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Inset games and strategies

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Abstract. This paper proposes a new class of combinatorial 2-player games with calculatable winning strategy. Our games are obtained as patchwork of the Sato-Welter games. This paper gives the explicit formula of the Sprague-Grundy functions of the games.

Keywords: d-complete poset, inset, combinatorial games.

1 Introduction

Since the beginning of combinatorial game theory, i.e. the 1930's [7][3], winning strategies have been studied. In this paper, we focus on impartial games [2], which are a sort of 2-player games with perfect information. Winning strategies of impartial games are analyzed by the Sprague-Grundy value. Since the Sprague-Grundy value is recursively defined, it is very difficult to calculate the Sprague-Grundy value for most games. Therefore, finding large classes for which it is possible to calculate the Sprague-Grundy value is a very important issue. Typical examples of games possible to calculate the Sprague-Grundy value by "good" algorithm are:

- Nim. See [1][2] for details.
- Sato-Welter game. This game is introduced by M. Sato [6] and C. P. Welter [8] independently. See section 3 for definition and properties.
- Turning turtles. See section 4 for definition and properties. See [1] for further details.

In [4], N. Kawanaka generalize these examples to d-complete posets, which are introduced by R. A. Proctor [5]. In his setting,

- Sato-Welter game is a game on a shape (a Young diagram).
- Turning turtles is a game on a shifted shape (a shifted Young diagram).

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He has given a formula which calculates the Sprague-Grudy function over his games. Unfortunately, his formula does not give a formula for individual d-complete posets.

In this paper, we focus on the class of insets, which is one of the classes of d-complete posets. We define a game over insets — which we call the *inset games* — and a closed formula for their Sprague-Grundy function. Our games are costructed as patchworks of the Sato-Welter games.

This paper is organized as follows: Section 2 explains fundamental notions and definitions of games and the Sprague-Grundy values. Section 3 explains precise definition of the Sato-Welter games. Section 4 explains precise definition of the turning turtles. Section 5 explains our main results.

2 Preliminaries

2.1 Games

We begin by defining the games we concern with.

Definition 2.1. *Let P be a set, and $\to$ a binary relation over P. For an element $p \in P$, we put $\alpha(p) := \{ q \in P \mid p \to q \}$. The pair $(P; \to)$ is called a game if it satisfies:*

1. *For any $p \in P$, the set $\alpha(p)$ is finite;*
2. *There exists no infinite sequence*

$$p_0, p_1, p_2, p_3, \dots \quad p_i \in P$$

with

$$p_i \to p_{i+1}, \quad i = 0, 1, 2, 3, \dots$$

We call an element p of P a *position*, $\alpha(p)$ the *option set* at the position p. If $\alpha(p) = \varnothing$, then we say p is an *ending position*. Any position $p = p_0 \in P$ can be interpreted as an opening position of a 2-player game (in the usual sense of the word); two players alternatively choose positions:

$$\begin{array}{ll} p_0 \to p_1 & \text{(the first player's move)}, \\ p_1 \to p_2 & \text{(the second player's move)}, \\ p_2 \to p_3 & \text{(the first player's move)}, \\ & \cdots\cdots \end{array}$$

until one of them reaches an ending position p_n. If n is odd (resp. even), we say the first (resp. second) player *wins*. If $(P; \to)$ and $(Q; \to)$ are isomorphic to each other as digraphs, then we say $(P; \to)$ is game-isomorphic to $(Q; \to)$.

Example 2.2 (1-heap nim)**.** *Denote by* $\mathbb{N}$ *the set of nonnegative integers. Then, the pair* $(\mathbb{N}; >)$ *is a game, where* $>$ *denotes the ordinary order relation 'greater than'. This game is called the* 1-heap nim*.*

According to [7][3], we define:

Definition 2.3. *For a game* $(P; \to)$*, let* $\mathrm{SG} = \mathrm{SG}_P : P \to \mathbb{N}$ *be the map defined by*

$$\mathrm{SG}(p) = \min\left(\mathbb{N} \setminus \left\{ \mathrm{SG}(q) \in \mathbb{N} \,\middle|\, p \to q \right\}\right), \qquad (p \in P).$$

