

Springer fibers and the Delta Conjecture at $t = 0$

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Abstract. We define a family of varieties $Y_{n,\lambda,s}$ generalizing the type A Springer fibers, whose cohomology rings have the structure of an S_n -module. We give an explicit presentation for the cohomology ring $H^*(Y_{n,\lambda,s}; \mathbb{Q})$, and we find an affine paving of $Y_{n,\lambda,s}$ that is in bijection with a collection of partial row-strict fillings of a partition shape. We also prove that the top cohomology groups of $Y_{n,\lambda,s}$ give a generalization of the type A Springer correspondence to the setting of induced Specht modules. Furthermore, the special case $Y_{n,(1^k),k}$ of our variety gives a new geometric realization of the representation corresponding to the expression in the Delta Conjecture when $t = 0$.

Résumé. Nous introduisons une famille de variétés $Y_{n,\lambda,s}$ qui généralise les fibres de Springer pour SL_n , dont les anneaux de cohomologie sont des représentations de S_n . Nous fournissons une présentation explicite pour ces anneaux de cohomologie, ainsi qu'une pavage d'affines en bijection avec une collection de tableaux strictement décroissants dans les lignes. Nous démontrons aussi que les groupes de cohomologie de degré maximum de $Y_{n,\lambda,s}$ sont les représentations induites des représentations irréductibles de S_n ; ceci généralise la correspondance de Springer. De plus, pour $\lambda = 1^k$, la variété $Y_{n,(1^k),k}$ fournit une réalisation géométrique nouvelle de la représentation correspondant à l'expression de la conjecture delta pour $t = 0$.

Keywords: Springer fiber, cohomology, affine paving, Specht module, symmetric function, ordered set partition

1 Introduction

In this article, we introduce a family of varieties generalizing the Springer fibers. We find an explicit presentation of their cohomology rings generalizing the one given by Tanisaki for the cohomology ring of a Springer fiber [14]. This presentation coincides with the graded ring introduced and studied by the first author [7]. As a special case, our construction gives a new *compact* geometric realization of the expression in the Delta

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Conjecture in the case $t = 0$. We also prove a version of the Springer correspondence for this family of varieties, showing that their top cohomology groups have the S_n -module structure of an induced Specht module.

In the seminal work [13, 12], T.A. Springer introduced a family of varieties associated to any complete flag variety G/B , called Springer fibers, that have remarkable connections to the representation theory of the corresponding Weyl group. Springer proved that although the Weyl group does not act on a Springer fiber, it does act nontrivially on the cohomology ring of a Springer fiber. In type A, the Weyl group is the symmetric group S_n . Springer proved that the highest degree nonzero cohomology group of a Springer fiber is an irreducible representation of S_n , and every irreducible representation appears this way. This is known as the *Springer correspondence*. We note that the S_n -action discussed in this paper differs from Springer's original construction by tensoring with the sign representation.

The graded S_n -module type of the cohomology ring of a Springer fiber was discovered by Hotta and Springer [10]. Under the Frobenius characteristic map Frob that associates a symmetric function to each S_n -module, the cohomology ring of a Springer fiber is sent to the *modified Hall–Littlewood symmetric function*

$$\text{Frob}(H^*(\mathcal{B}^\lambda; \mathbb{Q}); q) = \tilde{H}_\lambda(x; q^2), \quad (1.1)$$

where the q on the left-hand side keeps track of the grading of the cohomology ring. The cohomology ring of a Springer fiber was made explicit by De Concini–Procesi [4] and Tanisaki [14], who found quotient ring presentations for $H^*(\mathcal{B}^\lambda; \mathbb{Q})$. These presentations generalize Borel's theorem that the cohomology of $\text{Fl}(n)$ is the type A coinvariant ring. A detailed analysis of the graded Frobenius characteristic of the cohomology ring was then given by Garsia and Procesi [5], who were inspired by these presentations.

