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Completing and extending shellings of vertex decomposable complexes

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Abstract. We say that a pure d-dimensional simplicial complex Δ on n vertices is *shelling completable* if Δ can be realized as the initial sequence of some shelling of $\Delta_{n-1}^{(d)}$, the d-skeleton of the $(n-1)$-dimensional simplex. A well-known conjecture of Simon posits that any shellable complex is shelling completable. We prove that vertex decomposable complexes are shelling completable. In fact we show that if Δ is a vertex decomposable complex then there exists an ordering of its ground set V such that adding the revlex smallest missing $(d+1)$-subset of V results in a complex that is again vertex decomposable. We explore applications to matroids, shifted complexes, as well as k-vertex decomposable complexes. We also show that if Δ is a d-dimensional complex on at most $d+3$ vertices then the notions of shellable, vertex decomposable, shelling completable, and extendably shellable are all equivalent.

Keywords: extendable shellability, vertex decomposability, Simon's Conjecture

1 Introduction

A pure simplicial complex Δ is *shellable* if there exists an ordering of its facets $F_1, F_2, \ldots, F_s$ such that each F_i intersects the previous facets in a pure codimension one complex (see Definition 2.1). Shellability is an important combinatorial tool that has consequences for the topology of Δ as well as algebraic properties of its Stanley–Reisner ring. Examples of shellable simplicial complexes include the independence complexes of matroids [16], boundary complexes of simplicial polytopes [6], as well as the skeleta of shellable complexes [4]. In particular for any $k = 1, 2, \ldots, n-1$ the k-skeleton of a simplex on vertex set $[n]$, which we denote $\Delta_{n-1}^{(k)}$, is shellable.

Given a shellable complex a natural question to ask is whether one can get 'stuck' in the process of building a shelling order. A shellable complex Δ is said to be *extendably shellable* if any shelling of a subcomplex of Δ can be extended to a shelling of Δ. Here a subcomplex of Δ is a simplicial complex on the same vertex set as Δ, whose set of facets is a subset of the facets of Δ.

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Although shellable complexes arise naturally in many contexts, it seems that extendably shellable complexes are harder to come by. Results of Danaraj and Klee [10] imply that any 2-dimensional triangulated sphere (which is necessarily polytopal) is extendably shellable, and Kleinschmidt [14] has shown that any d-dimensional sphere with $d+3$ vertices is extendably shellable. Björner and Eriksson [3] proved that independence complexes of rank 3 matroids are extendably shellable. On the other hand Ziegler [20] has shown that there exist simplicial 4-polytopes that are not extendably shellable. The motivation for much of our work will be the following question posed by Simon [17].

Conjecture 1.1 (Simon's Conjecture). *[17] The complex $\Delta_{n-1}^{(k)}$ is extendably shellable.*

The $k = 2$ case of Simon's Conjecture follows from [3] by considering the uniform matroid of rank 3, and Bigdeli, Yazdan Pour, and Zaare-Nahandi [1] have established the $k \geq n-3$ cases (a simpler proof was provided independently by the second author [11] based on results of Culbertson, Guralnik, and Stiller [9]). In [8] it is shown that if Δ is a d-dimensional simplicial complex on at most $d+3$ vertices, then in fact the notions of shellable and extendably shellable are equivalent. This implies the $k = n-3$ case of Simon's conjecture and also provides a generalization of Kleinschmidt's results. This result is also best possible in the sense that there are 2-dimensional complexes on 6 vertices that are not extendably shellable ([15], [2]).

Inspired by these results and observations, in this paper we consider a related notion.

Definition 1.2. *A pure d-dimensional simplicial complex Δ on n vertices is said to be* shelling completable *if there exists a shelling $F_1, F_2, \dots, F_s$ of Δ that can be taken as the initial sequence of some shelling of $\Delta_{n-1}^{(d)}$.*

In particular, a shelling completable complex is shellable. Note that if Δ is shelling completable then any shelling of Δ can be completed to a shelling of $\Delta_{n-1}^{(d)}$. Also note that Simon's conjecture is equivalent to the statement that any pure shellable complex is shelling completable. From this perspective it is of interest to find a large class of shellable complexes that are shelling completable.

