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Skew characters and cyclic sieving (extended abstract)

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Abstract. In 2010, B. Rhoades proved that promotion on rectangular standard Young tableaux together with the associated fake-degree polynomial shifted by an appropriate power, provides an instance of the cyclic sieving phenomenon.

Motivated in part by this result, we show that we can expect a cyclic sieving phenomenon for m-tuples of skew standard Young tableaux of the same shape and the m^{th} power of the associated fake-degree polynomial, for fixed m, under mild and easily checked conditions. However, we are unable to exhibit an appropriate group action explicitly.

Put differently, we determine in which cases the m^{th} tensor power of a skew character of the symmetric group carries a permutation representation of the cyclic group.

To do so, we use a method proposed by N. Amini and the first author, which amounts to establishing a bound on the number of border-strip tableaux of skew shape.

Finally, we apply our results to the invariant theory of tensor powers of the adjoint representation of the general linear group. In particular, we prove the existence of a bijection between permutations and J. Stembridge's alternating tableaux, which intertwines rotation and promotion.

Keywords: Cyclic sieving, Skew characters, Border strip tableaux, Adjoint representation

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1 Introduction

This is an extended abstract of [2]. We determine which tensor powers of a skew character $\chi^{\lambda/\mu}$ of the symmetric group $\mathfrak{S}_n$ carry a permutation representation of the cyclic group of order n.

This problem can be rephrased in terms of V. Reiner, D. Stanton & D. White's *cyclic sieving phenomenon* [10]. Let $\mathrm{SYT}(\lambda/\mu)$ be the set of standard Young tableaux of skew shape λ/μ, and let $f^{\lambda/\mu}(q)$ be G. Lusztig's fake degree polynomial for $\chi^{\lambda/\mu}$. Then there *exists* an action ρ of the cyclic group of order $n = |\lambda/\mu|$ such that

$$\Big(\underbrace{\mathrm{SYT}(\lambda/\mu) \times \cdots \times \mathrm{SYT}(\lambda/\mu)}_{m}, \langle\rho\rangle, f^{\lambda/\mu}(q)^m\Big)$$

exhibits the cyclic sieving phenomenon, if and only if $f^{\lambda/\mu}(q)^m$ evaluates to nonnegative integers at n^{th} roots of unity.[1] If m is even this is always the case. If m is odd, this is the case if and only if there exists a tiling of λ/μ with border-strips of size k of even height for every $k \mid n$, see Theorem 4.3. Remarkably, a theorem of Swanson [16, thm. 1.5] implies that in this case, and provided that μ is the empty partition, the cyclic group action has a free orbit.

We also show that for any skew shape λ/μ and any integer $s > 0$ there is an action τ of the cyclic group of order s on *stretched shapes* such that

$$\Big(\mathrm{SYT}(s\lambda/s\mu), \langle\tau\rangle, f^{s\lambda/s\mu}(q)\Big)$$

exhibits the cyclic sieving phenomenon, see Theorem 4.7.

At this point we are unable to present ρ and τ explicitly for general skew shapes λ/μ. Instead, we use a characterization of P. Alexandersson & N. Amini [1], which says that $f \in \mathbb{N}_0[q]$ is a cyclic sieving polynomial for a group action of the cyclic group of order n, if and only if for a primitive n^{th} root of unity ξ and all $k \mid n$ we have that $f(\xi^k) \in \mathbb{N}_0$ and

$$\sum_{d \mid k} \mu(k/d) f(\xi^d) \geq 0,$$

where μ is the number-theoretic Möbius function.

To apply this result, we establish a new bound for the absolute value of the skew character evaluated at a power of the long cycle. More precisely, Theorem 3.8 implies that, for any $k \mid n$,

$$|f^{\lambda/\mu}(\xi^k)| \geq \sum_{d \mid k, d<k} |f^{\lambda/\mu}(\xi^d)|$$

[1] In fact, in the spirit of Rhoades' result [11], it is easy to see that this equivalence also holds for $q^k f^{\lambda/\mu}(q)^m$, where k is any integer.

provided $|f^{\lambda/\mu}(\xi^k)| \geq 2$.