The Delta Conjecture of Haglund–Remmel–Wilson [8], recently proved (in the rise case) by D'Adderio and Mellit [3] and by Blasiak–Haiman–Morse–Pun–Seelinger [1], gives a combinatorial formula for a symmetric function, $\Delta'_{e_{k-1}} e_n(q, t)$, coming from the theory of Macdonald polynomials. Since $\Delta'_{e_{k-1}} e_n(q, t)$ is conjectured to be Schur-positive, there is much interest in a natural algebraic or geometric construction of a (bigraded) S_n -module whose Frobenius characteristic is $\Delta'_{e_{k-1}} e_n(q, t)$. Haglund–Rhoades–Shimozono [9] did this in the case $t = 0$ by constructing a graded ring $R_{n,k}$ with a suitable S_n -action whose graded Frobenius characteristic is $\Delta'_{e_{k-1}} e_n(q, 0)$ (after a minor twist). The ring $R_{n,k}$ is defined as

$$R_{n,k} := \frac{\mathbb{Q}[x_1, \dots, x_n]}{\langle x_1^k, \dots, x_n^k, e_n, e_{n-1}, \dots, e_{n-k+1} \rangle} \quad (1.2)$$

and specializes to the type A coinvariant ring in the case $n = k$.

Pawlowski and Rhoades [11] gave a parallel geometric interpretation by exhibiting a variety whose cohomology ring is $R_{n,k}$. They defined the space of *spanning line arrange-*

ments, n -tuples of lines in \mathbb{C}^k that span \mathbb{C}^k ,

$$X_{n,k} := \{(L_1, \dots, L_n) \in (\mathbb{P}^{k-1})^n : L_1 + \dots + L_n = \mathbb{C}^k\}. \quad (1.3)$$

Unlike Springer fibers, which are closed subvarieties of $\text{Fl}(n)$, the space $X_{n,k}$ is an open subvariety of the product space $(\mathbb{P}^{k-1})^n$, and hence $X_{n,k}$ is smooth and noncompact in general. Pawlowski and Rhoades proved that

$$H^*(X_{n,k}; \mathbb{Q}) \cong R_{n,k}, \quad (1.4)$$

thus giving a connection between the expression in the Delta Conjecture at $t = 0$ and geometry.

Since the Poincaré series recording the graded dimensions of the ring $R_{n,k}$ is not symmetric, then by Poincaré duality, any complex variety whose cohomology ring is isomorphic to $R_{n,k}$ must either be noncompact or singular. In this article, we introduce a compact and singular variety $Y_{n,(1^k),k}$, similar to a Springer fiber, whose cohomology ring is the Haglund–Rhoades–Shimozono ring $R_{n,k}$. The space $Y_{n,(1^k),k}$ is constructed as a closed subvariety of a partial flag variety, and it naturally embeds into an iterated projective bundle whose cohomology ring is isomorphic to the cohomology ring of the product space $(\mathbb{P}^{k-1})^n$. Furthermore, the variety $Y_{n,(1^k),k}$ fits into a larger family of varieties $Y_{n,\lambda,s}$, indexed by integers n and s and partitions λ of size $k \leq n$, that contains the Springer fiber \mathcal{B}^λ as a special case.

As our main result, we prove an explicit presentation of the ring $H^*(Y_{n,\lambda,s}; \mathbb{Q})$ as a quotient of a polynomial ring, generalizing Tanisaki’s presentation for the cohomology ring of a Springer fiber [14]. This presentation coincides with the graded ring $R_{n,\lambda,s}$ recently introduced by the first author [7]. As a consequence, we see that the cohomology ring of $Y_{n,\lambda,s}$ has a graded S_n -module structure generalizing the classical one in the Springer fiber case.

One of our main tools for proving our presentation of the cohomology ring is an affine paving of $Y_{n,\lambda,s}$, which allows us to compute the ranks of the cohomology groups. The cells in our affine paving are in bijection with ordered set partitions $(B_1 | \dots | B_s)$ of $[n]$ such that $|B_i| \geq \lambda_i$. Equivalently, the cells are in bijection with certain row strict fillings whose shape contains λ . The irreducible components of the variety $Y_{n,\lambda,s}$ are the cell closures corresponding to row strict fillings in which the restriction of the labeling to λ standardizes to a standard Young tableau and such that all of the labels not in λ are in the lowest row.

We use results of the first author to prove a generalization of the Springer correspondence to the setting of induced Specht modules. We show that for general s , the top cohomology group of $Y_{n,\lambda,s}$ is isomorphic as an S_n -module to the irreducible S_k -module S^λ induced up to S_n .

In Section 2, we recall the Springer fibers and their connections to representation theory. In Section 3, we define the variety $Y_{n,\lambda,s}$ and state our main theorem, which

is an explicit presentation of the cohomology ring of $Y_{n,\lambda,s}$ which allows us to define an S_n -action on the cohomology ring. In Section 4, we construct an affine paving of $Y_{n,\lambda,s}$, which is one of our main tools in proving the presentation of the cohomology ring. In Section 5, we prove our generalization of the Springer correspondence, and we characterize the irreducible components of the variety $Y_{n,\lambda,s}$. Finally, in Section 6 we list some open problems.

