Séminaire Lotharingien de Combinatoire **85B** (2021)
Article #70, 12 pp.

Proceedings of the 33rd Conference on Formal Power Series and Algebraic Combinatorics (Ramat Gan)

Orbit harmonics and cyclic sieving: a survey

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Abstract. Orbit harmonics is a tool in combinatorial representation theory which promotes the (ungraded) action of a linear group G on a finite set X to a graded action of G on a polynomial ring quotient. The cyclic sieving phenomenon is a notion in enumerative combinatorics which encapsulates the fixed-point structure of the action of a finite cyclic group C on a finite set X in terms of root-of-unity evaluations of an auxiliary polynomial $X(q)$. In this survey, we present a variety of sieving results obtained by applying orbit harmonics.

Keywords: orbit harmonics, cyclic sieving, point locus

1 Introduction

We survey recent works ([16], [14], [15]) on the application of orbit harmonics to cyclic sieving phenomena. The orbit harmonics comes from the idea to model a finite set X geometrically as a finite point locus in a complex space. The relevant algebra has roots in (at least) the work of Kostant [11]. The idea of providing CSPs using orbit harmonics has been implicitly used in [2], [5], and [20]. Rhoades and the author gave this idea in an explicit and unified way in [16]. The author applied this idea to the complex reflection group $G(r, 1, n)$ to obtain sieving results concerning twisted rotations on colored words [14]. Extending the work in [16] to diagonal orbit harmonics, the author has provided a 'tri-CSP' for matrices [15] by using the work of Garsia and Haiman [7].

Let X be a finite set with an action of a finite cyclic group $C = \langle c \rangle$ and $\omega = \exp(2\pi i/|C|)$. Let $X(q) \in \mathbb{Z}_{\geq 0}[q]$ be a polynomial with nonnegative integer coefficients. The triple $(X, C, X(q))$ *exhibits the cyclic sieving phenomenon* [19] if for all $r \geq 0$ we have

$$|X^{c^r}| = |\{x \in X : c^r \cdot x = x\}| = X(\omega^r) = [X(q)]_{q=\omega^r}. \tag{1.1}$$

More generally, if we have a finite set X with a action of product $C_1 \times \cdots \times C_m$ of m cyclic groups, and a polynomial $X(q_1, \ldots, q_m)$ of m variables satisfy similar condition in Equation (1.1), then we say the triple $(X, C_1 \times \cdots \times C_m, X(q_1, \ldots, q_m))$ *exhibits the m-ary cyclic sieving phenomenon* (biCSP for $m = 2$ and triCSP for $m = 3$).

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Although ostensibly in the domain of enumerative combinatorics, the most desired proofs of CSPs are algebraic. In this survey, we present a systematic way to prove sieving results using orbit harmonics and provide various instances of CSPs.

The remainder of this survey is organized as follows. In **Section 2** we give background. In **Section 3**, we describe how orbit harmonics gives a new perspective on the classical results of Springer and Morita-Nakajima. In **Section 4** we provide a sieving generating theorem (Theorem 4.1) and apply this to various combinatorial loci to prove instances of CSPs. In **Section 5**, we consider diagonal orbit harmonics and obtain a new sieving result regarding $(q,t)-$Kostka polynomials and enumeration of matrices with certain symmetries.

2 Background

2.1 Symmetric Functions

We denote by $\Lambda = \bigoplus_{d\geq 0}\Lambda_d$ the graded ring of symmetric functions in an infinite variable set $\mathbf{x} = (x_1, x_2, \dots)$ over the ground field $\mathbb{C}(q,t)$. Here Λ_d consists of symmetric functions of homogeneous degree d. Two important elements of Λ_d are the *homogeneous* and *elementary* symmetric functions

$$h_d(\mathbf{x}) := \sum_{i_1\leq\cdots\leq i_d} x_{i_1}\cdots x_{i_d} \qquad \text{and} \qquad e_d(\mathbf{x}) := \sum_{i_1<\cdots<i_d} x_{i_1}\cdots x_{i_d}.$$

By restricting $h_d(\mathbf{x})$ and $e_d(\mathbf{x})$ to a finite variable set $\mathbf{x}_n = \{x_1, \dots, x_n\}$, we obtain the homogeneous and elementary symmetric polynomials $h_d(\mathbf{x}_n)$ and $e_d(\mathbf{x}_n)$.