To prove this inequality, we note that $|f^{\lambda/\mu}(\xi^d)| = |\chi^{\lambda/\mu}((m^d))| = |\mathrm{BST}(\lambda/\mu, m)|$, the number of border-strip tableaux of shape λ/μ with strips of size m, extending the theorems for partitions by T. Springer [14] and G. James & A. Kerber [6]. We then approximate the number of border-strip tableaux using a bound by S. Fomin & N. Lulov [5].

Our main motivation is an implication for the invariant theory of the general linear group, as we now explain. Let $\mathfrak{gl}_r$ be the adjoint representation of GL_r, and consider its n^{th} tensor power $\mathfrak{gl}_r^{\otimes n}$. The symmetric group $\mathfrak{S}_n$ acts on this space by permuting tensor positions. Thus, using Schur–Weyl duality, we can determine the subspace of GL_n-invariants of $\mathfrak{gl}_r^{\otimes n}$, regarded as a representation of $\mathfrak{S}_n$. This representation turns out to be isomorphic to

$$\bigoplus_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq r}} S_\lambda \otimes S_\lambda,$$

where the direct sum is over all partitions of n into at most r parts, and S_λ is the irreducible representation of $\mathfrak{S}_n$ corresponding to λ. In particular, for $r \geq n$, the dimension of the space of invariants equals the size of $\mathfrak{S}_n$.

A fundamental question of invariant theory is to find an explicit basis of the space of invariants, which, if possible, enjoys further desirable properties. One such property is invariance under rotation of tensor positions, following G. Kuperberg's idea of web bases [8].

An elegant and useful solution would be to describe a set of permutations in $\mathfrak{S}_n$, and a bijection from these to the basis elements which intertwines rotation of permutations (that is, conjugation with the long cycle) and rotation of tensor positions. It would be even nicer if this set of permutations for the invariants of $\mathfrak{gl}_r^{\otimes n}$ were a subset of the set of permutations for the invariants of $\mathfrak{gl}_{r+1}^{\otimes n}$.

Although it appears to be difficult to exhibit such an intertwining bijection explicitly, our results, combined with previous work of S. Pfannerer, M. Rubey & B. Westbury [9], implies that such a solution must exist, see Theorem 5.8.

The existence of such an intertwining bijection is closely related to the existence of a rotation invariant statistic st mapping permutations to partitions, such that $|\{\sigma \in \mathfrak{S}_n : \mathrm{st}(\sigma) = \lambda\}| = |\mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda)|$, see Theorem 5.5.

2 Cyclic group actions and cyclic sieving

In this section we recall V. Reiner, D. Stanton & D. White's cyclic sieving phenomenon, characters of cyclic group actions and a result of P. Alexandersson & N. Amini characterizing characters arising from cyclic group actions.

Definition 2.1 ([10])**.** Let X be a finite set and let ρ be a generator of an action of the cyclic group of order n on X.

Given a polynomial $f(q) \in \mathbb{N}_0[q]$ we say that the triple $(X, \langle\rho\rangle, f(q))$ *exhibits the cyclic sieving phenomenon* if for all $d \in \mathbb{Z}$

$$\#\{x \in X : \rho^d \cdot x = x\} = f(\xi^d),$$

where ξ is a primitive n^{th} root of unity. In this case $f(q)$ is a *cyclic sieving polynomial* for the group action.

In particular, the cardinality of X is given by $f(1)$. More generally, realizing the cyclic group of order n as the group of n^{th} roots of unity and identifying its ring of characters with $\mathbb{Z}[q]/(q^n - 1)$, the cyclic sieving polynomial $f(q)$ modulo $q^n - 1$ reduces to the character of the group action.

Much attention has been given to prove cyclic sieving phenomena on certain families of tableaux. Most famously, B. Rhoades [11] showed that $\mathrm{SYT}(a^b)$, together with promotion and the fake degree polynomial $f^\lambda(q)$ associated with $\lambda = (a^b)$, exhibits the cyclic sieving phenomenon.