Our first examples of shelling completable complexes come from the class of shifted complexes. Recall that a simplicial complex is *shifted* if there exists an ordering of its vertex set $V = \{1, 2, \dots, n\}$ such that for any face $\{v_1, v_2, \dots, v_k\}$ replacing any v_i with a smaller vertex results in a k-set that is also a face. Note that if Δ is a d-dimensional shifted complex according to some ordering on its vertex set then adding the reverse-lexicographically (revlex) smallest missing $(d+1)$-subset F again results in a shifted complex (see Proposition 3.1 below). Since shifted complexes are known to be shellable [5], this implies that shifted complexes are shelling completable.

Shifted complexes are examples of the more general class of *vertex decomposable complexes* (see Definition 2.4), and our first result says that in fact all such complexes admit an ordering of its vertex set with this property.

Theorem 3.2. *Suppose Δ is a d-dimensional vertex decomposable simplicial complex on ground set V. Then either Δ is full over V or there exists a linear order on V such that if F is the revlex smallest $(d+1)$-subset of V not contained in Δ then the simplicial complex generated by $\Delta \cup \{F\}$ is vertex decomposable.*

As a corollary we obtain a large class of shelling completable complexes, providing a positive answer to a weakened version of Simon's Conjecture.

Corollary 3.4. *Vertex decomposable complexes are shelling completable.*

Vertex decomposable complexes include pure shifted complexes and (independence complexes of) matroids. Theorem 3.2 implies that there exists an ordering of the ground set of these complexes with the property that adding the revlex smallest missing k-subset results in a vertex decomposable complex. In the context of shifted complexes we have seen that the natural ordering of the ground set satisfies this property. For the case of matroids we prove that such *decomposing orders* (see Definition 4.2) are easy to come by.

Proposition 4.4. *Let $\mathcal{M}$ be a rank d matroid on ground set V. Then any ordering $v_1, v_2, \ldots, v_n$ of V with the property that $\{v_1, v_2, \ldots, v_d\} \in \mathcal{M}$ is a decomposing order.*

In particular for a rank d matroid $\mathcal{M}$ it is 'easy' to find a d-subset F of the ground set with the property that $\mathcal{M} \cup \{F\}$, while no longer a matroid, is still vertex decomposable.

In the last part of the paper we consider shelling completable complexes with few vertices (relative to dimension). We exploit a connection between chordal graphs and certain shellable complexes to prove the following.

Theorem 5.2. *Suppose Δ is a shellable d-dimensional simplicial complex on $d+3$ vertices. Then Δ is vertex decomposable (and hence shelling completable).*

This theorem, along with results from [8], imply that for these complexes the notions of vertex decomposable, shellable, shelling completable, and extendably shellable are all equivalent.

The rest of this extended abstract is organized as follows. In Section 2 we recall some necessary definitions. In Section 3 we discuss the ideas behind Theorem 3.2 along with relevant corollaries. In Section 4 we discuss the notion of decomposing orders in the context of matroids and Theorem 4.4. In Section 5 we consider d-dimensional complexes on at most $d+3$ vertices and sketch the proof of Proposition 5.2. We end in Section 6 with some discussion and open questions. Throughout the paper we provide only the main ideas involved in our arguments, we refer to [7] for complete proofs.

2 Vertex decomposable complexes

A *simplicial complex* Δ on a finite ground set V is a collection of subsets of V that is closed under taking subsets, so that if $\sigma \in \Delta$ and $\tau \subset \sigma$ then $\tau \in \Delta$. The elements of Δ are called *faces*. Note that we do not require $\{v\} \in \Delta$ for all $v \in V$. The elements $v \in V$ such that $\{v\} \in \Delta$ will be called the *vertices* of Δ, whereas elements $w \in V$ that are not vertices will be called *loops*. In particular the vertex set of Δ can be a proper subset of its ground set. As in [13] we adopt the convention that the *void complex* $\emptyset$ is a simplicial complex, distinct from the *empty complex* $\{\emptyset\}$. We will sometimes refer to the simplicial complex *generated by* a collection of subsets $S \subset \mathcal{P}(V)$, by which we mean the smallest simplicial complex containing S.