Bases of Λ_n are indexed by partitions of n. For a partition $\lambda \vdash n$, we let

$$h_\lambda(\mathbf{x}), \quad e_\lambda(\mathbf{x}), \quad s_\lambda(\mathbf{x}), \quad \widetilde{Q}_\lambda(\mathbf{x};q) \quad \text{and} \quad \widetilde{H}_\lambda(\mathbf{x};q,t)$$

denote the associated *homogeneous symmetric function*, *elementary symmetric function*, *Schur function*, *Hall–Littlewood symmetric function*, and *Macdonald symmetric function*. For any partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ of n the h-, e- and Schur functions are defined by

$$h_\lambda(\mathbf{x}) := \prod_{i\geq 1} h_{\lambda_i}(\mathbf{x}), \quad e_\lambda(\mathbf{x}) := \prod_{i\geq 1} e_{\lambda_i}(\mathbf{x}) \quad \text{and} \quad s_\lambda(\mathbf{x}) := \sum_{T\in \mathrm{SSYT}(\lambda)} \mathbf{x}^T,$$

where $\mathrm{SSYT}(\lambda)$ denotes the set of semistandard tableaux of shape λ. The (modified) Macdonald polynomial $\widetilde{H}_\lambda(\mathbf{x};q,t)$ are the basis of Λ_n defined by so called *triangularity* and *normalization* axioms (see [9] for example). The (modified) Hall–Littlewood symmetric function $\widetilde{Q}_\lambda(\mathbf{x};q)$ is given by a specialization $\widetilde{Q}_\lambda(\mathbf{x};q) = \widetilde{H}_\lambda(\mathbf{x};0,q)$.

2.2 Representation theory of $\mathfrak{S}_n$

Irreducible representations of the symmetric group $\mathfrak{S}_n$ are in one-to-one correspondence with partitions $\lambda \vdash n$. We let S^λ denote the irreducible module corresponding to λ. If V is any finite-dimensional $\mathfrak{S}_n$-module, there are unique multiplicities c_λ so that $V \cong \bigoplus_{\lambda \vdash n} c_\lambda S^\lambda$. The *Frobenius image* of V is the symmetric function

$$\mathrm{Frob}(V) := \sum_{\lambda \vdash n} c_\lambda s_\lambda(\mathbf{x}).$$

If V is graded (or bigraded) $\mathfrak{S}_n$-module as $V = \bigoplus_{d \geq 0} V_d$ (or $V = \bigoplus_{d,e \geq 0} V_{d,e}$) the graded Frobenius image is the symmetric function over $\mathbb{C}(q,t)$ given by

$$\mathrm{grFrob}(V;q) := \sum_{d \geq 0} \mathrm{Frob}(V_d) q^d \quad (\text{or } \mathrm{grFrob}(V;q,t) := \sum_{d,e \geq 0} \mathrm{Frob}(V_{d,e}) q^d t^e).$$

2.3 Complex reflection groups

A finite subgroup $W \subseteq \mathrm{GL}_n(\mathbb{C})$ is a *reflection group* if it is generated by reflections. A complex reflection group W acts on the polynomial ring $\mathbb{C}[\mathbf{x}_n] := \mathbb{C}[x_1, \ldots, x_n]$ by linear substitutions. Let $\langle \mathbb{C}[\mathbf{x}_n]_+^W \rangle \subseteq \mathbb{C}[\mathbf{x}_n]$ be the ideal generated by the subspace of W-invariants with vanishing constant term. The *coinvariant ring* attached to W is the quotient $R_W := \mathbb{C}[\mathbf{x}_n] / \langle \mathbb{C}[\mathbf{x}_n]_+^W \rangle$. The ring R_W also has a graded W-module structure.

For any irreducible W-module U, the *fake degree polynomial* $f^U(q)$ is the graded multiplicity of U in the coinvariant ring. That is, we define

$$f^U(q) := \sum_{d \geq 0} m_{U,d} q^d$$

where $m_{U,d}$ is the multiplicity of U in the degree d piece $(R_W)_d$ of R_W.

An element $c \in W$ is *regular* if it possesses an eigenvector $v \in \mathbb{C}^n$ which has full W-orbit. Such an eigenvector v is called a *regular eigenvector*. For example when $W = \mathfrak{S}_n$, an element in W is a regular if and only if it is a power of an n-cycle or an $(n-1)$-cycle.