In many cases, the only known way to prove that a given triple $(X, \langle\rho\rangle, f(q))$ exhibits the cyclic sieving phenomenon is to enumerate the number of fixed points of the group action, and then verify that the evaluation of the polynomial yields the same number. A different strategy is to exhibit a representation of the symmetric group such that the given group action is the restriction of the representation to the cyclic group generated by the long cycle.

Our main result, Theorem 4.3, is of a rather different flavour. Instead of computing the character of a given cyclic group action, we consider a specific representation of the symmetric group, and determine whether its restriction to the cyclic group generated by the long cycle is isomorphic to a cyclic group action. Put differently, we determine whether this restriction is a permutation representation of the cyclic group. To do so, we use the following criterion.

Theorem 2.2 ([1, Thm. 2.7])**.** *Let $f(q) \in \mathbb{N}_0[q]$ and suppose that $f(\xi^d) \in \mathbb{N}_0$ for all $d \in \{1, \dots, n\}$, where ξ is a primitive n^{th} root of unity. Let X be any set of size $f(1)$.*

Then there exists a cyclic group action ρ of order n such that $(X, \langle\rho\rangle, f(q))$ exhibits the cyclic sieving phenomenon if and only if for every $k \mid n$,

$$\sum_{d|k} \mu(k/d) f(\xi^d) \geq 0,$$

where μ is the number-theoretic Möbius function.

Note that, except for its size, the nature of the set X is irrelevant in this theorem.

Remark 2.3. If $(X, \langle\rho\rangle, f(q))$ exhibits the cyclic sieving phenomenon, the expression

$$\sum_{d|k} \mu(k/d) f(\xi^d)$$

is the number elements in orbits of size k of the group action. Hence, it must be nonnegative and divisible by k.

We conclude this section by recalling a fact that makes cyclic group actions special. Suppose we are given a linear representation and a group action of a finite group which are isomorphic as linear representations. Then, trivially, there is a basis of the invariant space of the representation such that the restriction of the representation to this basis is isomorphic to the group action. However, two non-isomorphic group actions may be isomorphic as linear representations. This is not the case for group actions of a cyclic group, as R. Brauer's permutation lemma shows:

Theorem 2.4 ([4, 7])**.** *Two cyclic group actions are isomorphic if and only if they are isomorphic as linear representation, that is, their characters coincide.*

3 Character values and border strip tableaux

Definition 3.1. Given a skew standard Young tableau T with n cells, a label j with $1 \leq j < n$ is a *descent* of T, if the label $j+1$ appears in a row strictly below that of j. The *major index* of T, denoted $\operatorname{maj}(T)$, is the sum of the descents of T. The *(skew) fake-degree polynomial* $f^{\lambda/\mu}(q)$ associated with a skew Young diagram λ/μ is the generating function for the major index:

$$f^{\lambda/\mu}(q) := \sum_{T \in \mathrm{SYT}(\lambda/\mu)} q^{\operatorname{maj}(T)}.$$

In this section we give two alternative expressions for the fake-degree polynomial evaluated at a primitive n^{th} root of unity. The first is an evaluation of a skew character at a power of the *long cycle* $(1, \ldots, n) \in \mathfrak{S}_n$, generalizing a result by T. Springer [14]. The second one as the cardinality of a certain set of border strip tableaux times a sign.

Definition 3.2. A *border-strip* (or *ribbon* or *skew hook*) is a connected non-empty skew Young diagram containing no 2×2-square of cells, as in Figure 1a. The *height* of a border-strip is one less than the number of rows it spans. Its *tail* is the leftmost cell in the last row of a border-strip.

Let λ/μ be a skew shape. The *size* of λ/μ is its number of cells, denoted $|\lambda/\mu|$. Suppose that $\nu = (\nu_1, \ldots, \nu_\ell)$ is a partition of $|\lambda/\mu|$. A *border-strip tableau* of shape λ/μ and *type* ν is a tiling of the Young diagram of λ/μ with labeled border-strips $B_1, \ldots, B_\ell$ with the following properties:

- the border-strip B_j has label j and size ν_j,
- labeling all cells in B_j with j results in a labeling of the diagram λ/μ where labels in every row and every column are weakly increasing.