A *facet* of Δ is an element that is maximal under inclusion. The *dimension* of Δ is the largest cardinality (minus 1) of any facet. A simplicial complex Δ is *pure* if all facets have the same cardinality. For $W \subset V$ with $|W| = d + 1$, we let 2^W denote the set of all subsets of W and refer to it as a *d-simplex* (on W). With these notions we can state the definition of a shellable complex.

Definition 2.1. *A pure d-dimensional simplicial complex Δ is said to be* shellable *if there exists an ordering of its facets $F_1, F_2, \ldots, F_s$ such that for all $k = 2, 3, \ldots, n$ the simplicial complex generated by*

$$\Big(\bigcup_{i=1}^{k-1} F_i\Big) \cap F_k$$

is pure of dimension $d - 1$. By convention the void complex $\emptyset$ and the empty complex $\{\emptyset\}$ are both shellable.

Note that a shellable complex is connected as long as $d \geq 1$, and a 1-dimensional simplicial complex (a graph) is shellable if and only if it is connected. We next recall the notion of link, star, and deletion of a face in a simplicial complex.

Definition 2.2. *Suppose Δ is a simplicial complex on ground set V and let $F \in \Delta$ be a face. The* link, star *and the* deletion *of F are defined as*

$$\ell k_\Delta(F) := \{G \in \Delta : G \cap F = \emptyset, G \cup F \in \Delta\},$$

$$star_\Delta(F) := \{G \in \Delta : F \subset G\},$$

$$del_\Delta(F) := \{G \in \Delta : F \not\subseteq G\}.$$

The ground set of $star_\Delta(F)$ is V, and the ground sets of $\ell k_\Delta(F)$ and $del_\Delta(F)$ are given by $V \backslash F$.

We note that shellability is preserved by taking links, a fact that will be useful later.

Lemma 2.3 ([21], Lemma 8.7)**.** *If Δ is a shellable simplicial complex and $F \in \Delta$ is any face, then the link $\ell k_\Delta(F)$ is shellable.*

We next define the class of vertex decomposable simplicial complexes recursively as follows.

Definition 2.4. *A simplicial complex Δ is* vertex decomposable *if Δ is a simplex (including $\emptyset$ and $\{\emptyset\}$), or Δ contains a vertex v such that*

1. *both $del_\Delta(v)$ and $\ell k_\Delta(v)$ are vertex decomposable, and*
2. *any facet of $del_\Delta(v)$ is a facet of Δ.*

A vertex v that satisfies the second condition is called a shedding vertex *of Δ. We will call a vertex v that satisfies both conditions a* decomposing vertex.

Vertex decomposable complexes were introduced in the pure setting by Provan and Billera [16] and extended to non-pure complexes by Björner and Wachs [4]. It is known that any vertex decomposable complex is shellable, a fact implied by the following result of Wachs [19].

Lemma 2.5 ([19], Lemma 6)**.** *Suppose Δ is a simplicial complex with shedding vertex v. If both $del_\Delta(v)$ and $\ell k_\Delta(v)$ are shellable then Δ is shellable.*

3 Shelling completions

We next turn to the question of shelling completions for vertex decomposable complexes. For this we will need the following concepts. Suppose Δ is a d-dimensional simplicial complex on ground set V. We say that Δ is *full (over V)* if it is the d-skeleton of the simplex over the vertex set V, i.e. it consists of all $d+1$-subsets of V. Note that a d-simplex is full if and only if $|V| = d+1$.

We will also need the notion of reverse lexicographic (revlex) order on k-subsets of an ordered ground set. For this recall that if $V = \{1, 2, \dots, n\}$ is a linearly ordered set, then $\{v_1 < v_2 < \cdots < v_k\}$ is *revlex smaller* than $\{w_1 < w_2 < \cdots < w_k\}$ if for the largest j with $v_j \neq w_j$ we have $v_j < w_j$. Note that if one adds a new element $n+1$ to the set V then any k-subset that contains $n+1$ will be revlex larger than any k-subset that does not contain $n+1$.

As mentioned in Section 1 , we can use revlex orders to build new shifted complexes from existing ones. More precisely we have the following.