2.4 Orbit harmonics

Let $X \subseteq \mathbb{C}^n$ be a finite set of points which is closed under the action of $W \times C$ where $W \subseteq \mathrm{GL}_n(\mathbb{C})$ is a (finite) complex reflection group, and C is a finite cyclic group acting on $\mathbb{C}^n$ by root-of-unity scaling. Let $\mathbf{I}(X) := \{f \in \mathbb{C}[\mathbf{x}_n] : f(v) = 0 \text{ for all } v \in X\}$ be the ideal of polynomials in $\mathbb{C}[\mathbf{x}_n]$ which vanish on X. Since X is finite, Lagrange Interpolation affords a $\mathbb{C}$-algebra isomorphism

$$\mathbb{C}[X] \cong \mathbb{C}[\mathbf{x}_n] / \mathbf{I}(X) \tag{2.1}$$

where $\mathbb{C}[X]$ is the algebra of all functions $X \to \mathbb{C}$. Since X is $W \times C$-stable, (2.1) is also an isomorphism of ungraded $W \times C$-modules.

For any nonzero polynomial $f \in \mathbb{C}[\mathbf{x}_n]$, let $\tau(f)$ be the highest degree component of f. That is, if $f = f_d + \cdots + f_1 + f_0$ with f_i homogeneous of degree i and $f_d \neq 0$, we set $\tau(f) := f_d$. Given our locus X with ideal $\mathbf{I}(X)$ as above, we define a homogeneous ideal $\mathbf{T}(X)$ by

$$\mathbf{T}(X) := \langle \tau(f) \,:\, f \in \mathbf{I}(X),\ f \neq 0 \rangle \subseteq \mathbb{C}[\mathbf{x}_n].$$

The ideal $\mathbf{T}(X)$ is the *associated graded* ideal of $\mathbf{I}(X)$ and is often denoted $\operatorname{gr}\mathbf{I}(X)$. From the construction, $\mathbf{T}(X)$ is homogeneous and stable under $W \times C$. The isomorphism (2.1) extends to an isomorphism of $W \times C$-modules

$$\mathbb{C}[X] \cong \mathbb{C}[\mathbf{x}_n]/\mathbf{I}(X) \cong \mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X) \tag{2.2}$$

where $\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X)$ has the additional structure of a graded $W \times C$-module on which C acts by scaling in each fixed degree.

The procedure $X \rightsquigarrow \mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X)$ which promotes the (ungraded) locus X to the graded module $\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X)$ is known as *orbit harmonics*. The terminology comes from the seminal work of Garsia and Haiman [6] in which they studied point sets that were 'orbits' of reflection group actions where there is a correspondence between coinvariant and 'harmonic' space. The general procedure of $X \rightsquigarrow \mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X)$ is known as *deformation to the normal cone* and this also can be applied to prove sieving results as noted in Vic Reiner's talk [18]. Nevertheless, we prefer to call this procedure orbit harmonics because we choose finite set X to be orbits of reflection group actions.

3 Theorems of Springer and Morita–Nakajima

Before applying orbit harmonics to prove sieving results, we state representation-theoretic results of Springer [21] and Morita-Nakajima [13], which will be useful in our combinatorial work. We explain how orbit harmonics may be used to prove these results.

3.1 Springer's theorem on regular elements

Let $W \subseteq GL_n(\mathbb{C})$ be a complex reflection group and let $c \in W$ be a regular element with regular eigenvector $v \in \mathbb{C}^n$ whose eigenvalue is $\omega \in \mathbb{C}^\times$. Let $C = \langle c \rangle$ be the cyclic subgroup of W generated by c. We regard the coinvariant ring R_W as a graded $W \times C$-module, where W acts by linear substitutions and the generator $c \in C$ sends each variable x_i to ωx_i.

Theorem 3.1 (Springer [21])**.** *Consider the action of $W \times C$ on W given by $(u, c^r) \cdot w := uwc^{-r}$. Then $\mathbb{C}[W]$ is isomorphic to R_W as an ungraded $W \times C$-module.*

Proof. We describe an argument of Kostant [11] using orbit harmonics. We let C act on $\mathbb{C}^n$ by the rule $c \circ v' := \omega^{-1} v'$ for all $v' \in \mathbb{C}^n$. The corresponding action of C on $\mathbb{C}[\mathbf{x}_n]$ by linear substitutions sends x_i to ωx_i for all i. We may regard $\mathbb{C}^n$ as a $W \times C$-module in this way.

Define the *Springer locus* to be the W-orbit of the regular eigenvector v of c:

$$W \cdot v := \{w \cdot v : w \in W\} \subseteq \mathbb{C}^n$$

The locus $W \cdot v$ is closed under the action of W and $\circ$-action of C. We may regard $W \cdot v$ as a $W \times C$-set. The regularity of v shows that the map $w \mapsto w \cdot v$ furnishes a $W \times C$-equivariant bijection

$$W \xrightarrow{\sim} W \cdot v$$

where the action of $W \times C$ on W is as in Theorem 3.1.