The map SG *is called the* Sprague-Grundy function *of* P*. The value* $\mathrm{SG}(p)$ *is called the* Sprague-Grundy number *(or* Sprague-Grundy value*) of* p*.*

Example 2.4 (1-heap nim)**.** *The Sprague-Grundy function* SG *of the* 1*-heap nim* $(\mathbb{N}; >)$ *is the identity map:*

$$\mathrm{SG}(x) = x \ \ (x \in \mathbb{N}).$$

Propsition 2.5 ([7, 3])**.** *Let* $(P; \to)$ *be a game and* $p \in P$*. Then we have:*

1. *If* $\mathrm{SG}_P(p) = 0$*, then, for any* $q \in \alpha(p)$*, we have* $\mathrm{SG}_P(q) > 0$*.*
2. *If* $\mathrm{SG}_P(p) > 0$*, then, for some* $q \in \alpha(p)$*, we have* $\mathrm{SG}_P(q) = 0$*.*
3. *If* p *is an ending position, then we have* $\mathrm{SG}_P(p) = 0$*.*

Remark 2.6. *For a position* $p \in P$*, the following two conditions are equivalent:*

1. *the position* p *has a winning strategy.*
2. $\mathrm{SG}_P(p) > 0$*.*

Indeed, if the first player is at the position p *with* $\mathrm{SG}_P(p) > 0$*. Then there exists a next position* $q \in \alpha(p)$ *with* $\mathrm{SG}_P(q) = 0$*. The strategic move* $p \to q$ *leads the first player to win, because any next position* $r \in \alpha(q)$ *chosen by the second player must be satisfying* $\mathrm{SG}_P(r) > 0$*.*

2.2 Nim-Addition

Denote by $\mathbb{Z}$ the set of integers. We shall write the binary expression of an integer $a \in \mathbb{Z}$ as

$$a = [a_i] = [a_i]_{i \in \mathbb{N}} = [\dots, a_i, \dots, a_3, a_2, a_1, a_0].$$

For example, we have:

$$\begin{aligned} 11 &= 1 + 2 + \ 0 \ + 2^3 + \ 0 \ + \ 0 \ + \cdots = [\cdots 001011], \\ -1 &= 1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + \cdots = [\cdots 111111], \\ -12 &= 0 + 0 + 2^2 + \ 0 \ + 2^4 + 2^5 + \cdots = [\cdots 110100]. \end{aligned}$$

We recall the definition of Nim-addition $\oplus$ in $\mathbb{Z}$. For $a = [a_i]$, $b = [b_i]$, and $c = [c_i]$ in $\mathbb{Z}$, we write

$$a \oplus b = c$$

if

$$a_i + b_i \equiv c_i \pmod 2, \qquad i \in \mathbb{N}.$$

For example, we have

$$3 \oplus 5 = [\cdots 00011] \oplus [\cdots 00101] = [\cdots 00110] = 6.$$

The system $(\mathbb{Z}; \oplus)$ forms an abelian group with

$$a \oplus a = 0, \quad \text{for any } a \in \mathbb{Z}.$$

Note that $\mathbb{N}$ is an index 2 subgroup of $(\mathbb{Z}; \oplus)$. We have

$$(-1) \oplus a = -a - 1.$$

Here, the symbol $-$ denotes the inverse on the usual addition (the binary operation $+$). For $a \in \mathbb{Z}$, we put

$$\mathrm{N}(a) = a \oplus (a-1).$$

We also put, for $a, b \in \mathbb{Z}$,

$$\big(a \,\big|\, b\big) = \mathrm{N}(a \oplus b).$$

We have the following ([2, ch.13], [6], [8]).

Lemma 2.7. *Let $a, b, c \in \mathbb{Z}$. We have:*

1. *If a is a multiple of 2^t $(t \in \mathbb{N})$, and is not a multiple of 2^{t+1}, then*

$$\mathrm{N}(a) = [\cdots 0\overbrace{11\cdots 1}^{t+1}] = 2^{t+1} - 1.$$

2. *$\mathrm{N}(a)$ is negative if and only if $a = 0$.*

3. *$\big(a \,\big|\, b\big) = \mathrm{N}(a - b)$.*

4. *$\big(a \,\big|\, b\big) = \big(a + c \,\big|\, b + c\big) = \big(a \oplus c \,\big|\, b \oplus c\big)$.*

5. *If $c > 0$ (resp. $c < 0$), then we have*

$$a \oplus \sum_{h=0}^{c-1}{}^{\oplus} \big(a \,\big|\, h\big) = a - c \quad \left(\text{resp. } a \oplus \sum_{h=c}^{-1}{}^{\oplus} \big(a \,\big|\, h\big) = a - c\right),$$

where the symbol $\sum^{\oplus}$ denotes Nim-summation.

