^{(1)} \oplus \cdots \oplus \alpha^{(m)}$, where $\alpha^{(i)} \models n_i$ for positive integers $n_i$'s with $n_1 + \cdots + n_m = n$. The *generalized ribbon diagram* $\mathtt{rd}(\boldsymbol{\alpha})$ of $\boldsymbol{\alpha}$ is a skew diagram whose connected components are $\mathtt{rd}(\alpha^{(1)}), \ldots, \mathtt{rd}(\alpha^{(m)})$ such that $\mathtt{rd}(\alpha^{(i+1)})$ is strictly to the northeast of $\mathtt{rd}(\alpha^{(i)})$ for $i = 1, \ldots, m-1$. We define $[\boldsymbol{\alpha}]$ to be the set consisting of $2^{m-1}$ compositions of the form

$$\alpha^{(1)} \square \alpha^{(2)} \square \cdots \square \alpha^{(m)}, \quad \text{for } \square \in \{\odot, \cdot\}.$$

Here $\odot$ and $\cdot$ mean *near concatenation* and *concatenation* respectively. For instance, if $\boldsymbol{\alpha} = (1,2) \oplus (3)$, then

$\mathtt{rd}(\boldsymbol{\alpha}) =$ 

In this case, $[\boldsymbol{\alpha}] = \{(1,2,3), (1,5)\}$.

## 2.2 Standard ribbon tableaux and projective $H_n(0)$-modules

**Definition 2.1.** For a generalized composition $\boldsymbol{\alpha} \models n$, a *standard ribbon tableau* (SRT) of shape $\boldsymbol{\alpha}$ is a filling of $\mathtt{rd}(\boldsymbol{\alpha})$ by $1, 2, \ldots, n$ without repetition such that every row increases from left to right and every column increases from top to bottom.

Let $\mathrm{SRT}(\boldsymbol{\alpha})$ denote the set of all standard ribbon tableaux of shape $\boldsymbol{\alpha}$. In [7], Huang introduced an $H_n(0)$-action on the $\mathbb{C}$-span of $\mathrm{SRT}(\boldsymbol{\alpha})$ using generators $\overline{\pi}_i(:= \pi_i - 1)$'s and denoted the resulting $H_n(0)$-module by $\mathbf{P}_{\boldsymbol{\alpha}}$. It turns out that $\mathbf{P}_\alpha(\alpha \models n)$ form a complete list of inequivalent projective indecomposable $H_n(0)$-modules. We here define a new $H_n(0)$-action on the $\mathbb{C}$-span of $\mathrm{SRT}(\boldsymbol{\alpha})$ by

$$\pi_i \cdot T = \begin{cases} T, & \text{if } i \text{ is in a strictly above row of } T \text{ than } i+1, \\ 0, & \text{if } i \text{ and } i+1 \text{ are in the same row of } T, \\ s_i \cdot T, & \text{if } i \text{ is in a strictly below row of } T \text{ than } i+1, \end{cases} \tag{2.1}$$

for $1 \le i \le n-1$ and $T \in \mathrm{SRT}(\boldsymbol{\alpha})$. Here $s_i \cdot T$ is obtained from $T$ by swapping $i$ and $i+1$. Let us denote the resulting $H_n(0)$-module by $\overline{\mathbf{P}}_{\boldsymbol{\alpha}}$. In particular, in case where $\alpha \models n$, $\overline{\mathbf{P}}_{\alpha^c}$ is isomorphic to $\mathbf{P}_\alpha$ as an $H_n(0)$-module (see [4]). It should be pointed out that $\overline{\mathbf{P}}_\alpha$ is more adequate than $\mathbf{P}_\alpha$ in the sense that the $H_n(0)$-actions in our concern are described in terms of $\pi_i$'s.

**Theorem 2.2** ([7, Theorem 3.3])**.** *For a generalized composition $\boldsymbol{\alpha}$ of $n$, $\overline{\mathbf{P}}_{\boldsymbol{\alpha}}$ is projective and is isomorphic to $\oplus_{\beta \in [\boldsymbol{\alpha}]} \overline{\mathbf{P}}_\beta$.*

Let $T_0$ be the tableau obtained by filling $\mathtt{rd}(\boldsymbol{\alpha})$ with entries $1, 2, \ldots, n$ from top to bottom and from left to right. Then $\mathbf{P}_{\boldsymbol{\alpha}}$ and $\overline{\mathbf{P}}_{\boldsymbol{\alpha}}$ are generated by $T_0$.

## 2.3 Standard permuted composition tableaux and the $H_n(0)$-action on them

Standard permuted composition tableaux were introduced and studied intensively by Tewari and van Willigenburg [13].

