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Gröbner geometry for skew-symmetric matrix Schubert varieties

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Abstract. Matrix Schubert varieties are the orbit closures of $B \times B$ acting on all $n \times n$ matrices, where B is the group of invertible lower triangular matrices. We define skew-symmetric matrix Schubert varieties to be the orbit closures of B acting on all $n \times n$ skew-symmetric matrices. In analogy with Knutson and Miller's work on matrix Schubert varieties, we describe a natural generating set for the prime ideals of these varieties. We then compute a related Gröbner basis. Using these results, we identify a primary decomposition for the corresponding initial ideals involving certain fpf-involution pipe dreams, analogous to the pipe dreams of Bergeron and Billey. We show that these initial ideals are the Stanley–Reisner ideals of shellable simplicial complexes. As an application, we give a geometric proof of an explicit generating function for symplectic Grothendieck polynomials.

Keywords: Schubert varieties, Gröbner bases, Grothendieck polynomials, simplicial complexes

1 Introduction

Let $\mathbb{K}$ be an algebraically closed field and write $B_n \subseteq \mathrm{GL}_n := \mathrm{GL}_n(\mathbb{K})$ for the Borel group of $n \times n$ invertible lower triangular matrices over $\mathbb{K}$. An $n \times n$ matrix A is *skew-symmetric* if $A_{ij} = -A_{ji}$ and $A_{ii} = 0$ for all $i, j \in [n]$; note that the second condition is redundant if $\mathrm{char}(\mathbb{K}) \neq 2$. Let $\mathrm{Mat}_n^{\mathrm{ss}} := \mathrm{Mat}_n^{\mathrm{ss}}(\mathbb{K})$ denote the set of such matrices.

Consider the B_n-action on $\mathrm{Mat}_n^{\mathrm{ss}}$ defined by $g \cdot A = gAg^T$. The orbits for this action are *skew-symmetric matrix Schubert cells*, and their Zariski closures are the *skew-symmetric matrix Schubert varieties*. We write X_A^{ss} for the closure of the B_n-orbit of $A \in \mathrm{Mat}_n^{\mathrm{ss}}$.

Identify the coordinate ring $\mathbb{K}[\mathrm{Mat}_n^{\mathrm{ss}}]$ with $\mathbb{K}[u_{ij} : n \geq i > j \geq 1]$, and write $\mathcal{U}^{\mathrm{ss}}$ for the $n \times n$ skew-symmetric matrix with $\mathcal{U}_{ij}^{\mathrm{ss}} = u_{ij} = -\mathcal{U}_{ji}^{\mathrm{ss}}$ for $i > j$ and $\mathcal{U}_{ii}^{\mathrm{ss}} = 0$ for all i. If M is an $n \times n$ matrix and $R, C \subseteq [n]$, we write M_{RC} for the submatrix of M in rows R and columns C. One has $M \in X_A^{\mathrm{ss}}$ if and only if $\mathrm{rank}\, M_{[i][j]} \leq \mathrm{rank}\, A_{[i][j]}$ for all $i, j \in [n]$. Accordingly, X_A^{ss} is the zero locus of the family consisting of all minors $\det(\mathcal{U}_{RC}^{\mathrm{ss}})$ for

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all $R \subseteq [i]$ and $C \subseteq [j]$ with $|R| = |C| = \operatorname{rank} A_{[i][j]} + 1$ for some $i, j \in [n]$. These polynomials do not always generate the prime ideal $I(X^{\text{ss}}_A)$ of the variety X^{ss}_A, however. There is nevertheless a natural generating set for $I(X^{\text{ss}}_A)$, which we describe as follows:

Theorem 1.1 (See Theorem 3.4)**.** For each $A \in \text{Mat}^{\text{ss}}_n$, the collection of Pfaffians $\text{pf}(\mathcal{U}^{\text{ss}}_{RR})$, as R ranges over all even sized subsets of $[n]$ such that $R \subseteq [i]$ and $|R \cap [j]| > \operatorname{rank} A_{[i][j]}$ for some $i, j \in [n]$ with $i \geq j$, generate the (prime) ideal $I(X^{\text{ss}}_A)$.