2.3 Animating Functions

Following Conway [2], we call a function $f : \mathbb{Z} \to \mathbb{Z}$ of the form

$$f(x) = \cdots ((((x \oplus a) + b) \oplus c) + d) \oplus \cdots$$

an *animating function*. Clearly an animating function is a bijective map from $\mathbb{Z}$ to $\mathbb{Z}$, and its inverse is animating again. As is shown in [2], a function f is animating if and only if it can be written as

$$f(x) = x \oplus \sum_{i=1}^{r}{}^{\oplus} \left(x \,\middle|\, \alpha_i\right) \oplus \beta \tag{2.1}$$

with some $\alpha_i, \beta \in \mathbb{Z}$. Moreover, the expression (2.1) is unique as long as $\alpha_1, \alpha_2, \ldots, \alpha_r$ are distinct. We denote by $\mathrm{Anim}(\mathbb{Z})$ the set of animating functions.

Some of the fundamental properties of animating functions are listed in the following:

Lemma 2.8 (Sato [6] and Conway [2])**.** *Let $x, y \in \mathbb{Z}$.*

1. *If f and g are elements of $\mathrm{Anim}(\mathbb{Z})$, then the composition $f \circ g$ and the inverse f^{-1} are elements of $\mathrm{Anim}(\mathbb{Z})$. (Hence $(\mathrm{Anim}(\mathbb{Z}), \circ)$ forms a group.)*

2. *If f is an element of $\mathrm{Anim}(\mathbb{Z})$, then we have*

$$\left(f(x) \,\middle|\, f(y)\right) = \left(x \,\middle|\, y\right), \quad (x, y \in \mathbb{Z}).$$

3. *If $y = f(x)$ with $f(x)$ given by (2.1), then the inverse $x = f^{-1}(y)$ is given by*

$$f^{-1}(y) = y \oplus \sum_{i=1}^{r}{}^{\oplus} \left(y \,\middle|\, f(\alpha_i)\right) \oplus \beta.$$

Definition 2.9. *A multivariate function $E : \mathbb{Z}^n \to \mathbb{Z}$ is said to be* animating *if*

1. *$E_i(x_i) := E(x_1, \cdots, x_i, \cdots, x_n)$ is animating for each x_i,*

2. *E is symmetric in $x_1, \cdots, x_n$.*

The set of animating functions over $\mathbb{Z}^n$ is denoted by $\mathrm{Anim}(\mathbb{Z}^n)$.

3 The Sato-Welter Game

Definition 3.1. *Fix an integer $n \geq 1$. Put*

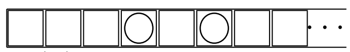
$$P_n := \left\{ \mathbf{x} \subseteq \mathbb{N} \;\middle|\; |\mathbf{x}| = n \right\}.$$

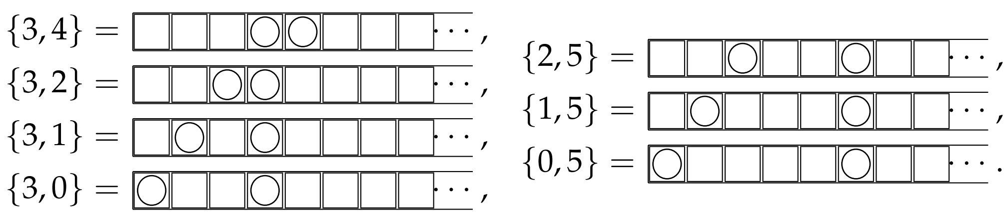
For $\mathbf{x}, \mathbf{y} \in P_n$, we denote $\mathbf{x} \rightarrow_{\mathrm{A}} \mathbf{y}$ if the following two coditions hold:

- $|\mathbf{x} \cap \mathbf{y}| = n - 1$;
- *if $x \in \mathbf{x} \setminus \mathbf{y}$ and $y \in \mathbf{y} \setminus \mathbf{x}$, then $y < x$.*

We call the game $(P_n; \rightarrow_{\mathrm{A}})$ the Sato-Welter game with n balls.