The *height* of a border-strip tableau T, or any tiling of a tableau with border-strips, is the sum of the heights of the border-strips in the partition. We let $\mathrm{BST}(\lambda/\mu, \nu)$ denote the set of all such border-strip tableaux.

We use the convention to place the label of a strip in its tail, as done in Figure 1b.

Remark 3.3. $\mathrm{BST}(\lambda/\mu, 1^n)$ can be identified with the set of standard Young tableaux of shape λ/μ.

(a) A border-strip of height 3. Its tail is labelled with x.

(b) A border-strip tableaux in $\mathrm{BST}((9^2,6^3,4,1)/(2,1^3),3)$ of height 13. The labels are placed in the tails of each strip.

Figure 1: A border-strip and a border-strip tableaux.

Finally, let us recall that the *skew character* $\chi^{\lambda/\mu}$ corresponding to a skew shape λ/μ can be defined as

$$\chi^{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu,\nu} \chi^{\nu},$$

where the Littlewood–Richardson coefficients $c^{\lambda}_{\mu,\nu}$ are the structure constants for the expansion $s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu,\nu} s_{\nu}$ of a skew Schur function into Schur functions, among many other interpretations.

We can now state the Murnaghan–Nakayama rule.

Theorem 3.4 (Murnaghan–Nakayama, see [15, Cor. 7.17.5])**.** *The skew characters are given by the signed sum*

$$\chi^{\lambda/\mu}(\nu) = \sum_{B \in \mathrm{BST}(\lambda/\mu,\nu)} (-1)^{\mathrm{height}(B)}.$$

In the remainder of the paper, we shall only concern ourselves with border-strip tableaux where all strips have the same size m, which we denote by $\mathrm{BST}(\lambda/\mu, m)$. In this case, the parity of the height depends only on the shape λ/μ. It follows that the signed sum in the Murnaghan–Nakayama rule, Theorem 3.4, is cancellation-free. In particular, we have:

Corollary 3.5. *Let λ/μ be a skew shape of size $n = dm$. Then*

$$f^{\lambda/\mu}(\xi^d) = \chi^{\lambda/\mu}((m^d)) = \varepsilon|\mathrm{BST}(\lambda/\mu, m)|,$$

where ξ is a primitive n^{th} root of unity and $\varepsilon = (-1)^{\mathrm{height}(B)}$ for any $B \in \mathrm{BST}\,(\lambda/\mu, m)$.

Remark 3.6. For the non-skew case, the first equation is due to T. Springer [14, Prop. 4.5], whereas the second was proved by G. James and A. Kerber [6, Thm. 2.7.27] using the abacus model for the quotient of a partition.

In order to apply Theorem 2.2, we need bounds on the number of border-strip tableaux. We shall first recall the following theorem by S. Fomin and N. Lulov, for partitions.

Theorem 3.7 ([5]). *For any partition $\lambda \vdash n$, we have*

$$|\mathrm{BST}(\lambda, d)| \leq \sqrt[d]{\frac{d^n}{\binom{n}{n/d,\dots,n/d}}} \cdot |\mathrm{BST}(\lambda, 1)|^{1/d} \tag{3.1}$$

From this we establish a new bound for the number of border-strip tableaux of skew shapes.

Theorem 3.8. *Let λ/μ be a skew shape with n cells and let k be a positive integer with $k \mid n$. Suppose that $|\mathrm{BST}(\lambda/\mu, k)| \geq 2$. Then*

$$|\mathrm{BST}(\lambda/\mu, k)| \geq \sum_{d \mid \frac{n}{k}, d>1} |\mathrm{BST}(\lambda/\mu, dk)|.$$

Additionally, the inequality holds if n/k is a prime number.