**Definition 2.3.** Given $\alpha \models n$ and $\sigma \in \mathfrak{S}_{\ell(\alpha)}$, a *standard permuted composition tableau* (SPCT) of shape $\alpha$ and type $\sigma$ is a filling $\tau$ of $\mathtt{cd}(\alpha)$ with entries in $\{1, 2, \ldots, n\}$ such that the following conditions hold.

(1) The entries are all distinct.
(2) The standardization of the word obtained by reading the first column from top to bottom is $\sigma$.
(3) The entries along the rows decrease weakly when read from left to right.
(4) (The *triple condition*) If $i < j$ and $\tau_{i,k} > \tau_{j,k+1}$, then $(i, k+1) \in \mathtt{cd}(\alpha)$ and $\tau_{i,k+1} > \tau_{j,k+1}$.

Let $\mathrm{SPCT}^\sigma(\alpha)$ be the set of all SPCTs of shape $\alpha$ and type $\sigma$. Given a composition $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_\ell)$ and $\sigma \in \mathfrak{S}_\ell$, we say that $\alpha$ *is compatible with* $\sigma$ if $\alpha_i \ge \alpha_j$ whenever $i < j$ and $\sigma(i) > \sigma(j)$. In fact, this condition is equivalent to $\mathrm{SPCT}^\sigma(\alpha) \neq \emptyset$. Hereafter, we will only deal with pairs $(\alpha, \sigma)$ such that $\alpha$ is compatible with $\sigma$.

For $\tau \in \mathrm{SPCT}^\sigma(\alpha)$, an integer $1 \le i \le n-1$ is a *descent* of $\tau$ if $i+1$ lies weakly right of $i$ in $\tau$. Let $\mathrm{Des}(\tau)$ be the set of all descents of $\tau$ and $\mathrm{comp}(\tau)$ the composition corresponding to $\mathrm{Des}(\tau)$. For $1 \le i < j \le n$, we say that $i$ and $j$ are *attacking* if either

(i) $i$ and $j$ are in the same column in $\tau$, or

(ii) $i$ and $j$ are in adjacent columns in $\tau$, with $j$ positioned lower-right of $i$.

Let $i$ be a descent of $\tau$. In case where $i$ and $i+1$ are attacking (resp. nonattacking), we simply say that $i$ is an *attacking descent* (resp. *nonattacking descent*).

In [13], the authors define an $H_n(0)$-action on the $\mathbb{C}$-span of $\mathrm{SPCT}^\sigma(\alpha)$ by

$$\pi_i \cdot \tau = \begin{cases} \tau & \text{if } i \text{ is not a descent,} \\ 0 & \text{if } i \text{ is an attacking descent,} \\ s_i \cdot \tau & \text{if } i \text{ is a nonattacking descent} \end{cases} \tag{2.2}$$

for $1 \le i \le n-1$ and $\tau \in \mathrm{SPCT}^\sigma(\alpha)$. Here $s_i \cdot \tau$ is obtained from $\tau$ by swapping $i$ and $i+1$. Denote the resulting $H_n(0)$-module by $\mathbf{S}^\sigma_\alpha$.

Given $\alpha \models n$ and $\sigma \in \mathfrak{S}_{\ell(\alpha)}$, define an equivalence relation $\sim_\alpha$ on $\mathrm{SPCT}^\sigma(\alpha)$ by defining $\tau_1 \sim_\alpha \tau_2$ if and only if the entries in each column of $\tau_1$ are in the same relative order as the entries in the corresponding column of $\tau_2$. Let $\mathcal{E}^\sigma(\alpha)$ be the set of all equivalence classes under $\sim_\alpha$. For each $E \in \mathcal{E}^\sigma(\alpha)$, let $\mathbf{S}^\sigma_{\alpha,E}$ be the $\mathbb{C}$-span of $E$. It is known that $\mathbf{S}^\sigma_{\alpha,E}$ is an $H_n(0)$-submodule of $\mathbf{S}^\sigma_\alpha$ and

$$\mathbf{S}^\sigma_\alpha \cong \bigoplus_{E \in \mathcal{E}^\sigma(\alpha)} \mathbf{S}^\sigma_{\alpha,E}. \tag{2.3}$$

**Definition 2.4.** An SPCT $\tau$ is said to be a *source tableau* if, for every $i \notin \mathrm{Des}(\tau)$ where $i \neq n$, we have that $i+1$ lies to the immediate left of $i$.