The Sato-Welter game with n balls can be visually interpreted as follows: n balls are lined up. At each move, a player moves one ball "◯" leftwards to any empty box. The player to make the last move wins.

Example 3.2. *For a position $\mathbf{x} = \{3, 5\} =$*  *$\in P_2$ of the Sato-Welter game of 2 balls, the elements of the option set $\alpha(\mathbf{x})$ are:*



Remark 3.3. *By regarding a position $\mathbf{x} \in P_n$ of the Sato-Welter game as a beta number, we can also regard $\mathbf{x}$ as Young diagrams. For exmaple, a position $\mathbf{x} = \{3, 5\} \in P_2$ is regarded as a Young diagram with partition $(4, 3)$.*

Remark 3.4. *The Sato-Welter game P_1 of 1 ball is game-isomorphic to the 1-heap nim.*

For $\mathbf{x} = \{x_1, x_2, \cdots, x_n\} \in P_n$, we put

$$\varphi_n(\mathbf{x}) := \sum_i^{\oplus} x_i \oplus \sum_{i<j}^{\oplus} \left(x_i \;\middle|\; x_j \right).$$

For $\mathbf{x} \in P_n$, we have $\varphi_n(\mathbf{x}) \geq 0$. Since φ is a symmetric function in $x_1, \cdots, x_n$, we denote $\varphi_n(\mathbf{x}) = \varphi_n(x_1, x_2, \cdots, x_n)$.

Theorem 3.5 (Sato [6], and Welter [8])**.** *The Sprague-Grundy function of the Sato-Welter game $(P_n; \rightarrow_{\mathrm{A}})$ is given by*

$$\varphi_n(\mathbf{x}), \quad \mathbf{x} \in P_n.$$

Example 3.6. *For $\mathbf{x} = \{3, 5\} \in P_2$, the Sprague-Grundy value $\varphi_2(\mathbf{x})$ is*

$$\varphi_2(\mathbf{x}) = 3 \oplus 5 \oplus \left(3 \;\middle|\; 5 \right) = 5 \neq 0.$$

Hence, the position $\mathbf{x}$ has a winning strategy. The (unique) winning move is:

$$\{3, 5\} \rightarrow_{\mathrm{A}} \{3, 2\}.$$

4 The Turning turtles

Definition 4.1. *Put*

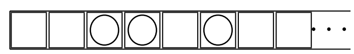
$$P_{odd} := \bigcup_{n:odd} P_n = \bigcup_{n:odd} \{\, \mathbf{x} \subseteq \mathbb{N} \mid |\mathbf{x}| = n \,\}. \tag{4.1}$$

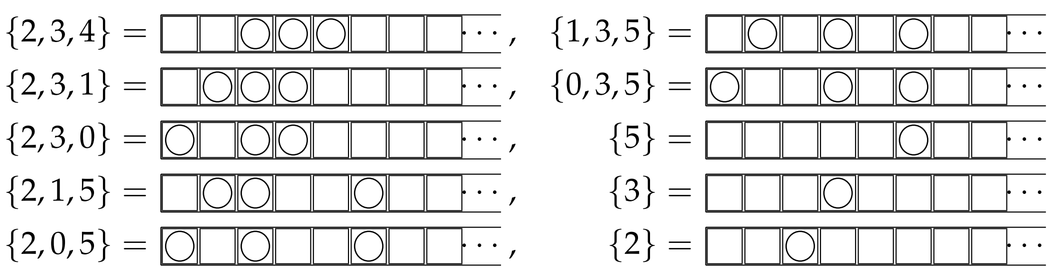
For $\mathbf{x} \in P_n$ *and* $\mathbf{y} \in P_m$*, we denote* $\mathbf{x} \rightarrow_{\mathrm{D}} \mathbf{y}$ *if*

- $n = m$ and $\mathbf{x} \rightarrow_{\mathrm{A}} \mathbf{y}$; *or*
- $n = m + 2$ and $\mathbf{x} \supset \mathbf{y}$.