2 Springer fibers

Given a partition λ of n , let N be a $n \times n$ nilpotent matrix whose Jordan block sizes are recorded by λ . The **Springer fiber** associated to λ is

$$\mathcal{B}^\lambda := \{V_\bullet \in \text{Fl}(n) : NV_i \subseteq V_i \text{ for all } i \leq n\}. \quad (2.1)$$

Springer proved that although S_n does not act on \mathcal{B}^λ , it does act on the cohomology ring of \mathcal{B}^λ . We note that in this article, the action on the cohomology ring we consider differs from the one originally constructed by Springer by tensoring with the sign representation.

A remarkable property of this action is that the top nonzero cohomology groups of \mathcal{B}^λ give a geometric construction of the irreducible Specht modules.

Theorem 2.1 (Springer). *In Lie type A there is a bijection, known as the Springer correspondence, between Springer fibers and the irreducible S_n -modules, up to isomorphism. Specifically, the top nonzero cohomology group of \mathcal{B}^λ as an S_n -module is*

$$H^{2n(\lambda)}(\mathcal{B}^\lambda; \mathbb{Q}) \cong S^\lambda, \quad (2.2)$$

where

$$n(\lambda) := \sum_i \binom{\lambda'_i}{2} = \dim_{\mathbb{C}}(\mathcal{B}^\lambda). \quad (2.3)$$

Hotta and Springer [10] proved that the map on cohomology induced by the inclusion $\mathcal{B}^\lambda \subseteq \text{Fl}(n)$,

$$H^*(\text{Fl}(n)) \rightarrow H^*(\mathcal{B}^\lambda), \quad (2.4)$$

is surjective. Hence, the cohomology ring $H^*(\mathcal{B}^\lambda)$ is generated by the cohomology classes $x_i = -c_1(\tilde{V}_i/\tilde{V}_{i-1})$. Here, we are abusing notation and writing \tilde{V}_i for the restriction of this vector bundle to \mathcal{B}^λ .

There is an explicit presentation of $H^*(\mathcal{B}^\lambda)$ as a quotient ring extending Borel's theorem that the cohomology ring of $\text{Fl}(n)$ is isomorphic to the type A coinvariant ring [4,

14]. For all $i \leq n$, let $p_i(\lambda) = \lambda'_n + \lambda'_{n-1} + \cdots + \lambda'_{n-i+1}$, where $\lambda'_i = 0$ for all $i > \lambda_1$. Given $S \subseteq \{x_1, \dots, x_n\}$, let $e_d(S)$ be the sum of all square-free products of variables in S of degree d . The ideal I_λ and quotient ring R_λ are defined as follows,

$$I_\lambda := \langle e_d(S) : |S| \geq d > |S| - p_{|S|}(\lambda) \rangle, \quad (2.5)$$

$$R_\lambda := \mathbb{Q}[x_1, \dots, x_n] / I_\lambda. \quad (2.6)$$

Tanisaki [14] proved that there is an isomorphism of graded rings

$$H^*(\mathcal{B}^\lambda; \mathbb{Q}) \cong R_\lambda \quad (2.7)$$

given by identifying the cohomology class x_i with the variable x_i . For example, when $\lambda = (2, 1)$, then $p_1(\lambda) = 0$, $p_2(\lambda) = 1$, and $p_3(\lambda) = 3$. Therefore, I_λ is generated by $e_d(S)$ where $3 \geq d > 0$ and $|S| = 3$, or $2 \geq d > 1$ and $|S| = 2$, so

$$I_{(2,1)} = \langle e_1(x_1, x_2, x_3), e_2(x_1, x_2, x_3), e_3(x_1, x_2, x_3), e_2(x_1, x_2), e_2(x_1, x_3), e_2(x_2, x_3) \rangle. \quad (2.8)$$

and $H^*(\mathcal{B}^{(2,1)}; \mathbb{Q}) \cong \mathbb{Q}[x_1, x_2, x_3] / I_{(2,1)}$.

3 The variety $Y_{n,\lambda,s}$

The variety $Y_{n,\lambda,s}$ is defined as follows. Fix integers $0 \leq k \leq n$ (with $n > 0$), a partition $\lambda \vdash k$, and an integer $s \geq \ell(\lambda)$. Define $m = k + s(n - k)$, and let $\text{Fl}_{1,\dots,n}(\mathbb{C}^m)$ be the partial flag variety of ascending chains of subspaces $V_1 \subset V_2 \subset \cdots \subset V_n \subset \mathbb{C}^m$ with $\dim_{\mathbb{C}}(V_i) = i$ for all i .