Each class $E \in \mathcal{E}^\sigma(\alpha)$ has a unique source tableau $\tau_E$ and $\mathbf{S}^\sigma_{\alpha,E}$ is generated by $\tau_E$.

# 3 Structures of $\mathbf{S}^\sigma_\alpha$

## 3.1 The indecomposability of $\mathbf{S}^\sigma_\alpha$

Recall the decomposition (2.3). When $\sigma = \mathrm{id}$, König [8] proved that $\mathbf{S}^{\mathrm{id}}_{\alpha,E}$ is indecomposable for every $E \in \mathcal{E}^{\mathrm{id}}(\alpha)$. We extend this result to all $\sigma \in \mathfrak{S}_{\ell(\alpha)}$.

**Theorem 3.1** ([4, Theorem 3.1])**.** *Let $\alpha \models n$ and $\sigma \in \mathfrak{S}_{\ell(\alpha)}$. For every $E \in \mathcal{E}^\sigma(\alpha)$, the $H_n(0)$-module $\mathbf{S}^\sigma_{\alpha,E}$ is indecomposable.*

We say that $\mathbf{S}^\sigma_\alpha$ is *SPCT-cyclic* if it is generated by a single SPCT. Combining (2.3) with Theorem 3.1 shows that $\mathbf{S}^\sigma_\alpha$ is SPCT-cyclic if and only if it is indecomposable. In case where $\sigma = \mathrm{id}$, the classification of these modules was done by Tewari and van Willigenburg. They call a composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ *simple* if there exists an integer $i < k < j$ such that $\alpha_k = \alpha_j - 1$ whenever $\alpha_i \geq \alpha_j \geq 2$ and $1 \leq i < j \leq \ell$.

**Theorem 3.2** ([12, Theorem 7.6])**.** $\mathbf{S}_\alpha^{\mathrm{id}}$ *is indecomposable if and only if $\alpha$ is a simple composition.*

We will extend this theorem to all $\sigma \in \mathfrak{S}_{\ell(\alpha)}$.

**Definition 3.3.** Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be compatible with $\sigma$ for $\sigma \in \mathfrak{S}_\ell$.

(a) For $i < j$, if $\sigma(i) < \sigma(j)$ and $\alpha_i \geq \alpha_j \geq 2$, then $(i, j)$ is called *a permutation-ascending composition-descending (PACD) pair attached to the pair* $(\alpha\,;\sigma)$.

(b) We say that $\alpha$ is *$\sigma$-simple* if every PACD pair $(i, j)$ attached to $(\alpha\,;\sigma)$ satisfies one of the following conditions:

**C1.** There exists $i < k < j$ such that $\sigma(i) < \sigma(k) < \sigma(j)$ and $\alpha_k = \alpha_j - 1$.
**C2.** There exists $k > j$ such that $\sigma(i) < \sigma(k) < \sigma(j)$ and $\alpha_k = \alpha_j$.

If $\sigma = \mathrm{id}$, then the terminology "$\sigma$-simple" is identical to "simple". For example, if $\alpha = (3, 1, 2, 3)$, then one can see that $\alpha$ is compatible with $\sigma \in \{1234, 2134, 3124, 4123\}$. In case where $\sigma = 1234$ or $4123$, one sees that $\alpha$ is $\sigma$-simple. Otherwise, that is, $\sigma = 2134$ or $3124$, $\alpha$ is not $\sigma$-simple. With this preparation, we can state the following classification theorem.

**Theorem 3.4** ([4, Theorem 3.12 and Corollary 3.14])**.**

(a) *For $\alpha \models n$ and $\sigma \in \mathfrak{S}_{\ell(\alpha)}$, $\mathbf{S}_\alpha^\sigma$ is SPCT-cyclic if and only if $\alpha$ is $\sigma$-simple.*

(b) *Let $w_0$ be the longest element in $\mathfrak{S}_\ell$. For a composition $\alpha$ of length $\ell$, the $H_n(0)$-module $\mathbf{S}_\alpha^{w_0}$ is SPCT-cyclic if and only if $\alpha$ is a partition of $n$.*