Remark 3.9. Our proof of this theorem is relatively complicated. We use Equation (3.1) to prove it in the case of partitions and $k = 1$. Then, using the Littlewood–Richardson rule, we extend this to the case of skew shapes and $k = 1$. Finally, we deduce the general case using the abacus model for the quotient of a skew shape.

Example 3.10. For $\lambda = (10,1^2) \vdash 12$ we have

$$|\mathrm{BST}(\lambda, 3)| = 1, \qquad |\mathrm{BST}(\lambda, 6)| + |\mathrm{BST}(\lambda, 12)| = 2.$$

Therefore, the condition that $|\mathrm{BST}(\lambda, k)| \geq 2$ is necessary.

Remark 3.11. For $k = 1$, apart from the single row and single column partitions, there are only three shapes λ/μ where equality is attained: (2^2), (3^2) and (2^3). Other than that, the minimal difference between the two sides of the inequality is attained for hooks of the form $(n-1,1)$. In this case it equals $n - \tau(n)$, where $\tau(n)$ is the number of divisors of n.

4 Cyclic sieving for skew standard tableaux

In this section we apply the bounds established in the previous section and Theorem 2.2 to prove the existence of several cyclic sieving phenomena for various families of skew standard Young tableaux.

From Theorem 3.8 and Corollary 3.5 we obtain:

Proposition 4.1. *Let λ/μ be a non-empty skew shape with n cells, let $m \in \mathbb{N}_0$ and let k be a positive integer with $k \mid n$. Then*

$$\sum_{d|k} \mu(k/d) |f^{\lambda/\mu}(\xi^d)|^m \geq 0,$$

for a primitive n^{th} root of unity ξ.

Remark 4.2. One might think that $|f^{\lambda/\mu}(\xi^d)|$ could be the number of fixed points of a group action, despite the fact that $|f^{\lambda/\mu}(q)|$ is not a polynomial. However, this is not the case. For example, consider $\lambda = (2,1)$. Then $f^\lambda(q) = q + q^2$ and, for a 3^{rd} root of unity ξ, we have $|f^\lambda(\xi^3)| = |\mathrm{BST}(\lambda, 1)| = 2$ and $|f^\lambda(\xi)| = |\mathrm{BST}(\lambda, 3)| = 1$. This is incompatible with the possible orbit sizes of a group action on a set with two elements. Indeed, for $k = 3$ we obtain

$$\frac{1}{k} \sum_{d|k} \mu(k/d) |f^\lambda(\xi^d)| = \frac{1}{3}(-1 + 2),$$

which, by Remark 2.3, would have to be an integer.

Taking into account the previous remark, it makes sense to look for shapes λ/μ such that the character $f^{\lambda/\mu}$ evaluated at roots of unity is nonnegative.

Theorem 4.3. *Let λ/μ be a skew shape with n cells and let m be a positive integer. Then there is a cyclic group action ρ of order n such that*

$$\Bigg(\underbrace{\mathrm{SYT}(\lambda/\mu) \times \cdots \times \mathrm{SYT}(\lambda/\mu)}_{m}, \langle \rho \rangle, f^{\lambda/\mu}(q)^m \Bigg)$$

exhibits the cyclic sieving phenomenon if and only if m is even, or m is odd and for each positive integer k with $k \mid n$, every tiling of λ/μ with strips of size k has even height.

Remark 4.4. The case $m = 2$ of this theorem does not extend to squares of arbitrary representations of the symmetric group. For example, consider the representation with character $\chi^{(4)} + \chi^{(2,1^2)}$. Its fake degree polynomial is $f(q) = 1 + q^3 + q^4 + q^5$. Then we obtain, for a primitive fourth root of unity ξ, that $f(\xi)^2 = 4$ and $f(\xi^2)^2 = 0$. This violates the condition in Theorem 2.2 for $k = 2$, because $\mu(2)f(\xi)^2 + \mu(1)f(\xi^2)^2 = -4$.