We are also interested in the problem of describing Gröbner bases for the ideals $I(X^{\text{ss}}_A)$ with respect to the *(graded) reverse lexicographic order* on $\mathbb{K}[\text{Mat}^{\text{ss}}_n]$. Recall that a generating set S of an ideal I is a *Gröbner basis* of I if the leading terms of the polynomials in S generate the *initial ideal* generated by the leading terms of all elements of I. These definitions are reviewed more carefully in §2.

The generating set in the preceding theorem is generally *not* a Gröbner basis for $I(X^{\text{ss}}_A)$; see Example 3.6. Our second main result resolves this problem. Specifically, in Theorem 3.7 we show that a Gröbner basis for $I(X^{\text{ss}}_A)$ with respect to the reverse lexicographic term order is provided by the Pfaffians of the block diagonal matrices

$$\begin{bmatrix} \mathcal{U}^{\text{ss}}_{CC} & \mathcal{U}^{\text{ss}}_{CR} \\ \mathcal{U}^{\text{ss}}_{RC} & 0 \end{bmatrix}$$

for certain subsets $R, C \subseteq [n]$. Experimental evidence suggests that these Pfaffians may also form a Gröbner basis for other so-called *antidiagonal* term orders, but this more general claim does not follow from our present methods.

Suppose n is even and $z \in S_n$ is a fixed-point-free involution, that is, a permutation with $z(z(i)) = i \neq z(i)$ for all $i \in [n]$. Associated to such a permutation is a set $\mathcal{FP}(z)$ of *fpf-involution pipe dreams*, whose elements are certain subsets of $\{(i, j) \in [n] \times [n] : i > j\}$; see Definition 4.1 for the full details. We let $X^{\text{ss}}_z := X^{\text{ss}}_A$ where A is the skew-symmetric $n \times n$ matrix with $A_{ij} = 1$ if $z(j) = i < j = z(i)$ and $A_{ij} = -1$ if $z(j) = i > j = z(i)$. Whenever we write $\text{in}(I(X^{\text{ss}}_z))$, we mean the initial ideal under the reverse lexicographic term order defined in Example 2.2.

The varieties X^{ss}_z as z ranges over the fixed-point-free involutions in S_n are exactly the B_n-orbit closures in $\text{Mat}^{\text{ss}}_n \cap \text{GL}_n$, which is nonempty only if n is even. For simplicity, we only consider these varieties here, but our results can be extended to the full family of skew-symmetric matrix Schubert varieties in Mat^{ss}_n for any n.

Theorem 1.2 (See Theorem 4.2)**.** For each fixed-point-free involution $z \in S_n$, the initial ideal of $I(X^{\text{ss}}_z)$ has primary decomposition $\text{in}(I(X^{\text{ss}}_z)) = \bigcap_{D \in \mathcal{FP}(z)} (u_{ij} : (i, j) \in D)$, where $(u_{ij} : (i, j) \in D)$ denotes the ideal in $\mathbb{K}[\text{Mat}^{\text{ss}}_n]$ generated by u_{ij} for all $(i, j) \in D$.

If a subvariety $X \subseteq \text{Mat}^{\text{ss}}_n(\mathbb{C})$ is invariant under the left B_n-action, then it defines a class $[X]_{B_n}$ in the equivariant cohomology ring $H^*_{B_n}(\text{Mat}^{\text{ss}}_n(\mathbb{C})) \simeq \mathbb{Z}[x_1, \ldots, x_n]$. The polynomial $[X]_{B_n}$ can also be computed algebraically as the *multidegree* of the ideal $I(X)$, and this definition works more generally over any algebraically closed field $\mathbb{K}$.