Proposition 3.1. *Suppose Δ is a pure shifted d-dimensional simplicial complex with respect to some linear order on its ground set V, and assume that Δ is not full on V. Let F be the revlex smallest $(d+1)$-subset of V satisfying $F \notin \Delta$. Then the complex generated by $\Delta \cup \{F\}$ is again a shifted simplicial complex.*

Proof. One can see that replacing any element $x \in F$ with some $y \in V$ satisfying $y < x$ results in a $(d+1)$-subset $F' = (F \setminus \{x\}) \cup \{y\}$ that is revlex smaller than F. Hence F' is a facet of $\Delta \cup \{F\}$, implying that $\Delta \cup \{F\}$ is shifted. □

Our main result generalizes this observation for the class of vertex decomposable complexes.

Theorem 3.2. *Suppose Δ is a d-dimensional vertex decomposable simplicial complex on ground set V. Then either Δ is full over V or there exists a linear order on V such that if F is the revlex smallest $(d+1)$-subset of V not contained in Δ then the simplicial complex generated by $\Delta \cup \{F\}$ is vertex decomposable.*

Proof. (Sketch) We use induction on d, then on $n = |V|$. The base cases can be checked by hand so we assume $d \geq 1$ and $n \geq d+2$. First if Δ is full over its vertex set W we pick any ordering on the ground set V so that all elements in W are smaller than all elements in $V \setminus W$. One can check that adding the revlex smallest missing $(d+1)$-subsets results in a vertex decomposable complex. If Δ is not full on its vertex set we can continue the induction by either picking v to be a loop or a decomposing vertex. If v is a loop we consider the complex Δ on ground set $V \setminus \{v\}$ and use induction. If v is a decomposing vertex we have two subcases to consider.

If $\mathrm{del}_\Delta(v)$ is not full on its vertex set $V \setminus \{v\}$ then we consider the complex $\mathrm{del}_\Delta(v)$, and use induction on n to extend the given ordering $V \setminus \{v\}$. If F is the revlex smallest missing $(d+1)$-subset $F \subset V \setminus \{v\}$ can be added added to $\mathrm{del}_\Delta(v)$ to obtain another vertex decomposable complex, one can check that F is revlex smallest among the missing facets of Δ, and that the complex generated by $\Delta \cup \{F\}$ is also vertex decomposable (with decomposing vertex v). If $\mathrm{del}_\Delta(v)$ is full on $V \setminus \{v\}$ then we consider the complex $\ell\mathrm{k}_\Delta(v)$ and use induction on d. We let G denote the revlex smallest missing d-set of $\ell\mathrm{k}_\Delta(V)$. One can show that $\{G \cup \{v\}\}$ is the revlex smallest missing $(d+1)$-subset of Δ and also that the complex generated by $\Delta \cup \{G \cup \{v\}\}$ is vertex decomposable. The result follows. □

For an illustration of the various steps in the above proof, we refer to Example 4.3. To establish our desired corollary we will next need the following observation.

Lemma 3.3. *Suppose Δ is a shellable d-dimensional simplicial complex on ground set V and suppose F is a $(d+1)$-subset of V with the property that the complex Δ' generated by $\Delta \cup \{F\}$ is again shellable. Then any shelling of Δ can be extended to a shelling of Δ' by adding F as the last facet.*

As a corollary we get a large class of complexes that are shelling completable, and hence we obtain a weakened form of Simon's conjecture.

Corollary 3.4. *Vertex decomposable complexes are shelling completable.*

Proof. Suppose Δ is a pure d-dimensional vertex decomposable complex on ground set V, where $|V| = n$. Let m be the number of $(d+1)$-subsets of V that are missing as facets in Δ. If $m = 0$ then $\Delta = \Delta_{n-1}^{(d)}$ is full and we are done. Otherwise by Theorem 3.2 we have some $(d+1)$-subset $F \subset V$ such that $F \notin \Delta$ with $\Delta \cup \{F\}$ vertex decomposable, and hence shellable. From Lemma 3.3 we know that any shelling order of Δ can be extended to a shelling of $\Delta \cup \{F\}$. The result follows by induction on m. □

4 Decomposing orders and matroid complexes

Recall that a simplicial complex $\mathcal{M}$ is a *matroid* if it is pure and its set of facets satisfy the following exchange property: If F and G are facets of $\mathcal{M}$ then for any $x \in F \backslash G$ there exists some $y \in G \backslash F$ such that $(F \backslash \{x\}) \cup \{y\}$ is a facet of $\mathcal{M}$. The facets of $\mathcal{M}$ are usually called *bases* in this theory. Also note that in some contexts this simplicial complex is called the *independence complex* of $\mathcal{M}$ but we will simply refer it to as the matroid itself. It is well known that matroids are vertex decomposable [16] and hence Corollary 3.4 implies the following.