For $\alpha \models n$ and $\sigma \in \mathfrak{S}_{\ell(\alpha)}$, consider the tableau whose $\alpha_{\sigma^{-1}(i)}$th row $(1 \leq i \leq k)$ consists of the entries $\sum_{j=0}^{i-1} \alpha_{\sigma^{-1}(j)} + 1\,, \sum_{j=0}^{i-1} \alpha_{\sigma^{-1}(j)} + 2\,, \ldots\,, \sum_{j=0}^{i-1} \alpha_{\sigma^{-1}(j)} + \alpha_{\sigma^{-1}(i)}$ in the decreasing order from left to right. Here $\alpha_{\sigma^{-1}(0)}$ is set to be 0. Let $C$ be the equivalence class containing this tableau, denoted by $\tau_C$. By definition, $\tau_C$ is a source tableau. Let us call $C$ the *canonical class* in $\mathcal{E}^\sigma(\alpha)$, $\tau_C$ the *canonical source tableau of shape $\alpha$ and type $\sigma$*, and $\mathbf{S}_{\alpha,C}^\sigma$ the *canonical submodule* of $\mathbf{S}_\alpha^\sigma$. It follows from Theorem 3.4 that $\mathbf{S}_\alpha^\sigma = \mathbf{S}_{\alpha,C}^\sigma$ for every $\sigma$-simple composition $\alpha$.

## 3.2 Characteristic relations and the quasi-Schur expansion of $\mathrm{ch}([\mathbf{S}_\alpha^\sigma])$

It was shown in [12] that the image of the quasisymmetric characteristic of $\mathbf{S}_\alpha^{\mathrm{id}}$ turns out to be the *quasisymmetric Schur function* $\mathcal{S}_\alpha$ introduced by Haglund *et al.*, that is,

$$\mathrm{ch}([\mathbf{S}_\alpha^{\mathrm{id}}]) = \sum_{\tau \in \mathrm{SPCT}^{\mathrm{id}}(\alpha)} F_{\mathrm{comp}(\tau)} = \mathcal{S}_\alpha.$$

However, in case where $\sigma \neq \mathrm{id}$, $\mathrm{ch}([\mathbf{S}^\sigma_\alpha])$ has not been well studied yet. In this subsection we provide a recursive relation for $\mathrm{ch}([\mathbf{S}^\sigma_\alpha])$. Using this, we expand $\mathrm{ch}([\mathbf{S}^\sigma_\alpha])$ in terms of quasisymmetric Schur functions.

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, we define a right $H_n(0)$-action on $\mathbb{C}[\mathbb{Z}^n]$ as follows: For each $1 \leq i \leq n-1$,

$$\alpha \bullet \pi_i = \begin{cases} \alpha \cdot s_i & \text{if } \alpha_i < \alpha_{i+1}, \\ \alpha & \text{otherwise.} \end{cases} \tag{3.1}$$

Here $\alpha \cdot s_i$ is obtained from $\alpha$ by swapping $\alpha_i$ and $\alpha_{i+1}$. Obviously this action restricts to the $\mathbb{C}$-span of compositions of length $n$.

Suppose that there is an index $1 \leq i \leq \ell(\alpha) - 1$, which will be fixed in the rest of this subsection, such that $\ell(\sigma s_i) < \ell(\sigma)$. By the $\sigma$-compatibility of $\alpha$, we have that $\alpha_i \geq \alpha_{i+1}$. Define a map $\psi^{(i)} : \mathrm{SPCT}^\sigma(\alpha) \to \coprod_{\alpha = \beta \bullet \pi_i} \mathrm{SPCT}^{\sigma s_i}(\beta)$ as follows:

- In case where $\tau_{i+1,j} < \tau_{i,j+1}$ for all $1 \leq j \leq \alpha_i$, define $\psi^{(i)}(\tau)$ to be the SPCT of shape $\alpha \cdot s_i$ and type $\sigma s_i$ obtained from $\tau$ by swapping the $i$th row and the $(i+1)$st row.
- Otherwise, let $j_0$ be the smallest integer such that $\tau_{i+1,j_0} > \tau_{i,j_0+1}$. In this case, define $\psi^{(i)}(\tau)$ to be the SPCT of shape $\alpha$ and type $\sigma s_i$ obtained from $\tau$ by swapping $\tau_{i,j}$ and $\tau_{i+1,j}$ for all $1 \leq j \leq j_0$.

**Theorem 3.5** ([4, Theorem 4.7 and Corollary 4.8]). *Let $\alpha \models n$ and $\sigma \in \mathfrak{S}_{\ell(\alpha)}$.*

(a) *If there is $1 \leq i \leq \ell(\alpha) - 1$ such that $\ell(\sigma s_i) < \ell(\sigma)$, then* $\mathrm{ch}([\mathbf{S}^\sigma_\alpha]) = \sum_{\alpha = \beta \bullet \pi_i} \mathrm{ch}([\mathbf{S}^{\sigma s_i}_\beta])$.

(b) $\mathrm{ch}([\mathbf{S}^\sigma_\alpha]) = \sum_{\alpha = \beta \bullet \pi_\sigma} \mathcal{S}_\beta$.