Fix a nilpotent endomorphism N of \mathbb{C}^m whose matrix has Jordan type $\Lambda = (n - k + \lambda_1, n - k + \lambda_2, \dots, n - k + \lambda_s)$, where $\lambda_i := 0$ for $i > \ell(\lambda)$, see Figure 1. Define $\mathbb{C}^\lambda := \text{im}(N^{n-k})$, which is a subspace on which N has Jordan type λ . We define the following variety, which is our main object of study

$$Y_{n,\lambda,s} := \{V_\bullet \in \text{Fl}_{1,\dots,n}(\mathbb{C}^m) : NV_i \subseteq V_{i-1} \text{ for all } i, \text{ and } V_n \supseteq \mathbb{C}^\lambda\}. \quad (3.1)$$

Throughout the paper, we fix a basis of \mathbb{C}^m ,

$$\{f_{i,j} : i \leq s, j \leq \lambda_i + n - k\}, \quad (3.2)$$

such that $Nf_{i,j} = f_{i,j-1}$ for all $j > 1$ and $Nf_{i,1} = 0$. The set of vectors $\{f_{i,j}\}$ is called a **generalized eigenbasis of N** , and the span of the vectors $f_{i,j}$ for a fixed i is called a **generalized eigenspace of N** . Note that $\mathbb{C}^\lambda = \text{span}\{f_{i,j} : j \leq \lambda_i\}$.

Example 3.1. When $k = n$, then it can be checked that $Y_{n,\lambda,s} = \mathcal{B}^\lambda$ for any $s \geq \ell(\lambda)$, so these varieties generalize the Springer fibers. \square

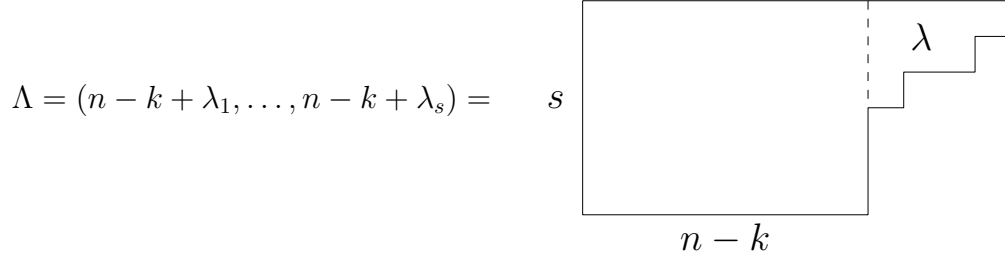


Figure 1: The partition Λ representing the Jordan type of N .

Example 3.2. Let $n = 5$, $\lambda = (2, 1)$, and $s = 3$. Then the generalized eigenbasis for the nilpotent operator N can be visualized with the following diagram, where we label the cells in the i th row of Λ by the vectors $f_{i,n-k+\lambda_i}, \dots, f_{i,1}$.

$f_{1,4}$	$f_{1,3}$	$f_{1,2}$	$f_{1,1}$
$f_{2,3}$	$f_{2,2}$	$f_{2,1}$	
$f_{3,2}$	$f_{3,1}$		

$$\begin{aligned}
 f_{1,4} &\xrightarrow{N} f_{1,3} \xrightarrow{N} f_{1,2} \xrightarrow{N} f_{1,1} \xrightarrow{N} 0 \\
 f_{2,3} &\xrightarrow{N} f_{2,2} \xrightarrow{N} f_{2,1} \xrightarrow{N} 0 \\
 f_{3,2} &\xrightarrow{N} f_{3,1} \xrightarrow{N} 0
 \end{aligned}$$

Notably, $\mathbf{C}^\lambda = \text{span}\{f_{1,1}, f_{1,2}, f_{2,1}\}$, the span of the vectors contained in the bolded cells with partition shape $(2, 1)$. Then $Y_{5,(2,1),3}$ is the variety of partial flags $V_\bullet = (V_1, V_2, V_3, V_4, V_5) \in \text{Fl}_{1,2,3,4,5}(\mathbf{C}^9)$ such that the following conditions hold:

$$NV_i \subseteq V_{i-1} \text{ for } i \leq 5, \quad (3.3)$$

$$V_5 \supseteq \text{span}\{f_{1,1}, f_{1,2}, f_{2,1}\}. \quad (3.4)$$

For example, the partial flag

$$\langle f_{1,1} \rangle \subset \langle f_{1,1}, f_{1,2} \rangle \subset \langle f_{1,1}, f_{1,2}, f_{3,1} \rangle \subset \langle f_{1,1}, f_{1,2}, f_{3,1}, f_{2,1} \rangle \subset \langle f_{1,1}, f_{1,2}, f_{3,1}, f_{2,1}, f_{1,3} \rangle.$$

is in $Y_{5,(2,1),3}$. □

Our main result is an explicit presentation of the cohomology ring $H^*(Y_{n,\lambda,s}; \mathbf{Q})$, which coincides with the quotient ring $R_{n,\lambda,s}$ introduced and studied by the first author in [7], which we describe next. Given n , λ , and s as above, define

$$I_{n,\lambda,s} := \langle e_d(S) : |S| \geq d > |S| - p_{|S|}(\lambda) \rangle + \langle x_i^s : 1 \leq i \leq n \rangle, \quad (3.5)$$

$$R_{n,\lambda,s} := \mathbf{Q}[x_1, \dots, x_n] / I_{n,\lambda,s}. \quad (3.6)$$

Example 3.3. For example, when $n = 4$, $\lambda = (2, 1)$, and $s = 2$, then $I_{4,(2,1),2}$ is generated by x_i^2 for $i = 1, 2, 3, 4$ and the polynomials $e_d(S)$ for $S \subseteq \{x_1, \dots, x_4\}$ such that

$$\begin{aligned}
 d = 2 \text{ and } |S| = 4, & & d = 3 \text{ and } |S| = 4, \\
 d = 4 \text{ and } |S| = 4, & & d = 3 \text{ and } |S| = 3.
 \end{aligned}$$

We have

$$\begin{aligned} I_{4,(2,1),2} &= \langle x_1^2, x_2^2, x_3^2, x_4^2, e_2, e_3, e_4, e_3(x_1, x_2, x_3), e_3(x_1, x_2, x_4), e_3(x_1, x_3, x_4), e_3(x_2, x_3, x_4) \rangle \\ &= \langle x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4, \\ &\quad x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4, \\ &\quad x_1x_2x_3x_4, x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4 \rangle. \end{aligned}$$

Observe, this is not a minimal set of generators for $I_{4,(2,1),2}$, since the e_3 and e_4 generators are redundant. \square

Theorem 3.4. We have an isomorphism of graded rings

$$R_{n,\lambda,s} \cong H^*(Y_{n,\lambda,s}; \mathbb{Q}), \quad (3.7)$$

given by identifying x_i with $-c_1(\tilde{V}_i/\tilde{V}_{i-1})$.

Corollary 3.5. We have an isomorphism of graded rings

$$R_{n,k} \cong H^*(Y_{n,(1^k),k}; \mathbb{Q}), \quad (3.8)$$

given by identifying x_i with $-c_1(\tilde{V}_i/\tilde{V}_{i-1})$.

Since the ring $R_{n,\lambda,s}$ is symmetric in the x_i variables, then by Theorem 3.4, the ring $H^*(Y_{n,\lambda,s}; \mathbb{Q})$ has the structure of a graded S_n -module where S_n acts by permuting the classes $-c_1(\tilde{V}_i/\tilde{V}_{i-1})$. This action is a generalization of Springer's representation on the cohomology ring of a Springer fiber up to tensoring with the sign representation. By Theorem 3.4, the graded Frobenius characteristic formulas for $R_{n,\lambda,s}$ in [7] also give the graded Frobenius characteristic of $H^*(Y_{n,\lambda,s}; \mathbb{Q})$.

Example 3.6. The graded Frobenius characteristic of $H^*(Y_{4,(2,1),2}; \mathbb{Q}) \cong R_{4,(2,1),2}$ is

$$\text{Frob}(H^*(Y_{4,(2,1),2}; \mathbb{Q}); q) = s_{(4)} + q(s_{(4)} + s_{(3,1)}) + q^2(s_{(3,1)} + s_{(2,2)}) \quad (3.9)$$

in terms of Schur functions. \square

4 Affine paving

In this section, we illustrate one of the key tools used in the proof of Theorem 3.4, that of an affine paving. We show that $Y_{n,\lambda,s}$ has an affine paving, and we use it to find the ranks of the cohomology groups of $Y_{n,\lambda,s}$.