Chevalley [4] proved that there exist algebraically independent W-invariant polynomials $f_1, \ldots, f_n$ of homogeneous positive degree such that $\mathbb{C}[\mathbf{x}_n]^W = \mathbb{C}[f_1, \ldots, f_n]$. Furthermore, we have isomorphisms of ungraded W-modules

$$R_W = \mathbb{C}[\mathbf{x}_n]/\langle f_1, \ldots, f_n \rangle \cong \mathbb{C}[W].$$

The W-invariance of f_i implies that $f_i - f_i(v) \in \mathbf{I}(W \cdot v)$ and taking the top degree component gives $f_i \in \mathbf{T}(W \cdot v)$. On the other hand,

$$\dim(\mathbb{C}[\mathbf{x}_n]/\langle f_1, \ldots, f_n \rangle) = \dim \mathbb{C}[W] = |W| = |W \cdot v| = \dim(\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(W \cdot v)),$$

so we have $\langle f_1, \ldots, f_n \rangle = \mathbf{T}(W \cdot v)$.

Finally, orbit harmonics furnishes isomorphisms of ungraded $W \times C$-modules

$$R_W = \mathbb{C}[\mathbf{x}_n]/\langle f_1, \ldots, f_n \rangle = \mathbb{C}[\mathbf{x}_n]/\mathbf{T}(W \cdot v) \cong \mathbb{C}[W \cdot v] \cong \mathbb{C}[W]$$

where the last isomorphism used the $W \times C$-equivariant bijection (3.1). □

3.2 A theorem of Morita–Nakajima via orbit harmonics

In this subsection we consider the case of the symmetric group $W = \mathfrak{S}_n$. We fix a weak composition $\mu = (\mu_1, \ldots, \mu_k)$ of n with k parts where $\mu_i = \mu_{i+a}$ for all i with subscripts interpreted modulo k. Let c be a fixed generator of the cyclic group $\mathbb{Z}_{k/a}$. Morita and Nakajima proved [13] a variant of Springer's theorem as follows.

Let W_μ be the family words $w_1 \ldots w_n$ over the alphabet $[k]$ in which the letter i appears μ_i times. The set W_μ carries an action of $\mathfrak{S}_n \times \mathbb{Z}_{k/a}$ where $\mathfrak{S}_n$ acts by subscript permutation and $\mathbb{Z}_{k/a}$ acts by $c : w_1 \cdots w_n \mapsto (w_1 + a) \cdots (w_n + a)$ where letter values are interpreted modulo k. Extending by linearity, the space $\mathbb{C}[W_\mu]$ is a $\mathfrak{S}_n \times \mathbb{Z}_{k/a}$-module.

Let $I_\mu \subseteq \mathbb{C}[\mathbf{x}_n]$ be the *Tanisaki ideal* attached to the composition μ and let $R_\mu := \mathbb{C}[\mathbf{x}_n]/I_\mu$ be the corresponding *Tanisaki quotient ring*. The ring R_μ is a graded $\mathfrak{S}_n$-module which has several descriptions ([22], [8]). One way to describe this module is via orbit harmonics as follows.

Let $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ be distinct complex numbers. We have $I_\mu = \mathbf{T}(W_\mu)$ as ideals in $\mathbb{C}[\mathbf{x}_n]$ by considering the set $W_\mu \subseteq \mathbb{C}^n$ as a point locus under the following identification

$$w_1 \ldots w_n \leftrightarrow (\alpha_{w_1}, \ldots, \alpha_{w_n}).$$

The orbit harmonics interpretation of R_μ was used by Garsia and Procesi [8] to derive

$$\mathrm{grFrob}(R_\mu; q) = \widetilde{Q}_\mu(\mathbf{x}; q).$$

Let $\omega := \exp(2a\pi i/k)$ be a primitive $(k/a)^{th}$ root-of-unity. We extend the graded $\mathfrak{S}_n$-action on R_μ to a graded $\mathfrak{S}_n \times \mathbb{Z}_{k/a}$-action by letting the distinguished generator $c \in \mathbb{Z}_{k/a}$ scale by ω^d in homogeneous degree d.

Theorem 3.2. *(Morita-Nakajima [13, Theorem 13]) We have an isomorphism of ungraded $\mathfrak{S}_n \times \mathbb{Z}_{k/a}$-modules*

$$\mathbb{C}[W_\mu] \cong R_\mu.$$

The proof in [13] involves tricky symmetric function manipulations involving the Hall–Littlewood polynomials $\widetilde{Q}_\mu(\mathbf{x}; q)$ when q is a root of unity, and relies on further intricate symmetric function identities due to Lascoux-Leclerc-Thibon [12]. Orbit harmonics gives an easier and more conceptual proof.