(c) *If $\alpha$ is a partition and $w_0$ is the longest element in $\mathfrak{S}_{\ell(\alpha)}$, then* $\mathrm{ch}([\mathbf{S}^{w_0}_\alpha]) = s_\alpha$. *Here $s_\alpha$ is the Schur function attached to $\alpha$.*

We close this subsection by providing a new $\mathbb{Z}$-basis of $\mathrm{QSym}_n$, the $n$th homogeneous component of QSym. For a partition $\lambda$ of $n$, let $(\mathfrak{S}_{\ell(\lambda)})_\lambda$ be the stabilizer subgroup of $\mathfrak{S}_{\ell(\lambda)}$ with respect to $\lambda$, that is, $(\mathfrak{S}_{\ell(\lambda)})_\lambda = \{\sigma \in \mathfrak{S}_{\ell(\lambda)} \mid \lambda \cdot \sigma = \lambda\}$. Denote by $\mathbf{I}_\lambda$ the set of minimal length coset representatives of $\mathfrak{S}_{\ell(\lambda)} / (\mathfrak{S}_{\ell(\lambda)})_\lambda$. Set

$$\mathbf{B}_n := \bigcup_{\lambda \vdash n} \left\{ \mathrm{ch}([\mathbf{S}^\sigma_\lambda]) \mid \sigma \in \mathbf{I}_\lambda \right\} .$$

**Theorem 3.6** ([4, Corollary 4.9]). *$\mathbf{B}_n$ is a $\mathbb{Z}$-basis of* $\mathrm{QSym}_n$.

**Remark 3.7.** It was shown in [1] the dual immaculate function $\mathfrak{S}^*_\alpha$ in [2] expands positively in terms of quasisymmetric Schur functions. Based on amount of Sage experiments we conjecture that $\mathfrak{S}^*_\alpha$ also expands positively in terms of the elements of $\mathbf{B}_n$.

## 3.3 The projective cover of $\mathbf{S}^{\sigma}_{\alpha,E}$ for an arbitrary class $E$

Let $R$ be a left artin ring and $A, B$ be finitely generated $R$-modules. An epimorphism $f : A \to B$ is called an *essential epimorphism* if a morphism $g : X \to A$ is an epimorphism whenever $f \circ g : X \to B$ is an epimorphism. A *projective cover* of $A$ is a pair $(P, f)$ of a projective $R$-module $P$ and an essential epimorphism $f : P \to A$. It plays an extremely important role in understanding the structure of $A$. The goal of this subsection is to find the projective cover of $\mathbf{S}^{\sigma}_{\alpha,E}$ for all $E \in \mathcal{E}^{\sigma}(\alpha)$.

Let $\tau_{E}$ be the source tableau of $E$ for $E \in \mathcal{E}^{\sigma}(\alpha)$. Let $\mathrm{Des}(\tau_{E}) = \{d_1 < d_2 < \cdots < d_m\}$, $d_0 := 0$, and $d_{m+1} := n$. We construct a generalized composition $\boldsymbol{\alpha}_E$ from $\tau_{E}$ as follows:

(i) Let $\boldsymbol{\alpha}^{(1)} = (1^{d_1})$.

(ii) For $1 < j \le m+1$, define

$$\boldsymbol{\alpha}^{(j)} = \begin{cases} \boldsymbol{\alpha}^{(j-1)} \odot (1^{d_j - d_{j-1}}) & \text{if } d_{j-1}+1 \text{ is weakly right of } d_{j+1} \text{ or} \\ & d_{j-1}+1 \text{ and } d_{j+1} \text{ are attacking in } \tau_{E}, \\ \boldsymbol{\alpha}^{(j-1)} \oplus (1^{d_j - d_{j-1}}) & \text{otherwise.} \end{cases} \tag{3.2}$$

(iii) Denote by $\boldsymbol{\alpha}_E := \boldsymbol{\alpha}^{(m+1)}$.

**Example 3.8.** Let $E$ and $F$ be the equivalence classes of the source tableaux

$$\tau = \begin{array}{|c|c|c|} \hline 1 & & \\ \hline 5 & 4 & 3 \\ \hline 7 & 6 & 2 \\ \hline \end{array} \qquad \text{and} \qquad \gamma = \begin{array}{|c|c|} \hline 4 & 3 \\ \hline 5 & 2 \\ \hline 1 & \\ \hline \end{array},$$

respectively. The entries in red denote descents. Applying (3.2), one can see that $\boldsymbol{\alpha}_E = (1) \oplus (2,1,2,1)$ and $\boldsymbol{\alpha}_F = (1) \oplus (2,2)$.