Corollary 4.5. *Let $\lambda = (a, 1^{n-a})$ be a hook-shaped partition of n. Then there is a cyclic group action ρ such that $\big(\mathrm{SYT}(\lambda), \langle\rho\rangle, f^{\lambda}(q)\big)$ exhibits the cyclic sieving phenomenon if and only if n and a are odd and $a-1 \mod m$ is even for $m \mid n$, $1 \leq m < a$.*

Remark 4.6. According to the previous theorem, for $\lambda = (3, 1^{n-3})$ a cyclic group action of order n with character $f^{\lambda}(q) = q^{(n-2)(n-3)/2}\frac{[n-1]_q[n-2]_q}{[2]_q}$ exists for all odd $n > 3$. In this case, there should be one singleton orbit and $(n-3)/2$ orbits of size n. Indeed, an appropriate group action can be constructed as follows:

Identify a tableau with the two labels $x < y$ different from 1 in the first row. Note that $y - x \in \{1, 2, \ldots, n-2\}$, and only the pair $(2, n)$ has difference $n-2$. We let the generator of the group action η act as follows:

$$\eta(x,y) := \begin{cases} (2,n) & \text{if } x = 2,\ y = n, \\ (x+2, y+2) & \text{if } 2 \leq x < y \leq n-2, \\ (2, x+1) & \text{if } y = n-1, \\ (3, x+1) & \text{if } x > 2,\ y = n. \end{cases}$$

We then note that if $(u, v) = \eta(x, y)$, then $v - u \in \{y - x, (n-2) - (y-x)\}$. This explains why there are $(n-3)/2$ orbits of length n. We leave the remaining details to the reader.

A different way to ensure positivity of the character $f^{\lambda/\mu}$ is to decrease the order of the cyclic group.

Theorem 4.7. *Let λ/μ be a skew shape such that every row contains a multiple of m cells. Then there is a cyclic group action ρ of order m such that*

$$\Big(\mathrm{SYT}(\lambda/\mu), \langle\rho\rangle, f^{\lambda/\mu}(q)\Big)$$

exhibits the cyclic sieving phenomenon.

5 Permutations and invariants of the adjoint representation of GL_n

In this section we apply our results to study the space of invariants of tensor powers of the adjoint representation $\mathfrak{gl}_r$ of the general linear group GL_r.

Definition 5.1. The *rotation* $\operatorname{rot}\sigma$ of a permutation $\sigma \in \mathfrak{S}_n$ is the permutation obtained by conjugating with the long cycle $(1, \ldots, n)$.

Remark 5.2. Equivalently, let D_σ be the chord diagram associated with σ, that is, the directed graph with vertices $\{1, \ldots, n\}$ arranged counterclockwise on a circle, and arcs $(i, \sigma(i))$. Then $D_{\operatorname{rot}\sigma}$ is the chord diagram obtained by rotating the graph clockwise. See Figure 2 for an illustration.

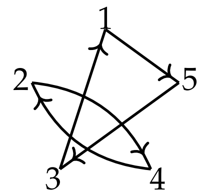
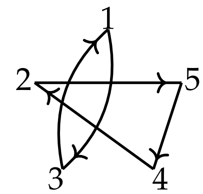
$[5{,}4{,}1{,}2{,}3] =$  $\overset{\text{rot}}{\mapsto}$  $= [3{,}5{,}1{,}2{,}4]$

Figure 2: Rotation of $\pi = [5{,}4{,}1{,}2{,}3]$ as rotation of the chord diagram. Note that $\mathrm{sh}([5{,}4{,}1{,}2{,}3]) = (3, 1^2)$ and $\mathrm{sh}([3{,}5{,}1{,}2{,}4]) = (3, 2)$.

The following theorem makes the character of rotation explicit.

Theorem 5.3 ([3, 12, 13])**.**

$$\left(\mathfrak{S}_n, \langle \mathrm{rot} \rangle, \sum_{\lambda \vdash n} f^\lambda(q)^2\right)$$

exhibits the cyclic sieving phenomenon.