4 Cyclic sieving generating theorem

The 'generating theorem' for sieving results in [16] is as follows. The heart of the proof comes from the isomphism (2.2) and Springer's theorem (Theorem 3.1).

Theorem 4.1. *Let $W \subseteq GL_n(\mathbb{C})$ be a complex reflection group, $C' = \langle c' \rangle$ be the subgroup of W generated by a regular element c', and $\omega := \exp(2\pi i/k)$. Let $C = \langle c \rangle$ be a cyclic group of order k and consider the action of $W \times C$ on $\mathbb{C}^n$ where c scales by ω and W acts by left multiplication. Let $X \subseteq \mathbb{C}^n$ be a finite point set that is closed under the action of $W \times C$.*

1. *Suppose that the isomorphism type of the degree $d \geq 0$ piece of $\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X)$ is given by*

$$(\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X))_d \cong \bigoplus_{U \in \mathrm{Irr}(W)} U^{\oplus m_{U,d}}.$$

The triple $(X, C \times C', X(q,t))$ exhibits the bicyclic sieving phenomenon where

$$X(q,t) = \sum_{U \in \mathrm{Irr}(W)} \sum_{d \geq 0} m_{U,d} q^d f^{U^*}(t).$$

2. *Let $G \subseteq W$ be a subgroup. The set X/G of G-orbits in X carries a natural C-action and the triple $(X/G, C, X(q))$ exhibits the cyclic sieving phenomenon where*

$$X(q) = \mathrm{Hilb}((\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X))^G; q).$$

Remark 4.2. *Note that if we have*

$$\mathrm{grFrob}(\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X); q) = \sum_{\lambda \vdash n} c_\lambda(q) s_\lambda,$$

then the Hilbert series of G-invariant space for $W = \mathfrak{S}_n$, for $G = C_n$ a cyclic group generated by a long cycle $(1, 2, \ldots, n)$, and for $G = H_{n/2}$ a hyperoctahedral group, respectively are given by

$$\begin{aligned} \mathrm{Hilb}((\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X)^{\mathfrak{S}_n}; q) &= c_{(n)}(q), \\ \mathrm{Hilb}((\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X))^{C_n}; q) &= \sum_{\lambda \vdash n} c_\lambda(q) a_{\lambda,0}, \\ \mathrm{Hilb}((\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X))^{H_{n/2}}; q) &= \sum_{\substack{\lambda \vdash n \\ \lambda: even}} c_\lambda(q), \end{aligned}$$

where $a_{\lambda,0}$ is the number of standard tableaux of shape λ with major index 0 modulo n.

4.1 The functional loci

The loci considered in this section correspond to arbitrary, injective, and surjective functions between finite sets inspired by Rota's *Twelvefold Way*.

Definition 4.3. *Given integers n and k, set $\omega := \exp(2\pi i/k)$. We define the following three point sets in $\mathbb{C}^n$ closed under the action of $\mathfrak{S}_n \times \mathbb{Z}_k$, where $\mathbb{Z}_k$ scales by ω:*

$$\begin{aligned} X_{n,k} &:= \{(a_1, \ldots, a_n) \,:\, a_i \in \{\omega, \omega^2, \ldots, \omega^k\}\} \\ Y_{n,k} &:= \{(a_1, \ldots, a_n) \in X_{n,k} \,:\, a_1, \ldots, a_n \text{ are distinct}\} \\ Z_{n,k} &:= \{(a_1, \ldots, a_n) \in X_{n,k} \,:\, \{a_1, \ldots, a_n\} = \{\omega, \omega^2, \ldots \omega^k\}\} \end{aligned}$$

The table below summarizes combinatorial objects for sieving results obtained by exploiting Theorem 4.1 of each functional locus $X_{n,k}$, $Y_{n,k}$, and $Z_{n,k}$.

	$X_{n,k}$	$Y_{n,k}$	$Z_{n,k}$
$G = \mathfrak{S}_n$	$WComp(n,k)$	$\binom{[n]}{k}$	$Comp(n,k)$
$G = C_n$	necklaces	necklaces with distinct letters	necklaces with all letters used
$G = H_{\frac{n}{2}}$	graphs	graphs with degree ≤ 1	graph without an isolated vertex

4.2 Other combinatorial loci

4.2.1 Springer locus

In this subsection we return to the setting of an arbitrary complex reflection group $W \subseteq \mathrm{GL}_n(\mathbb{C})$ acting on $\mathbb{C}^n$. We fix a regular element $c \in W$ with regular eigenvector $v \in \mathbb{C}^n$ and corresponding regular eigenvalue $\omega \in \mathbb{C}$, so that $c \cdot v = \omega v$. We also let $C := \langle c \rangle$ be the subgroup of W generated by c.