For $1 \le j \le m+1$, define $\mathtt{H}_j$ to be the horizontal strip in $\mathtt{cd}(\alpha)$ consisting of boxes occupied by the entries from $d_{j-1}+1$ to $d_j$ in $\tau_{E}$. For $T \in \mathrm{SRT}(\boldsymbol{\alpha}_E)$, let $\tau_T$ be the filling of $\mathtt{cd}(\alpha)$ such that $\mathtt{H}_j$ is filled with the entries of the $j$th column of $T$ in the decreasing order beginning from $j = 1$ and ending at $j = m+1$. Define a $\mathbb{C}$-linear map $\eta : \overline{\mathbf{P}}_{\boldsymbol{\alpha}_E} \to \mathbf{S}^{\sigma}_{\alpha,E}$ by

$$\eta(T) = \begin{cases} \tau_T & \text{if it is contained in } E, \\ 0 & \text{otherwise.} \end{cases}$$

**Example 3.9.** Continuing Example 3.8, let

$$T_0 = \begin{array}{cccc} & & & \boxed{6} \\ & & \boxed{3} & \boxed{7} \\ & & \boxed{4} & \\ & \boxed{2} & \boxed{5} & \\ \boxed{1} & & & \end{array}, \quad T_1 = \begin{array}{cccc} & & & \boxed{6} \\ & & \boxed{2} & \boxed{7} \\ & & \boxed{4} & \\ & \boxed{1} & \boxed{5} & \\ \boxed{3} & & & \end{array}, \quad T_2 = \begin{array}{cccc} & & & \boxed{6} \\ & & \boxed{2} & \boxed{7} \\ & & \boxed{3} & \\ & \boxed{1} & \boxed{5} & \\ \boxed{4} & & & \end{array} \in \mathrm{SRT}(\boldsymbol{\alpha}_E).$$

Then $\eta(T_0) = \tau$ and $\eta(T_1) = \begin{array}{ccc} 3 & & \\ 5 & 4 & 2 \\ 7 & 6 & 1 \end{array} \in E$. While, $\eta(T_2) = 0$ since $\tau_{T_2} = \begin{array}{ccc} 4 & \times & \\ 5 & 3 & 2 \\ 7 & 6 & 1 \end{array}$ is not an SPCT.

The map $\eta : \overline{\mathbf{P}}_{\boldsymbol{\alpha}_E} \to \mathbf{S}^{\sigma}_{\alpha,E}$ turns out to be a surjective $H_n(0)$-homomorphism. But it is far from being obvious that it is an essential epimorphism.

**Theorem 3.10** ([3, Theorem 5.11]). *For an arbitrary class $E \in \mathcal{E}^{\sigma}(\alpha)$, $(\overline{\mathbf{P}}_{\boldsymbol{\alpha}_E}, \eta)$ is the projective cover of $\mathbf{S}^{\sigma}_{\alpha,E}$.*

# 4 Connections to other $H_n(0)$-modules arising from tableaux

## 4.1 Standard immaculate tableaux and standard extended tableaux

**Definition 4.1.** Let $\alpha \models n$. A *standard immaculate tableau* (SIT) of shape $\alpha$ is a filling $\mathscr{T}$ of the composition diagram $\mathtt{cd}(\alpha)$ with $\{1, 2, \ldots, n\}$ such that the entries are all distinct, the entries in each row increase from left to right, and the entries in the first column increase from top to bottom.

We denote the set of all standard immaculate tableaux of shape $\alpha$ by $\mathrm{SIT}(\alpha)$. For each $i = 1, \ldots, n-1$, the action of $\pi_i \in H_n(0)$ on the $\mathbb{C}$-span of $\mathrm{SIT}(\alpha)$ is defined by

$$\pi_i \cdot \mathscr{T} = \begin{cases} \mathscr{T} & \text{if } i \text{ is weakly below } i+1 \text{ in } \mathscr{T}, \\ 0 & \text{if } i \text{ and } i+1 \text{ are in the first column of } \mathscr{T}, \\ s_i \cdot \mathscr{T} & \text{otherwise.} \end{cases} \tag{4.1}$$

Denote by $\mathcal{V}_\alpha$ the resulting $H_n(0)$-module. Let $\mathscr{T}_0 \in \mathrm{SIT}(\alpha)$ be the SIT obtained by filling $\mathtt{cd}(\alpha)$ with entries $1, 2, \ldots, n$ from left to right and from top to bottom.