Given a complex algebraic variety X , an **affine paving** of X is a sequence of closed subvarieties

$$X_0 \subseteq X_1 \subseteq \cdots \subseteq X_m = X \quad (4.1)$$

of X such that $X_i \setminus X_{i-1} = \bigsqcup_j A_{i,j}$, where $A_{i,j} \cong \mathbb{C}^{a_{i,j}}$ as varieties for some integers $a_{i,j} \geq 0$. The affine spaces $A_{i,j}$ are called the **cells** of the affine paving. When X is compact in the analytic topology, an affine paving gives us a way of computing the ranks of the cohomology groups.

Lemma 4.1. *Suppose X is a compact complex algebraic variety that has an affine paving. If $X_i \setminus X_{i-1} = \bigsqcup_{i,j} A_{i,j}$ is the decomposition of X into cells, then*

$$H^{2k}(X) \cong \mathbb{Z}^{\#\{(i,j) : \dim_{\mathbb{C}}(A_{i,j})=k\}} \quad (4.2)$$

$$H^{2k+1}(X) = 0, \quad (4.3)$$

for all $k \geq 0$.

In the case $X = Y_{n,\lambda,s}$, there is a filtration by closed subvarieties $Y_{n,\lambda,s}^1 \subseteq Y_{n,\lambda,s}^2 \subseteq \dots \subseteq Y_{n,\lambda,s}^s$, defined as follows. Observe that $\ker(N) = \text{span}_{\mathbb{C}}\{f_{1,1}, \dots, f_{s,1}\}$. Let F_{\bullet} be the complete flag of $\ker(N)$ given by $F_i = \text{span}_{\mathbb{C}}\{f_{1,1}, \dots, f_{i,1}\}$. For $i \leq s$, define

$$Y_{n,\lambda,s}^i := \{V_{\bullet} \in Y_{n,\lambda,s} : V_1 \subseteq F_i\}, \quad (4.4)$$

where $Y_{n,\lambda,s}^0 := \emptyset$. Since $V_1 \in \ker(N)$ for all $V_{\bullet} \in Y_{n,\lambda,s}$, we have $Y_{n,\lambda,s}^s = Y_{n,\lambda,s}$, so the subspaces $Y_{n,\lambda,s}^i$ form a filtration of $Y_{n,\lambda,s}$ by closed subvarieties. The filtration $Y_{n,\lambda,s}^{\bullet}$ is an affine paving, which follows by the next lemma and induction on n .

Lemma 4.2. *There is an isomorphism of varieties*

$$Y_{n,\lambda,s}^i \setminus Y_{n,\lambda,s}^{i-1} \cong \begin{cases} \mathbb{C}^{i-1} \times Y_{n-1,\lambda^{(i)},s} & \text{if } 1 \leq i \leq \ell(\lambda) \\ \mathbb{C}^{i-1} \times Y_{n-1,\lambda,s} & \text{if } \ell(\lambda) < i \leq s. \end{cases} \quad (4.5)$$

Here, $\lambda^{(i)}$ is the partition obtained from λ by subtracting 1 from the i th part of λ and then sorting the parts.

Let $\mathcal{T}_{n,\lambda,s}$ be the set of partial fillings of the Young diagram of $\Lambda = (n - k + \lambda_1, n - k + \lambda_2, \dots, n - k + \lambda_s)$ with the labels $[n]$ (without repetition), such that the labels in each row are right justified and decrease from left to right, and the i th row contains at least λ_i many labels. See Figure 2 for an example with $n = 5$, $\lambda = (2, 1)$, and $s = 3$.

Example 4.3. *Taking our basis of \mathbb{C}^k , the cell in Figure 2 is (under one convention for cell indexing) two-dimensional and given by the flags of the form*

$$\begin{aligned} \langle f_{11} \rangle &\subseteq \langle f_{11}, f_{12} \rangle \subseteq \langle f_{11}, f_{12}, f_{31} + \alpha_1 f_{21} + \alpha_2 f_{13} \rangle \\ &\subseteq \langle f_{11}, f_{12}, f_{31} + \alpha_2 f_{13}, f_{21} \rangle \\ &\subseteq \langle f_{11}, f_{21}, f_{31}, f_{21}, f_{13} \rangle, \end{aligned}$$

where α_1 and α_2 are arbitrary complex parameters. □

	5	2	1
		4	
	3		

Figure 2: An example of a filling in the set $\mathcal{T}_{5,(2,1),3}$, which indexes the cells in the affine paving of $Y_{5,(2,1),3}$. The cell corresponding to this filling is not of maximal dimension because the entries outside λ are not all in the bottom row and the λ entries are not column-strict.