We call the game $(P_{odd}; \rightarrow_{\mathrm{D}})$ *the* turning turtles*.*

The turning turtles can be visually interpreted as follows: Several turtles are lined up. An odd number of them are awake and the others sleeping. At each move, a player chooses two turtles with both hands and turns them over. The player is allowed to turn "◯"(awake turtle) into "empty"(sleeping turtle) and also "empty" into "◯". However, the turtle that he grabs with his right hand must be an awake turtle. The player to make the last move wins.

Example 4.2. *For a position* $\mathbf{x} = \{2, 3, 5\} =$  $\in P_{odd}$ *of the turning turtles, the elements of the option set* $\alpha(\mathbf{x})$ *are:*



Remark 4.3. *By regarding a position* $\mathbf{x} \in P_{odd}$ *of the turning tutles as a strict partition, we can also regard* $\mathbf{x}$ *as shifted Young diagrams. For exmaple, a position* $\mathbf{x} = \{2, 3, 5\} \in P_{odd}$ *is regarded as a shifted Young diagram with strict partition* $(5, 3, 2)$*.*

For $\mathbf{x} = \{x_1, x_2, \cdots, x_n\} \in P_{\mathrm{odd}}$, we put

$$\psi(\mathbf{x}) := x_1 \oplus x_2 \oplus \cdots \oplus x_n.$$

For $\mathbf{x} \in P_{\mathrm{odd}}$, we have $\psi(\mathbf{x}) \geq 0$. See [1] for further details.

Theorem 4.4. *The Sprague-Grundy function of the turning turltes* $(P_{odd}; \rightarrow_{\mathrm{D}})$ *is given by*

$$\psi(\mathbf{x}), \quad \mathbf{x} \in P_{odd}.$$

Example 4.5. *For* $\mathbf{x} = \{2, 3, 5\} \in P_{odd}$*, the Sprague-Grundy value* $\psi(\mathbf{x})$ *is*

$$\psi(\mathbf{x}) = 2 \oplus 3 \oplus 5 = 4 \neq 0.$$

Hence, the position $\mathbf{x}$ *has a winning strategy. The (unique) winning move is:*

$$\{2, 3, 5\} \rightarrow_{\mathrm{D}} \{2, 3, 1\}.$$

5 Main result

The definition 4.1 means the position set P_{odd} of turning turtles is a "patchwork" of position sets P_n of Sato-Welter games. This is our motivation of this study.

Definition 5.1. *Fix an integer $N \geq 2$. Put*

$$P_{N,1} := P_N \cup P_1.$$

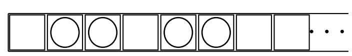
Let $n, m \in \{1, N\}$. For $\mathbf{x} \in P_n$ and $\mathbf{y} \in P_m$, we denote $\mathbf{x} \to \mathbf{y}$ if

- $n = m$ and $\mathbf{x} \to_{\mathrm{A}} \mathbf{y}$; *or*
- $n = N, m = 1$ and $\mathbf{x} \supset \mathbf{y}$.

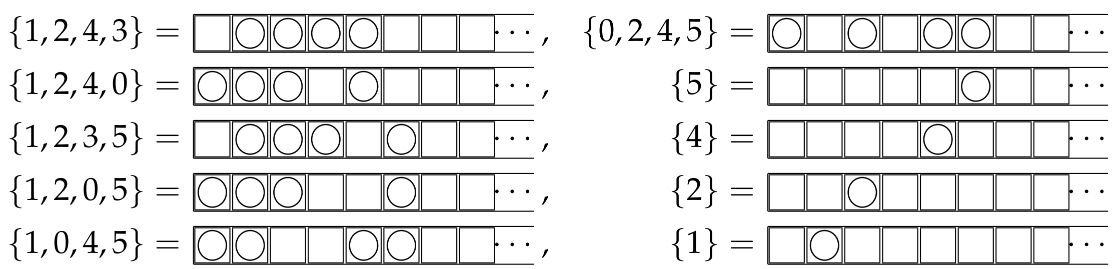
We call the game $(P_{N,1}; \to)$ the inset game.