**Theorem 4.2** ([2, Theorem 3.12]). *For $\alpha \models n$, $\mathcal{V}_\alpha$ is an indecomposable $H_n(0)$-module generated by $\mathscr{T}_0$ whose quasisymmetric characteristic is the dual immaculate function $\mathfrak{S}^*_\alpha$.*

**Definition 4.3.** Given $\alpha \models n$, a *standard extended tableau* (SET) of shape $\alpha$ is a filling $\mathsf{T}$ of the reverse composition diagram $\mathtt{cd}(\alpha^{\mathrm{r}})$ with $\{1, 2, \ldots, n\}$ such that the entries are all distinct, the entries in each row increase from left to right, and the entries in each column decrease from top to bottom.

We denote the set of all standard extended tableaux of shape $\alpha$ by $\mathrm{SET}(\alpha)$. For each $i = 1, \ldots, n-1$, the action of $\pi_i \in H_n(0)$ on the $\mathbb{C}$-span of $\mathrm{SET}(\alpha)$ is defined by

$$\pi_i \cdot \mathsf{T} = \begin{cases} \mathsf{T} & \text{if } i \text{ is strictly left of } i+1 \text{ in } \mathsf{T}, \\ 0 & \text{if } i \text{ and } i+1 \text{ are in the same column of } \mathsf{T}, \\ s_i \cdot \mathsf{T} & \text{if } i \text{ is strictly right of } i+1 \text{ in } \mathsf{T}. \end{cases} \tag{4.2}$$

Let $X_\alpha$ denote the resulting $H_n(0)$-module. Let $\mathsf{T}_0 \in \mathrm{SET}(\alpha)$ be the SET obtained by filling $\mathtt{rcd}(\alpha)$ with entries $1, 2, \ldots, n$ from left to right and from bottom to top.

**Theorem 4.4** ([11, Theorem 3.13]). *For $\alpha \models n$, $X_\alpha$ is an indecomposable $H_n(0)$-module generated by $\mathsf{T}_0$ whose quasisymmetric characteristic is the extended Schur function attached to $\alpha$.*

## 4.2 Connections among $\overline{\mathbf{P}}_\alpha$, $\mathcal{V}_\alpha$, $X_\alpha$ and $\mathbf{S}^\sigma_\alpha$

First, we define a $\mathbb{C}$-linear map $\Phi : \overline{\mathbf{P}}_{\alpha^c} \to \mathcal{V}_\alpha$ by $\Phi(T) = \mathscr{T}_T$ if $\mathscr{T}_T$ is an SIT, 0 otherwise. Here $\mathscr{T}_T$ is the filling of $\mathtt{cd}(\alpha)$ given by $(\mathscr{T}_T)_{i,j} = T^j_i$, the entry at the $j$th box from the top of the $i$th column from the left in $T$. Then the map $\Phi : \overline{\mathbf{P}}_{\alpha^c} \to \mathcal{V}_\alpha$ is a surjective $H_n(0)$-module homomorphism ([3, Theorem 3.2]).

Second, we define a $\mathbb{C}$-linear map $\Gamma : \mathcal{V}_\alpha \to X_\alpha$ by $\Gamma(\mathscr{T}) = \mathsf{T}_\mathscr{T}$ if $\mathsf{T}_\mathscr{T}$ is an SET, 0 otherwise. Here $\mathsf{T}_\mathscr{T}$ is the filling of $\mathtt{cd}(\alpha^{\mathrm{r}})$ given by $(\mathsf{T}_\mathscr{T})_{i,j} = \mathscr{T}_{\ell(\alpha)+1-i,j}$, the entry at the box in row $\ell(\alpha)+1-i$ from top to bottom and column $j$ from left to right in $\mathscr{T}$. Then the map $\Gamma : \mathcal{V}_\alpha \to X_\alpha$ is a surjective $H_n(0)$-module homomorphism ([3, Theorem 3.5]).

Let us recall the automorphism $\phi : H_n(0) \to H_n(0)$ defined by $\phi(\pi_i) = \pi_{w_0 s_i w_0} = \pi_{n-i}$ for all $1 \le i \le n-1$ (see [6]). Given an $H_n(0)$-module $M$, it induces another $H_n(0)$-action $\diamond$ on the vector space $M$ given by $\pi_i \diamond v := \phi(\pi_i) \cdot v$ for $1 \le i \le n-1$. We denote the resulting $H_n(0)$-module by $\phi[M]$ and call it *the $\phi$-twist of $M$*.