Lemma 4.4. *The cells in the affine paving, defined inductively from Lemma 4.2, are in bijection with $\mathcal{T}_{n,\lambda,s}$.*

With appropriate choices of the isomorphisms in Lemma 4.2 and indexing of the cells of $Y_{n,\lambda,s}$, it is possible to realize the affine paving as a restriction of the Schubert decomposition of $\text{Fl}_{1,\dots,n}(\mathbb{C}^m)$.

Lemma 4.5. *Under an appropriate total ordering of the basis $\{f_{i,j}\}$ of \mathbb{C}^m , the Schubert decomposition of $\text{Fl}_{1,\dots,n}(\mathbb{C}^m)$ restricts to an affine paving of $Y_{n,\lambda,s}$ upon intersecting the Schubert cells with $Y_{n,\lambda,s}$. Precisely, the nonempty intersections of the form $C_w \cap Y_{n,\lambda,s}$, where C_w is a Schubert cell, are the cells in an affine paving of $Y_{n,\lambda,s}$ that respects the isomorphisms in Lemma 4.2.*

Proof of Theorem 3.4 (Sketch). First, we use the affine paving combined with Lemma 4.1 to show that the dimensions of the graded pieces of $H^*(Y_{n,\lambda,s}; \mathbb{Q})$ and $R_{n,\lambda,s}$ are the same. Second, we show that the main theorem holds in the case $\lambda = \emptyset$,

$$H^*(Y_{n,\emptyset,s}; \mathbb{Q}) \cong R_{n,\emptyset,s} = \frac{\mathbb{Q}[x_1, \dots, x_n]}{\langle x_1^s, \dots, x_n^s \rangle}. \quad (4.6)$$

Third, we exhibit a surjective map on cohomology

$$\frac{\mathbb{Q}[x_1, \dots, x_n]}{\langle x_1^s, \dots, x_n^s \rangle} \cong H^*(Y_{n,\emptyset,s}; \mathbb{Q}) \twoheadrightarrow H^*(Y_{n,\lambda,s}; \mathbb{Q}). \quad (4.7)$$

Finally, by combining a dimension counting argument and results of Brundan and Ostrick [2] on Spaltenstein varieties, we show that the kernel of the surjection in (4.7) is generated by the polynomials $e_d(S) \in I_{n,\lambda,s}$. \square

5 A Springer correspondence for induced Specht modules

In this section, we characterize the irreducible components of $Y_{n,\lambda,s}$. In the case $s > \ell(\lambda)$, the number of irreducible components is equal to $\binom{n}{k} \cdot \#\text{SYT}(\lambda)$.

Given a subspace $W \subseteq \mathbb{C}^m$ such that $NW \subseteq W$, then $N(W \cap \mathbb{C}^\lambda) \subseteq W \cap \mathbb{C}^\lambda$. The nilpotent operator N thus induces a nilpotent operator on the quotient space $\mathbb{C}^\lambda / (W \cap \mathbb{C}^\lambda)$, which we denote by $N|_{\mathbb{C}^\lambda / (W \cap \mathbb{C}^\lambda)}$.

Suppose T is a filling of the Young diagram of λ with a k -element subset of $[n]$ that decreases from left to right across each row. Let $T|_{n, \dots, n-i+1}$ be the restriction of T to the cells containing the labels $n, \dots, n-i+1$. Furthermore, let $\text{sh}(T|_{n, \dots, n-i+1})$ be the partition obtained by recording the row sizes of $T|_{n, \dots, n-i+1}$ and then sorting them to a partition. Given such a filling T that also decreases down each column, define the following subset of $Y_{n, \lambda, s}$,

$$Y_{n, \lambda, s}^T = \{V \bullet \in Y_{n, \lambda, s} : N|_{\mathbb{C}^\lambda / (V_i \cap \mathbb{C}^\lambda)} \text{ has Jordan type } \text{sh}(T|_{n, \dots, n-i+1}) \text{ for all } i\}. \quad (5.1)$$

Lemma 5.1. *Each subvariety $Y_{n, \lambda, s}^T$ is an irreducible locally-closed union of cells from the affine paving. If $s > \ell(\lambda)$, then each $Y_{n, \lambda, s}^T$ is nonempty. If $s = \ell(\lambda)$, then $Y_{n, \lambda, s}^T$ is nonempty if and only if for all i , if T does not contain i as a label then the labels up to $i-1$ fill at least one row of λ . Furthermore, if $Y_{n, \lambda, s}^T$ is nonempty, then*

$$\dim_{\mathbb{C}}(Y_{n, \lambda, s}^T) = n(\lambda) + (n-k)(s-1). \quad (5.2)$$

The $Y_{n, \lambda, s}^T$ give a partition of $Y_{n, \lambda, s}$ as we consider all possible choices for T .