Using Theorem 1.2, one can show that $[X^{\mathrm{ss}}_z]_{B_n} = \sum_{D\in\mathcal{FP}(z)} \prod_{(i,j)\in D}(x_i + x_j)$ under the identification $H^*_{B_n}(\mathrm{Mat}^{\mathrm{ss}}_n) \simeq \mathbb{Z}[x_1, \ldots, x_n]$. This formula was proven combinatorially in [6]. It was also shown in [6] that the polynomials $[X^{\mathrm{ss}}_z]_{B_n}$ are the same as the *fpf-involution Schubert polynomials* introduced by Wyser and Yong [18], which represent the ordinary cohomology classes the orbit closures of the symplectic group $\mathrm{Sp}_n(\mathbb{C})$ acting on $\mathrm{GL}_n(\mathbb{C})/B_n$. Briefly, the connection to our situation is that $\mathrm{Sp}_n(\mathbb{C})$-orbits on $\mathrm{GL}_n(\mathbb{C})/B_n$ are in bijection with B_n-orbits on $\mathrm{GL}_n(\mathbb{C})/\,\mathrm{Sp}_n(\mathbb{C})$, which can be identified with $\mathrm{Mat}^{\mathrm{ss}}_n(\mathbb{C}) \cap \mathrm{GL}_n(\mathbb{C})$.

Finally, we relate the initial ideal of $I(X^{\mathrm{ss}}_z)$ to the geometry of simplicial complexes. We also use the next result to give a new geometric proof of a combinatorial formula [15, Thm. 4.5] for the B_n-equivariant K-theory representative of X^{ss}_z; see Theorem 4.10.

Theorem 1.3 (See Theorem 4.5)**.** For each fixed-point-free involution $z \in S_n$, the ideal $\mathrm{in}(I(X^{\mathrm{ss}}_z))$ is square-free and equal to the Stanley–Reisner ideal of a shellable simplicial complex.

One can equally well consider the orbits of $B_n \times B_n$ acting on the space of all $n \times n$ matrices by the formula $(g, h) \cdot A = gAh^t$. The closures of these orbits are called *matrix Schubert varieties*. These varieties have been studied by Fulton [5] and Knutson and Miller [10] among others, who proved results analogous to those described above. Indeed, our main goal in this work was to reproduce their results in the skew-symmetric setting.

Ideals generated by Pfaffians of a generic skew-symmetric matrix have been well-studied [3, 4, 7, 8, 17], and there is some overlap between our results and prior work. De Negri and Sbarra [4] consider a family of ideals which, translated into our language, turns out to be a subfamily of the ideals $I(X^{\mathrm{ss}}_A)$ for $A \in \mathrm{Mat}^{\mathrm{ss}}_n$. They observe that the Pfaffian generators of Theorem 1.1 need not form a Gröbner basis with respect to an antidiagonal term order, and proceed to classify the ideals which do enjoy this property. Raghavan and Upadhyay [17] study the same family of ideals, computing their initial ideals and realizing the latter as Stanley–Reisner ideals of shellable complexes, as we do in Theorems 1.2 and 1.3. Although similar in form, their results are in fact quite different from ours, because they use term orders which are far from antidiagonal.

In this extended abstract we have omitted most proofs and focused on new results. However, the techniques we use to prove the results above differ from those of Knutson and Miller, and a secondary goal of this work was to use these techniques to give new proofs of their results; see [12], of which this abstract is a condensed version.

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2 Preliminaries

Throughout, we write $\mathbb{N} = \{1, 2, 3, \dots\}$ for the set of natural numbers and $[n] = \{i \in \mathbb{Z} : 0 < i \leq n\}$ for the first n positive integers. We write $\mathcal{I}_n^{\mathsf{FPF}} = \{w \in S_n : w^2 = 1, w(i) \neq i \text{ for } i \in [n]\}$ for the set of fixed-point-free involutions in the symmetric group S_n. This set is nonempty only if n is even, in which case it is the S_n-conjugacy class of $(1,2)(3,4)\cdots(n-1,n)$.

Suppose $x_1, x_2, \dots, x_N$ are commuting variables, and consider the polynomial ring $\mathbb{K}[\mathbf{x}] := \mathbb{K}[x_1, x_2, \dots, x_N]$ over a field $\mathbb{K}$ (for now arbitrary, but in later sections algebraically closed). A *term order* on the polynomial ring $\mathbb{K}[\mathbf{x}] := \mathbb{K}[x_1, x_2, \dots, x_N]$ is a total order on the set of all monomials, such that 1 is the unique minimum and such that if $\mathsf{mon}_1, \mathsf{mon}_2, \mathsf{mon}_3$ are monomials and $\mathsf{mon}_1 \leq \mathsf{mon}_2$, then $\mathsf{mon}_1\mathsf{mon}_3 \leq \mathsf{mon}_2\mathsf{mon}_3$.