For any $\sigma \in \mathfrak{S}_{\ell(\alpha)}$ with $\widetilde{\alpha} = \alpha^{\mathrm{r}} \cdot \sigma$, we define a $\mathbb{C}$-linear map $\Upsilon : X_\alpha \to \phi[\mathbf{S}^\sigma_{\widetilde{\alpha},C}]$ by $\Upsilon(\mathsf{T}) = \tau_\mathsf{T}$ if $\tau_\mathsf{T} \in C$, 0 otherwise. Here the filling $\tau_\mathsf{T}$ of $\mathtt{cd}(\widetilde{\alpha})$ is obtained by $(\tau_\mathsf{T})_{i,j} = n+1-\mathsf{T}_{\sigma(i),j}$, that is, $n+1$ minus the entry at the box in row $\sigma(i)$ and column $j$ in $\mathsf{T}$. Then, for $\sigma \in \mathfrak{S}_{\ell(\alpha)}$ satisfying that $\widetilde{\alpha} = \alpha^{\mathrm{r}} \cdot \sigma$, the map $\Upsilon : X_\alpha \to \phi[\mathbf{S}^\sigma_{\widetilde{\alpha},C}]$ is a surjective $H_n(0)$-module homomorphism ([3, Theorem 3.9]).

Finally, we define a $\mathbb{C}$-linear map $\Psi^{(i)} : \mathbf{S}^\sigma_\alpha \to \mathbf{S}^{\sigma s_i}_{\alpha \cdot s_i}$ by $\Psi^{(i)}(\tau) = \psi^{(i)}(\tau)$ if $\psi^{(i)}(\tau) \in \mathrm{SPCT}^{\sigma s_i}(\alpha \cdot s_i)$, 0 otherwise. Here $\psi^{(i)}(\tau)$ is defined in Subsection 3.2. Let $\Psi^{(i)}_C = \mathbf{pr}_C \circ \Psi^{(i)} \circ \iota_C$, where $\iota_C : \mathbf{S}^\sigma_{\alpha,C} \to \mathbf{S}^\sigma_\alpha$ is the natural inclusion map and $\mathbf{pr}_C : \mathbf{S}^{\sigma s_i}_{\alpha \cdot s_i} \to \mathbf{S}^{\sigma s_i}_{\alpha \cdot s_i, C}$ is the natural projection. Then, for $1 \le i \le \ell(\alpha) - 1$ such that $\ell(\sigma s_i) < \ell(\sigma)$, the map $\Psi^{(i)}_C : \mathbf{S}^\sigma_{\alpha,C} \to \mathbf{S}^{\sigma s_i}_{\alpha \cdot s_i, C}$ is a surjective $H_n(0)$-module homomorphism ([3, Theorem 4.2]).

**Theorem 4.5** (cf. [3, Section 4 and 5]). *Let $\sigma \in \mathfrak{S}_{\ell(\alpha)}$ be a permutation satisfying $\widetilde{\alpha} = \alpha^{\mathrm{r}} \cdot \sigma$ and $s_{i_1} \cdots s_{i_k}$ any reduced expression of $\sigma$. Then we have the following series of surjections:*

$$\overline{\mathbf{P}}_{\alpha^c} \xrightarrow{\ \Phi\ } \mathcal{V}_\alpha \xrightarrow{\ \Gamma\ } X_\alpha \xrightarrow{\ \Upsilon\ } \phi[\mathbf{S}^\sigma_{\widetilde{\alpha},C}] \xrightarrow{\ \Psi^{(i_k)}_C\ } \phi[\mathbf{S}^{\sigma s_{i_k}}_{\widetilde{\alpha} \cdot s_{i_k},C}] \xrightarrow{\ \Psi^{(i_{k-1})}_C\ } \cdots \xrightarrow{\ \Psi^{(i_1)}_C\ } \phi[\mathbf{S}^{\mathrm{id}}_{\alpha^{\mathrm{r}},C}]$$

*In particular, if $\alpha$ is a partition, then we have*

$$\overline{\mathbf{P}}_{\alpha^c} \xrightarrow{\ \Phi\ } \mathcal{V}_\alpha \xrightarrow{\ \Gamma\ } X_\alpha \overset{\Upsilon}{\cong} \phi[\mathbf{S}^{w_0}_\alpha] \xrightarrow{\ \Psi^{(i_k)}_C\ } \phi[\mathbf{S}^{w_0 s_{i_k}}_{\alpha \cdot s_{i_k},C}] \xrightarrow{\ \Psi^{(i_{k-1})}_C\ } \cdots \xrightarrow{\ \Psi^{(i_1)}_C\ } \phi[\mathbf{S}^{\mathrm{id}}_{\alpha^{\mathrm{r}},C}].$$

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