Let W-orbit $W \cdot v = \{w \cdot v : w \in W\} \subseteq \mathbb{C}^n$ be the Springer locus. Subsection 3.1 shows that the Springer locus is closed under the action of the group $W \times C$, where W acts by its natural action on $\mathbb{C}^n$ and C acts by the rule $c : v' \mapsto \omega v'$ for all $v' \in \mathbb{C}^n$. (Note that this is different from the $\circ$-action $c \circ v' := \omega^{-1} v'$ of C considered in Subsection 3.1.) By the discussion in Subsection 3.1 and Theorem 4.1, we have the following.

Theorem 4.4. *Let $c, c' \in W$ be regular elements and let $C = \langle c \rangle, C' = \langle c' \rangle$ be the cyclic subgroups which they generate. The product of cyclic groups $C \times C'$ acts on W by the rule $(c, c') \cdot w := c'wc$. The triple $(W, C \times C', W(q,t))$ exhibits the bicyclic sieving phenomenon where*

$$W(q,t) := \sum_{U} f^{U}(q) f^{U^*}(t)$$

and the sum is over all (isomorphism classes of) irreducible W-modules U.

Theorem 4.4 is a result of Barcelo, Reiner, and Stanton [3, Thm. 1.4]. In [3] the polynomial $W(q,t)$ is referred to as a *bimahonian distribution*.

4.2.2 The Tanisaki Locus

Throughout this subsection, we fix a weak composition $\mu = (\mu_1, \ldots, \mu_k)$ of n into k parts which satisfies $\mu_i = \mu_{i+a}$ for all i, where indices are interpreted modulo k. If $\omega := \exp(2\pi i/k)$, define the *Tanisaki locus* $X_\mu \subseteq \mathbb{C}^n$ by

$$X_\mu := \{(\alpha_1, \ldots, \alpha_n) : \alpha_j = \omega^i \text{ for precisely } \mu_i \text{ values of } j\}.$$

As discussed in Subsection 3.2, Garsia-Procesi [8] proved that $\mathbf{T}(X_\mu)$ is the Tanisaki ideal and $\mathrm{grFrob}(\mathbb{C}[\mathbf{x}_n]/\mathbf{T}(X_\mu); q) = \widetilde{Q}_\mu(\mathbf{x}; q)$. We have the following bicyclic sieving result.

Theorem 4.5. *Let W_μ be the set of length n words $w_1 \ldots w_n$ of content μ. The set W_μ carries an action of $\mathbb{Z}_n \times \mathbb{Z}_{k/a}$, where $\mathbb{Z}_n$ acts by word rotation and $\mathbb{Z}_{k/a}$ acts by adding a to each letter modulo k. The triple $(W_\mu, \mathbb{Z}_n \times \mathbb{Z}_{k/a}, X(q,t))$ exhibits the bicyclic sieving phenomenon, where*

$$X_\mu(q,t) = \sum_{\lambda \vdash n} \widetilde{K}_{\lambda, \mathrm{sort}(\mu)}(q) f^{\lambda}(t)$$

and $\mathrm{sort}(\mu)$ *is the partition obtained by sorting the parts of μ in weakly decreasing order.*

Theorem 4.5 was proven in the unpublished work of Reiner and White. A proof of Theorem 4.5 in full generality using Theorem 3.2 is in [20].

4.2.3 Colored locus

The complex reflection group $G(r,1,n) \leq \mathrm{GL}_n(\mathbb{C})$ is a group of $n \times n$ monomial matrices whose nonzero entries are ζ^i for some i, where $\zeta := e^{\frac{2\pi i}{r}} \in \mathbb{C}$. Irreducible representations of $G(r,1,n)$ are in one-to-one correspondence with r-tuple $\lambda^\bullet = (\lambda^{(0)}, \ldots, \lambda^{(r-1)})$ of partitions with total size $|\lambda^{(0)}| + \cdots + |\lambda^{(r-1)}| = n$. We denote the irreducible representation corresponding to $\lambda^\bullet$ by $S^{\lambda^\bullet}$. Let $\Lambda^{G(r,1,n)} := \Lambda^{\otimes r}$ be r^{th} tensor of the symmetric function ring Λ (with variables $x^{(i)}$'s). For any finite-dimensional $G(r,1,n)$-module V, there extists unique multiplicity $c_{\lambda^\bullet}$ for each $\lambda^\bullet$ so that $V \cong \bigoplus_{\lambda^\bullet \vdash n} c_{\lambda^\bullet} S^{\lambda^\bullet}$. The *Frobenius image* of V is defined by