Definition 5.4. Recall that the *Robinson–Schensted correspondence* provides a bijection

$$\mathfrak{S}_n \leftrightarrow \{(P, Q) \in \mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda) : \lambda \vdash n\}.$$

The *shape* $\mathrm{sh}(\sigma)$ of a permutation σ is the common shape of the standard Young tableaux P and Q corresponding to σ under the Robinson–Schensted correspondence. We let R_λ denote the set of permutations of shape λ.

By Theorem 4.3 there exists an action of the cyclic group of order n on R_λ with character $(f^\lambda(q))^2$. Taking the direct sum of the underlying linear representations over all partitions of n we obtain by Theorem 5.3 and Theorem 2.4, the following.

Theorem 5.5. *Let P_n be the set of partitions of n. Then there exists a map $\mathrm{st} : \mathfrak{S}_n \to P_n$ which is invariant under rotation and equidistributed with the Robinson–Schensted shape. That is,*

$$\mathrm{st} \circ \mathrm{rot} = \mathrm{st} \qquad \text{and} \qquad \#\{\sigma \in \mathfrak{S}_n : st(\sigma) = \mu\} = \#\{\sigma \in \mathfrak{S}_n : sh(\sigma) = \mu\}$$

for all $\mu \vdash n$. Moreover, with $\mathfrak{S}_n^\lambda := \{\pi \in \mathfrak{S}_n : \mathrm{st}(\sigma) = \lambda\}$, the triple

$$(\mathfrak{S}_n^\lambda, \langle \mathrm{rot} \rangle, f^\lambda(q)^2)$$

exhibits the cyclic sieving phenomenon.

Remark 5.6. We stress that we are currently unable to present such a statistic explicitly.

We now turn to the connection with the invariants of tensor powers of the adjoint representation of GL_r, which is the original motivation for this article.

Let V be an r-dimensional complex vector space and let $\mathfrak{gl}_r = \mathrm{End}(V)$ be the adjoint representation $\mathrm{GL}_r \to \mathrm{End}(\mathfrak{gl}_r)$, $A \mapsto TAT^{-1}$. We consider the action of the symmetric group $\mathfrak{S}_n$ on $\mathfrak{gl}_r^{\otimes n}$ which permutes tensor positions. This action is inherited by the subspace $\left(\mathfrak{gl}_r^{\otimes n}\right)^{\mathrm{GL}_r}$ of GL_r-invariants. Using Schur–Weyl duality we obtain the following.

Lemma 5.7 ([13])**.** *Let $\mathfrak{gl}_r$ be the adjoint representation of GL_r, and, given a partition $\lambda \vdash n$, let S_λ be the corresponding irreducible representation of the symmetric group. Then there is an isomorphism*

$$\left(\mathfrak{gl}_r^{\otimes n}\right)^{\mathrm{GL}_r} \cong \bigoplus_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq r}} S_\lambda \otimes S_\lambda.$$

Theorem 5.8. *There is a basis of $\left(\mathfrak{gl}_r^{\otimes n}\right)^{\mathrm{GL}_r}$ which is preserved by the action of the long cycle. Moreover, this action is isomorphic to the action of rotation on*

$$\mathfrak{S}_n^{(r)} := \{\pi \in \mathfrak{S}_n : \ell(\mathrm{st}(\pi)) \leq r\} = \bigcup_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq r}} \mathfrak{S}_n^\lambda.$$

Proof. Theorem 5.5 together with Lemma 5.7 shows that the linear representation corresponding to the action of rotation on $\mathfrak{S}_n^{(r)}$ and

$$\bigoplus_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq r}} S_\lambda \otimes S_\lambda \downarrow_{\langle (1,\dots,n) \rangle}$$

have the same character. □

Remark 5.9. There is a notion of promotion for so called *alternating tableaux*, that is, highest weight words in the crystal for $\mathfrak{gl}_r^{\otimes n}$. It follows from our results and a theorem of Westbury, that this action is isomorphic to the action of rotation on $\mathfrak{S}_n^{(r)}$. For $r \geq n$ an appropriate bijection is given in [9].

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