Corollary 4.1. *Independence complexes of matroids are shelling completable.*

Given that any shelling of a rank d matroid can be completed to a shelling of the full skeleton $\Delta_n^{(d-1)}$, a natural question to ask is whether one can control which facet can be added in the next step of the completion. In the context of matroids, one expects some flexibility since matroids themselves admit many shelling orders. In particular recall that if $V = \{v_1, v_2, \dots, v_n\}$ is any ordering of the ground set of a rank d matroid $\mathcal{M}$, then both lexicographic ([2] Theorem 7.3.4) and reverse lexicographic ([12] Proposition 6.3) orderings of the facets (bases) of $\mathcal{M}$ give rise to a shelling of the complex $\mathcal{M}$. In fact matroids can be characterized by the property that any ordering of the ground set gives rise to such a shelling. For our purposes we will need the following notion.

Definition 4.2. *Suppose Δ is a a pure d-dimensional vertex decomposable simplicial complex. An ordering $v_1, v_2, \dots, v_n$ of its ground set is a* decomposing order *for Δ if the complex generated by $\Delta \cup \{F\}$ is again vertex decomposable, where F is the revlex smallest $(d+1)$-subset of V that is missing from Δ.*

Note that Theorem 3.2 says that any vertex decomposable complex admits a decomposing order. Also note that in the proof of Theorem 3.2 at each step in the induction we must choose a vertex v where we employ the inductive hypothesis on either the deletion $\mathrm{del}_\Delta(v)$ (in the first case) or the link $\ell\mathrm{k}_\Delta(v)$ (in the latter). In this way we can obtain a sequence of subcomplexes

$$\Delta = \Delta_n, \Delta_{n-1}, \dots, \Delta_s,$$

where n is the size of the ground set of Δ, and Δ_s is a simplex over some (possibly smaller) ground set. We illustrate this process below with a worked example.

Example 4.3. *Suppose Δ is the 3-dimensional complex on ground set $\{1, 2, \ldots, 7\}$ with facets*

$$\{1234, 1235, 1245, 1345, 2345, 1236, 1246, 1256, 2356, 1237, 2347\}.$$

Here we abuse notation and for example let 1245 denote the 4-subset $\{1, 2, 4, 5\}$. We use the natural ordering on the ground set and verify that it is a decomposing order.

First we define $\Delta_7 = \Delta$ and note that

$$del_{\Delta_7}(7) = \{1234, 1235, 1245, 1345, 2345, 1236, 1246, 1256, 2356\}$$

is not full (e.g. 1346 is missing). Hence at this step we consider the deletion of vertex 7 and define $\Delta_6 = del_{\Delta_7}(7)$. Next we note that $del_{\Delta_6}(6)$ is full so we now consider the link of 6 and define

$$\Delta_5 = \ell k_{\Delta_6}(6) = \{123, 124, 125, 235\}.$$

Continuing in this fashion we have that $del_{\Delta_5}(5)$ is not full so we define

$$\Delta_4 = del_{\Delta_5}(5) = \{123, 124\}.$$

Next we see that $del_{\Delta_4}(4) = \{123\}$ is full and so we consider the link of 4 and have

$$\Delta_3 = \ell k_{\Delta_4}(4) = \{12\}.$$

At this point we see that Δ_3 is a simplex (on ground set $\{1, 2, 3\}$) and hence we have reached a base case.