$$\mathrm{Frob}^{G(r,1,n)}(V) := \sum_{\lambda^\bullet \vdash n} c_{\lambda^\bullet} s_{\lambda^\bullet}(\mathbf{x}) \in \Lambda^{G(r,1,n)},$$

where $s_{\lambda^\bullet}(\mathbf{x}) = s_{\lambda^{(0)}}(\mathbf{x}^{(0)}) \cdots s_{\lambda^{(r-1)}}(\mathbf{x}^{(r-1)})$. We define $\mathrm{grFrob}^{G(r,1,n)}$ in a usual way.

Suppose that a finite set $X \subseteq \mathbb{C}^n$ is invariant under the action of $\mathfrak{S}_n \times C_k$, where $\mathfrak{S}_n$ acts on X by permuting coordinates and a generator $c \in C_k$ acts on X by k^{th} root of unity scaling. Then a *r-colored version* of X,

$$\mathrm{Col_r}(X) := \{(\zeta^{c_1} x_1^{\frac{1}{r}}, \ldots, \zeta^{c_n} x_n^{\frac{1}{r}}\} : (x_1, \ldots, x_n) \in X, c_1, \ldots, c_n \in \{0, 1, \ldots, r-1\}\}$$

is invariant under action of $G(r,1,n) \times C_{kr}$, where $G(r,1,n)$ acts by left multiplication and C_{kr} acts by scaling a kr^{th} root of unity. Then we have the following equivalence as ungraded $G(r,1,n) \times C_{kr}$-modules

$$\mathbb{C}[\mathrm{Col_r}(X)] \cong \mathbb{C}[x_1, \ldots, x_n] / \mathrm{Col_r}(T(X)),$$

where $\mathrm{Col_r}(T(X))$ is the image of $T(X)$ under r^{th} power ring homomorphism given by $x_k \mapsto x_k^r$. If we can calculate graded Frobenius image of $\mathbb{C}[\mathbf{x}_n] / \mathrm{Col_r}(T(X))$, then applying Theorem 4.1 gives a sieving result involving 'colored words'.

For example, let $X = \{(1, 1, \ldots, 1)\} \subseteq \mathbb{C}^n$ be a set with a single element (with $\mathfrak{S}_n \times C_1$-action). Then $\mathrm{Col}_2(X) = \{(a_1, \ldots, a_n) | a_i \in \{-1, 1\}\}$ can be regarded as the set $\mathrm{BW_n}$ of binary words of length n. By a graded Frobenius character formula (Proposition 9 [17]) and usual fact for a plethystic substitution, we have

$$\begin{aligned}\mathrm{grFrob}^{G(2,1,n)}(\mathbb{C}[\mathbf{x}_n] / \mathrm{Col}_2(T(X)); q) &= \mathrm{grFrob}(\mathbb{C}[\mathbf{x}_n] / T(X); q^2)[\mathbf{x}^{(0)} + q\mathbf{x}^{(1)}] \\ &= s_{(n)}[\mathbf{x}^{(0)} + q\mathbf{x}^{(1)}] = \sum_{k=0}^{n} q^k s_{(n-k)}(\mathbf{x}^{(0)}) s_{(k)}(\mathbf{x}^{(1)}).\end{aligned}$$

By Theorem 4.1 and a formula for fake degree polynomial $f^{\lambda^\bullet}(q)$ for $G(r,1,n)$ in [3], we conclude that the polynomial

$$\mathrm{BW_n}(q,t) := \sum_{k=0}^{n} q^k t^k \begin{bmatrix} n \\ k \end{bmatrix}_{t^2}$$

provides a biCSP for binary words where one cyclic group acts by twisted rotation $(a_1, \dots, a_n) \mapsto (-a_n, a_1, \dots, a_{n-1})$ and the other cyclic group $\mathbb{Z}_2$ acts by $(a_1, \dots, a_n) \mapsto (-a_1, \dots, -a_n)$. These two actions reflect the action of a regular element of $G(r, 1, n)$ and the action of scaling a root of unity on $\mathrm{Col}_2(X)$. This gives a desired representation theoretic proof of the sieving result for twisted rotation on binary words in [1]. More examples concerning sieving results coming from orbits of $G(r, 1, n)$ will be provided in [14].