Remark 5.2. *If $N = 2, 3$, then this game is not new (see subsection 5.1, 5.2). For $N \geq 4$, this game is new.*

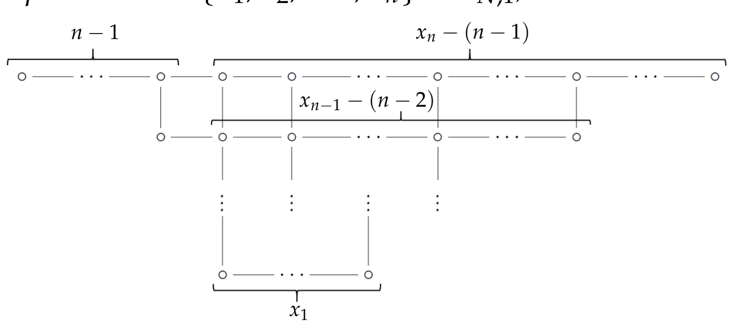
Example 5.3. *For a position*

$$\mathbf{x} = \{1,2,4,5\} =$$  $$\in P_{4,1}$$

of our game with $N = 4$, the elements of option set $\alpha(\mathbf{x})$ are:



Remark 5.4. *With a position $\mathbf{x} = \{x_1, x_2, \cdots, x_n\} \in P_{N,1}$, the inset*



can be associated, if $n = N$ and $x_1 < x_2 < \cdots < x_N$. If, on the other hand, $n = 1$, then we attach the x_1-chain.

For $\mathbf{x} = \{x_1, x_2, \cdots, x_n\} \in P_{N,1}$, we put

$$\varphi_{N,1}(\mathbf{x}) := \begin{cases} \sum_i^{\oplus} \left(-1 \,\middle|\, \varphi_{N-1}(\mathbf{x}^{(i)})\right) \oplus \varphi_N(\mathbf{x}) & n = N \\ x_1 & n = 1 \end{cases}$$

where $\mathbf{x}^{(i)} = \mathbf{x} \setminus \{x_i\}$. For $\mathbf{x} \in P_{N,1}$, we have

$$\varphi_{N,1}(\mathbf{x}) \geq 0. \tag{5.1}$$

Now we can state the main result:

Theorem 5.5. *The Sprague-Grundy function of the inset game $(P_{N,1}; \to)$ is given by*

$$\varphi_{N,1}(\mathbf{x}), \quad \mathbf{x} \in P_{N,1}.$$

A sketch of proof is given in the subsection 5.3.

Example 5.6. *For $\mathbf{x} = \{1, 2, 4, 5\} \in P_{4,1}$, the Sprague-Grundy value $\varphi_{4,1}(\mathbf{x})$ is*

$$\begin{aligned} \varphi_{4,1}(\mathbf{x}) &= \left(-1 \,\middle|\, \varphi_3(2,4,5)\right) \oplus \left(-1 \,\middle|\, \varphi_3(1,4,5)\right) \\ &\quad \oplus \left(-1 \,\middle|\, \varphi_3(1,2,5)\right) \oplus \left(-1 \,\middle|\, \varphi_3(1,2,4)\right) \oplus \varphi_4(1,2,4,5) \\ &= \left(-1 \,\middle|\, 0\right) \oplus \left(-1 \,\middle|\, 7\right) \oplus \left(-1 \,\middle|\, 1\right) \oplus \left(-1 \,\middle|\, 4\right) \oplus 6 = 10. \end{aligned}$$

Hence, the position $\mathbf{x}$ has a winning strategy. The (unique) winning move is:

$$\{1, 2, 4, 5\} \to \{0, 1, 4, 5\}.$$

5.1 Case $N = 2$

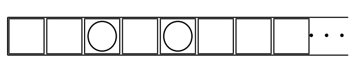
Define an injection $f : P_{2,1} \to P_2$ by

$$f(\{x_1, x_2\}) = \{x_1 + 1, x_2 + 1\}, \qquad f(\{x_1\}) = \{0, x_1 + 1\}.$$

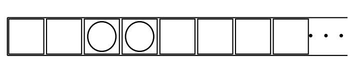
It is straightforward to see the case $N = 2$:

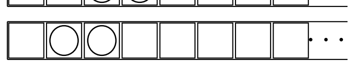
Propsition 5.7. *The map f is a game isomorphism from $P_{2,1}$ to the image $f(P_{2,1})$. In particular, for $\mathbf{x} \in P_{2,1}$, we have $\varphi_{2,1}(\mathbf{x}) = \varphi_2(f(\mathbf{x}))$.*