Reversing this process, we see that at each step the addition of a new facet F leads to a vertex decomposable complex. We begin with Δ_3 since it is full over its vertex set, and hence a base case of Theorem 3.2. We extend Δ_3 to Δ'_3 by noting that 3 is the smallest loop, and 2 is the largest vertex. Hence we add the facet $(12\backslash\{2\}) \cup \{3\} = 13$. Now in Δ_4 we replace $\ell k_{\Delta_4}(4) = \Delta_3$ with Δ'_3, which results in adding the facet 134 to obtain Δ'_4. In Δ_5 we replace $del_{\Delta_5}(5) = \Delta_4$ with Δ'_4 which results in adding the facet 134 to obtain Δ'_5. Next in Δ_6 we replace $\ell k_{\Delta_6}(6) = \Delta_5$ with Δ'_5, adding facet 1346 to obtain Δ'_6. Finally in Δ_7 we replace $del_{\Delta_7}(7) = \Delta_6$ with Δ'_6, adding the facet 1346. We note that 1346 is indeed the smallest element missing from $\Delta = \Delta_7$ among the revlex ordered 4-subsets of $\{1, \ldots, 7\}$

We will use the above observations to show that many orderings of the ground set of a matroid give rise to decomposing orders.

Proposition 4.4. *Let $\mathcal{M}$ be a rank d matroid and suppose $v_1, v_2, \ldots, v_n$ is any linear ordering of its ground set V with the property that $\{v_1, v_2, \ldots, v_d\} \in \mathcal{M}$. Then $v_1, v_2, \ldots, v_n$ is a decomposing order for $\mathcal{M}$.*

Proof. (Sketch) Suppose that $v_1, v_2, \ldots, v_n$ is such an ordering of V and let F be the revlex smallest $(d+1)$-subset of V that is missing from Δ. Let q be the smallest $i \geq 2$ such that $v_i \in F$ and $v_{i-1} \notin F$. Since $\mathcal{M}$ is a matroid we have that any element v is a loop or a decomposing vertex. From the proof of Theorem 3.2, we get that $v_1, v_2, \ldots, v_n$ is decomposing order for $\mathcal{M}$ if $v_1, v_2, \ldots, v_q$ is a decomposing order for

$$\Gamma := \ell\mathrm{k}_\Delta(F \cap \{v_{q+1}, \ldots, v_n\})|_{\{v_1,\ldots,v_q\}}.$$

We see that Γ is a matroid with the property that $\mathrm{del}_\Gamma(v_q)$ is full, which follows from the fact that F was the revlex smallest d-subset missing from Δ. One can check that any ordering of the ground set of Γ is a decomposing ordering, and the result follows. □

As a consequence of Proposition 4.4 we get the following.

Corollary 4.5. *Let $\mathcal{M}$ be a rank d matroid and suppose $v_1, v_2, \ldots, v_n$ is any linear ordering of its ground set V with the property that $\{v_1, v_2, \ldots, v_d\} \in \mathcal{M}$, and let F be the revlex smallest d-subset missing from $\mathcal{M}$. Then the complex generated by $\mathcal{M} \cup \{F\}$ is vertex decomposable.*

Remark 4.6. *Given a rank d matroid $\mathcal{M}$ on ground set V, a related question to ask is whether there exists a d-subset $F \subset V$ such that $\mathcal{M} \cup \{F\}$ is again a* matroid. *Truemper [18] has shown that if $\mathcal{M}$ is connected then this is the case if and only if F is a* circuit hyperplane *(that is, a circuit in $\mathcal{M}$ such that $V \backslash F$ is a circuit of the dual matroid $\mathcal{M}^*$).*

In light of Corollary 4.5 a natural question to ask is whether there exists an ordering of the ground set such that *all* missing facets can be added in reverse lexicographic order. The next example shows that an arbitrary ordering will not work.

Example 4.7. *Let $\mathcal{M}$ be the matroid on ground set $[6]$ generated by the facets*

$$\{1234, 1345, 2346, 3456\}.$$

We note that adding the revlex smallest missing 4-subset 1235 results in a shelling move but the shelling fails when we continue to add the next revlex smallest subsets

$$1235, 1245, 1236, 1246, 1256.$$

To see this let Δ denote the complex obtained by adding these 4-subsets and consider $F = 56$, a face of Δ. We note that $\ell k_\Delta(F) = \{34, 12\}$, which is 1-dimensional and disconnected and hence not shellable. By Lemma 2.3 we conclude that Δ is not shellable.