5 Diagonal orbit harmonics and cyclic sieving

In this section, we consider diagonal orbit harmonics. To be precise, suppose a finite set $X \subseteq \mathbb{C}^{2n}$ is invariant under the action of $\mathfrak{S}_n \times C \times C'$, where $\mathfrak{S}_n$ acts diagonally, i.e.

$$\sigma \cdot (x_1, \dots, x_n, y_1, \dots, y_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)}, y_{\sigma(1)}, \dots, y_{\sigma(n)}),$$

and the cyclic group C acts on x coordinates by scaling a root of unity and the cyclic group C' acts on y coordinates by scaling a root of unity. Let $\mathbf{I}(X)$ be the vanishing ideal in $\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]$ and $\mathbf{T}(X)$ be the homogeneous ideal obtained by applying top degree homogeneous part in both x and y variables,

$$\mathbf{T}(X) := \langle \tau_y(\tau_x(f)) : f \in \mathbf{I}(X), f \neq 0 \rangle.$$

Then we have the following equivalence as $W \times C \times C'$-modules.

$$\mathbb{C}[X] \cong \mathbb{C}[\mathbf{x}_n, \mathbf{y}_n] / \mathbf{T}(X) \tag{5.1}$$

A sieving generating theorem for diagonal orbit harmonics can be obtained by using a similar argument in the proof of Theorem 4.1 [15, Thm. 3.1]. We provide an application of this idea.

To each partition μ of n, one associates a bigraded $\mathfrak{S}_n$-module called the *Garsia–Haiman module* as follows. An *injective tableau* T of shape $\mu \vdash n$ is a filling of cells of μ by integers $1, 2, \dots, n$. For each injective tableau T, we assign a point $p_T \in \mathbb{C}^{2n}$ by letting i^{th} and $(n+i)^{th}$ coordinates of p_T record the position of i in T:

$$p_T = (\omega^{x_T(1)}, \dots, \omega^{x_T(n)}, \zeta^{y_T(1)}, \dots, \zeta^{y_T(n)}),$$

where $x_T(i)$ and $y_T(i)$ are x and y coordinates of i in T (in a French notation), ω is $\ell(\mu)^{th}$ root of unity and ζ is μ_1^{th} root of unity. The point locus X_μ possesses a natural $\mathfrak{S}_n$ action that acts diagonally on X_μ. Combining results in [7] and [10], we can conclude that the graded Frobenius image of $\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n] / \mathbf{T}(X_\mu)$ is the Macdonald polynomial $\widetilde{H}_\mu(\mathbf{x}; q, t)$.

To provide an instance of CSP, consider a rectangular partition $\mu = (a^b)$. The set $X_\mu \subseteq \mathbb{C}^{2n}$ has not only $\mathfrak{S}_n$ action but also $C \times C'$ action, where a cyclic group C of order a acts by scaling ω to the first n coordinates and a cyclic group C' of order b acts scaling ζ to the last n coordinates. Note that the set $X_{(a^b)}$ can be identified with the set of $b \times a$ matrices where each of $1, 2, \ldots, ab$ is used once as entry. Moreover, under this correspondence, $\mathfrak{S}_n \times C \times C'$ action on $X_{(a^b)}$ corresponds to the permutation action on entries, rotation action on columns, and rotation action on rows. Applying isomorphism (5.1) gives

$$\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]/\mathbf{T}(X_{(a^b)}) \cong \mathbb{C}[X_{(a^b)}]$$

as $\mathfrak{S}_n \times C \times C'$ modules and by applying Springer's theorem, we have the following.

Theorem 5.1. *Let $X_{(a^b)}$ be the set of $b \times a$ matrices of content of entries (1^{ab}). The product of cyclic groups $\mathbb{Z}_a \times \mathbb{Z}_b \times \mathbb{Z}_{ab}$ acts on X by column rotation, row rotation and adding 1 modulo ab to each entry. Then the triple $(X_{(a^b)}, \mathbb{Z}_a \times \mathbb{Z}_b \times \mathbb{Z}_{ab}, X_{(a^b)}(q,t,z))$ exhibits triCSP, where*

$$X_{(a^b)}(q,t,z) = \sum_{\lambda \vdash ab} \tilde{K}_{\lambda,(a^b)}(q,t) f^{\lambda}(z).$$

If we apply a similar argument in [20], one can obtain tri-cyclic sieving phenomena of $a \times b$ matrices of content ν with ν has a cyclic symmetry (See [15] for details).

Acknowledgements

The author is grateful to Brendon Rhoades for many helpful conversations and encouragement in writing this survey. We also thank referees for careful reading and helpful comments.

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