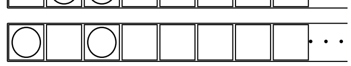
Example 5.8. *For a position*

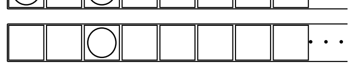
$\mathbf{x} = \{2, 4\} =$  $\in P_{2,1}$

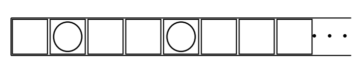
of our game with $N = 2$, the elements of option set $\alpha(\mathbf{x})$ are:

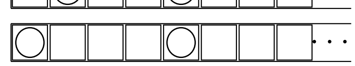
$\{2,3\} =$  ,

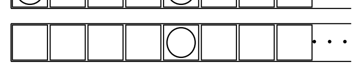
$\{2,1\} =$  ,

$\{2,0\} =$  ,

$\{2\} =$  ,

$\{1,4\} =$  ,

$\{0,4\} =$  ,

$\{4\} =$  .

These positions correspond to the positions in Example 3.2.

5.2 Case $N = 3$

Define an injection $f : P_{3,1} \to P_{\text{odd}}$ by

$$f(\{x_1, x_2, x_3\}) = \{x_1, x_2, x_3\}, \qquad f(\{x_1\}) = \{x_1\}.$$

It is straightforward to see the case $N = 3$:

Propsition 5.9. *The map f is a game isomorphism from $P_{3,1}$ to the image $f(P_{3,1})$. In particular, for $\mathbf{x} \in P_{3,1}$, we have $\varphi_{3,1}(\mathbf{x}) = \psi(f(\mathbf{x}))$.*

5.3 Sketch of proof of main theorem

We need the following two lemmata:

Lemma 5.10. *Let $\mathbf{x} \in P_{N,1}$ and $h \in \mathbb{N}$. Then we have*

1. *if $0 \le h < \varphi_{N,1}(\mathbf{x})$, then the number of $\mathbf{y} \in \alpha(\mathbf{x})$ with $\varphi_{N,1}(\mathbf{y}) = h$ is odd. In particular, there exists such a next position $\mathbf{y}$ of $\mathbf{x}$ with $\varphi_{N,1}(\mathbf{y}) = h$.*
2. *if $h \ge \varphi_{N,1}(\mathbf{x})$, then the number of $\mathbf{y} \in \alpha(\mathbf{x})$ with $\varphi_{N,1}(\mathbf{y}) = h$ is even (it may be zero).*

We omit the proof of Lemma 5.10.

Lemma 5.11. *Let $\mathbf{x} \in P_{N,1}$ and $\mathbf{y} \in \alpha(\mathbf{x})$. Then we have $\varphi_{N,1}(\mathbf{x}) \ne \varphi_{N,1}(\mathbf{y})$.*

Proof. Let $\mathbf{x} = \{x_1, \cdots, x_n\}$ and $\mathbf{y} = \{y_1, \cdots, y_m\}$.

Case $\mathbf{x} \in P_N$ and $\mathbf{y} \in P_N$: We may assume that $x_1 > y_1$ and $x_i = y_i$ $(2 \le i \le N)$. We have

$$\begin{aligned}
&\varphi_{N,1}(\mathbf{x}) \oplus \varphi_{N,1}(\mathbf{y}) \\
&= \sum_{i=2}^{N}{}^{\oplus} \begin{pmatrix} (-1 \mid \varphi_{N-1}(\{x_1, x_2, \cdots, x_N\} \setminus \{x_i\})) \\ \oplus (-1 \mid \varphi_{N-1}(\{y_1, x_2, \cdots, x_N\} \setminus \{x_i\})) \end{pmatrix} \oplus x_1 \oplus y_1 \oplus \sum_{i=2}^{N}{}^{\oplus} \left((x_1 \mid x_i) \oplus (y_1 \mid x_i) \right).
\end{aligned}$$