Theorem 5.2. *The space $Y_{n, \lambda, s}$ is equidimensional of dimension $n(\lambda) + (n-k)(s-1)$. In particular, the closed subvarieties $\overline{Y_{n, \lambda, s}^T}$ for which $Y_{n, \lambda, s}^T$ is nonempty (as described in Lemma 5.1) form a complete set of irreducible components. In the case $s > \ell(\lambda)$, there are $\binom{n}{k} \cdot \#\text{SYT}(\lambda)$ many irreducible components.*

For each cell in the affine paving, the filling T can be obtained from the partial filling of Λ by restricting the labeling to the upper right copy of λ contained in Λ , as depicted in Figure 1.

Example 5.3. *For the filling in Figure 2, the corresponding filling of λ is*

$$\begin{array}{|c|c|} \hline 2 & 1 \\ \hline 4 & \\ \hline \end{array},$$

and the Jordan types of the operators $N|_{\mathbb{C}^\lambda / (V_i \cap \mathbb{C}^\lambda)}$ for $i = 1, 2, 3, 4, 5$ are

$$\begin{array}{|c|} \hline \\ \hline \end{array} \supset \square \supset \square \supset \emptyset \supset \emptyset.$$

Notice that the filling was not column-strict; accordingly, the row lengths corresponding to the Jordan type of $N|_{\mathbb{C}^\lambda / (V_2 \cap \mathbb{C}^\lambda)}$ are $(0, 1)$ (corresponding to the entry $4 \in \lambda$), but become $(1, 0)$ after sorting. \square

In the case $s > \ell(\lambda)$, the irreducible components are naturally indexed by Standard Young Tableaux on $(n - k) \cup \lambda$. This indexing of irreducible components extends to a representation theory statement on the top cohomology group of $Y_{n,\lambda,s}$, generalizing Springer's theorem that the top cohomology group of a Springer fiber is a Specht module.

Theorem 5.4. *Let $d = \dim(Y_{n,\lambda,s}) = n(\lambda) + (n - k)(s - 1)$, and consider S_k as the subgroup of S_n permuting the elements of $[k]$. For $s > \ell(\lambda)$, we have an isomorphism of S_n -modules*

$$H^{2d}(Y_{n,\lambda,s}; \mathbb{Q}) \cong \text{Ind}_{S_k}^{\uparrow S_n} (S^\lambda). \quad (5.3)$$

For $s = \ell(\lambda)$, we have

$$H^{2d}(Y_{n,\lambda,s}; \mathbb{Q}) \cong S^{\Lambda/(n-k)^{s-1}}, \quad (5.4)$$

the Specht module of skew shape $\Lambda/(n - k)^{s-1}$.

In the case of $s > \ell(\lambda)$, the proof follows by combining Theorem 3.4 and the fact that, in this case, the top degree component of $R_{n,\lambda,s}$ is isomorphic to $\text{Ind}_{S_k}^{\uparrow S_n} (S^\lambda)$ [6, Corollary 3.3.15]. In the case of $s = \ell(\lambda)$, the proof follows by combining Theorem 3.4 with the formula [7, Theorem 5.13] for $\text{Frob}(R_{n,\lambda,s}; q)$ and then using bijective techniques to show that the top degree component of this symmetric function is the skew Schur function $S_{\Lambda/(n-k)^{s-1}}(x)$.

6 Future work

The S_n -module structure of the cohomology of a Springer fiber, $H^*(\mathcal{B}^\lambda; \mathbb{Q})$, has several constructions. One construction involves utilizing the Grothendieck–Springer resolution and intersection cohomology, and it is known that this action is the same as the one defined by permuting the cohomology classes $-c_1(\tilde{V}_i/\tilde{V}_{i-1})$.

Problem 6.1. *Find a generalization of the Grothendieck–Springer resolution, and realize the S_n -module structure on $H^*(Y_{n,\lambda,s}; \mathbb{Q})$ defined in this article as a specialization of an S_n -action on an intersection cohomology complex.*

Problem 6.2. *Find a generalization of the family of varieties $Y_{n,\lambda,s}$ to other Lie types such that the top cohomology groups gives an induced module analogue of the Springer correspondence in those types.*

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