5 Complexes with few vertices

In [8] it is shown that a d-dimensional complex Δ on $d+3$ vertices is extendably shellable if and only if Δ is shellable. In this section we show that these conditions are also equivalent to Δ being vertex decomposable. In what follows we will assume that our complexes have no loops, so that vertex set of Δ coincides with its ground set.

For our result we will exploit a connection between shellable complexes and the notion of a chordal graph. Recall that a simple graph G is *chordal* if it has no induced cycles of length 4 or more (so that all cycles of length 4 or more have a 'chord'). It is well known that any chordal graph admits a *simplicial vertex*, a vertex $v \in V(G)$ such that its neighborhood (the subgraph induced on the set of vertices adjacent to v) is a complete graph. From [11] we have the following result, adapted for our purposes.

Lemma 5.1. *Let K_n denote the complete graph on vertex set $[n] = \{1, 2, \ldots, n\}$. Suppose $\{e_1, e_2, \ldots, e_k\} \subset E(K_n)$ is a collection of edges and for each $j = 1, 2, \ldots, k$ let $F_j = [n] \backslash e_j$ denote the complementary $(n-2)$-subset. Then $K_n \backslash \{e_1, e_2, \ldots, e_k\}$ is a chordal graph if and only if the simplicial complex induced by $F_1, F_2, \ldots F_k$ is shellable.*

Theorem 5.2. *Suppose Δ is a shellable d-dimensional simplicial complex on $d+3$ vertices. Then Δ is vertex decomposable.*

Proof. (Sketch) We prove the statement by induction on d. If $d = 1$ the claim follows since a graph is connected if and only if it is vertex decomposable. Suppose $d \geq 1$ and let Δ be a d-dimensional complex with shelling order $F_1, F_2, \ldots, F_j$. Let G be the graph consisting of edges $\{V \backslash F_i : F_i \in \Delta\}$. From Lemma 5.1 we have that G is chordal and hence admits a simplicial vertex $v \in G$. We then have that $G \backslash \{v\}$ is chordal and hence $\text{del}_\Delta(v)$ is a shellable d-dimensional complex on at most $d+2$ vertices. It follows that $N(v)$ is vertex decomposable. We see that $\ell\text{k}_\Delta(v)$ is a shellable $(d-1)$-dimensional complex on at most $d+2$ vertices, which by induction is vertex decomposable. The fact that v is simplicial implies that v is a shedding vertex, and the result follows. □

Hence for a d-dimensional simplicial complex on at most $d+3$ vertices the concepts of shellable, extendably shellable, shelling completable, and vertex decomposable are all equivalent. We note that there exist 2-dimensional complexes on 6 vertices that are shellable but not vertex decomposable [15].

6 Completing k-decomposable complexes

Recall that Simon's Conjecture posits that all shellable complexes are shelling completable, and here we have shown the conjecture holds for the particular class of vertex decomposable complexes. We can ask the same question for complexes that properly sit in between these two classes. We first recall the relevant definition.

Definition 6.1. *A pure d-dimensional simplicial complex* Δ *is said to be k*-vertex decomposable *if* Δ *is a simplex, or* Δ *contains a face F such that*

1. $\dim(F) \leq k$
2. *both* $del_\Delta(F)$ *and* $\ell k_\Delta(F)$ *are k-vertex decomposable, and*
3. $del_\Delta(F)$ *is pure (and the dimensions stays same as that of* Δ*).*

The notion of k-vertex decomposable interpolates between the notion of vertex decomposable (which is equivalent to 0-vertex decomposable in this language) and shellable (which can be seen to coincide with d-vertex decomposable).

Example 6.2 (Example V6F10-6 from [15])**.** *Let* Δ *be the* 2*-dimensional complex with facets*

$$\{123, 124, 125, 134, 136, 245, 256, 346, 356, 456\}.$$

In [15] it is shown that Δ *is not vertex decomposable, but one can check that it is* 1*-decomposable using* 15 *as a shedding face.*

Our results imply that a 0-vertex decomposable complex is shelling completable, and Simon's conjecture posits that a d-vertex decomposable complex is shelling completable. As far we know it is an open question whether a 1-vertex decomposable complex is shelling completable.

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