Suppose $\varphi_{N,1}(\mathbf{x}) = \varphi_{N,1}(\mathbf{y})$. Then we have

$$\begin{aligned}
&\sum_{i=2}^{N}{}^{\oplus} (-1 \mid \varphi_{N-1}(\{x_1, x_2, \cdots, x_N\} \setminus \{x_i\})) \oplus x_1 \oplus \sum_{i=2}^{N}{}^{\oplus} (x_1 \mid x_i) \\
&= \sum_{i=2}^{N}{}^{\oplus} (-1 \mid \varphi_{N-1}(\{y_1, x_2, \cdots, x_N\} \setminus \{x_i\})) \oplus y_1 \oplus \sum_{i=2}^{N}{}^{\oplus} (y_1 \mid x_i).
\end{aligned}$$

Put

$$g_i(x) := \varphi_{N-1}(\{x, x_2, \cdots, x_N\} \setminus \{x_i\}), \quad (2 \le i \le N).$$

Then g_i is an animating function. Put

$$f(x) := \sum_{i=2}^{N}{}^{\oplus} \left(-1 \,\middle|\, g_i(x)\right) \oplus x \oplus \sum_{i=2}^{N}{}^{\oplus} \left(x \,\middle|\, x_i\right).$$

Then f is an animating function and $f(x_1) = f(y_1)$. Since f is bijective, we have $x_1 = y_1$. This contradicts our assumption. Hence, we have $\varphi_{N,1}(\mathbf{x}) \neq \varphi_{N,1}(\mathbf{y})$.

Case $\mathbf{x} \in P_N$ and $\mathbf{y} \in P_1$: Put

$$f(x) := x \oplus \sum_{j=1}^{N}{}^{\oplus} \left(x \,\middle|\, x_j\right) \oplus \varphi_N(\mathbf{x}), \quad (x \in \mathbb{Z}).$$

Then f is animating and we have

$$f^{-1}(y) = y \oplus \sum_{i=1}^{N}{}^{\oplus} \left(y \,\middle|\, f(x_i)\right) \oplus \varphi_N(\mathbf{x}), \quad (y \in \mathbb{Z}) \tag{5.2}$$

by Lemma 2.8 (3). Since

$$\begin{aligned}
\left(-1 \,\middle|\, \varphi_{N-1}(\mathbf{x}^{(i)})\right) &= \left(\left(x_i \,\middle|\, x_i\right) \,\middle|\, x_i \oplus \sum_{j\neq i}{}^{\oplus} \left(x_i \,\middle|\, x_j\right) \oplus \varphi_N(\mathbf{x})\right) \\
&= \left(0 \,\middle|\, x_i \oplus \sum_{j=1}^{N}{}^{\oplus} \left(x_i \,\middle|\, x_j\right) \oplus \varphi_N(\mathbf{x})\right) \\
&= \left(0 \,\middle|\, f(x_i)\right),
\end{aligned}$$

we have

$$\varphi_{N,1}(\mathbf{x}) = 0 \oplus \sum_{i=1}^{N}{}^{\oplus} \left(0 \,\middle|\, f(x_i)\right) \oplus \varphi_N(\mathbf{x}). \tag{5.3}$$

Hence, by (5.2) and (5.3), we have $\varphi_{N,1}(\mathbf{x}) = f^{-1}(0)$. Therefore,

$$0 = f(\varphi_{N,1}(\mathbf{x})) = \varphi_{N,1}(\mathbf{x}) \oplus \sum_{i=1}^{N}{}^{\oplus} \left(\varphi_{N,1}(\mathbf{x}) \,\middle|\, x_i\right) \oplus \varphi_N(\mathbf{x}).$$

Since the first term (by (5.1)) and the last terms are nonnegative in the right hand side and x_i's are distinct, there exists no i such that $\varphi_{N,1}(\mathbf{x}) = x_i$.

Case $\mathbf{x} \in P_1$ and $\mathbf{y} \in P_1$: Then we have

$$\varphi_{N,1}(\mathbf{x}) = x_1 > y_1 = \varphi_{N,1}(\mathbf{y}).$$

Hence, we have $\varphi_{N,1}(\mathbf{x}) \neq \varphi_{N,1}(\mathbf{y})$. □

By Lemma 5.10, Lemma 5.11 and induction, we have

$$\mathrm{SG}_{P_{N,1}}(\mathbf{x}) = \varphi_{N,1}(\mathbf{x}), \quad (\mathbf{x} \in P_{N,1}).$$

This proves Theorem 5.5.

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