Example 2.1. The *lexicographic term order* on $\mathbb{K}[\mathbf{x}]$ declares that $x_1^{a_1} \cdots x_N^{a_N} \leq x_1^{b_1} \cdots x_N^{b_N}$ whenever $(a_1, \dots, a_N) \leq (b_1, \dots, b_N)$ in lexicographic order. The *(graded) reverse lexicographic term order* declares that $x_1^{a_1} \cdots x_N^{a_N} \leq x_1^{b_1} \cdots x_N^{b_N}$ whenever $\sum_i a_i \leq \sum_i b_i$ and $(a_N, \dots, a_1) \geq (b_N, \dots, b_1)$ in lexicographic order; note the double reversal.

Fix a term order and suppose $f = \sum_{\mathsf{mon}} c_{\mathsf{mon}} \cdot \mathsf{mon} \in \mathbb{K}[\mathbf{x}]$ where the sum is over monomials mon and each $c_{\mathsf{mon}} \in \mathbb{K}$. If f is nonzero, then its *initial term* (or *leading term*) is the maximal monomial mon such that $c_{\mathsf{mon}} \neq 0$. If $f = 0$ then its initial term is also defined to be zero. In either case, we write $\mathrm{in}(f)$ for the corresponding initial term.

The *initial ideal* of an ideal I in $\mathbb{K}[\mathbf{x}]$ is then $\mathrm{in}(I) := \mathbb{K}\text{-span}\{\mathrm{in}(f) : f \in I\}$. This abelian group is itself an ideal in $\mathbb{K}[\mathbf{x}]$. A *Gröbner basis* G for an ideal $I \subseteq \mathbb{K}[\mathbf{x}]$, relative to a fixed term order, is a generating set whose set of initial terms $\{\mathrm{in}(g) : g \in G\}$ generates $\mathrm{in}(I)$.

Example 2.2. In our applications, we will usually take $x_1, x_2, \dots, x_N$ to be either the commuting variables u_{ij} indexed by all positions $(i, j) \in [n] \times [n]$ for some n, or the subset of these variables indexed by positions strictly below the main diagonal. Unless otherwise mentioned, we equip $\mathbb{K}[u_{ij} : i, j \in [n]]$ and its subrings with the (graded) reverse lexicographic order from Example 2.2 with u_{ij} identified with $x_{n(i-1)+j}$.

This means that we order the variables u_{ij} lexicographically, so that $u_{ij} < u_{i'j'}$ if $i < i'$ or if $i = i'$ and $j < j'$. Then, we declare that $\mathsf{mon}_1 < \mathsf{mon}_2$ if either $\deg(\mathsf{mon}_1) < \deg(\mathsf{mon}_2)$ or $\deg(\mathsf{mon}_1) = \deg(\mathsf{mon}_2)$ and the following holds: there is some variable u_{ij} whose exponent e_1 in mon_1 differs from its exponent e_2 in mon_2, and when u_{ij} is the (lexicographically) largest such variable one has $e_1 > e_2$. If mon_1 and mon_2 are both square-free of the same degree, then we have $\mathsf{mon}_1 < \mathsf{mon}_2$ if and only there is some variable u_{ij} that does not divide both monomials, and the largest such variable divides mon_1 but not mon_2.

3 Skew-symmetric matrix Schubert varieties

Fix a positive even integer n and an algebraically closed field $\mathbb{K}$. Recall that $\mathrm{Mat}^{\mathrm{ss}}_n$ denotes the set of $n \times n$ skew-symmetric matrices over $\mathbb{K}$.

Definition 3.1. Given an involution $z \in \mathcal{I}^{\mathsf{FPF}}_n$, the associated $n \times n$ *skew-symmetric matrix Schubert cell* $\mathring{X}^{\mathrm{ss}}_z$ and $n \times n$ *skew-symmetric matrix Schubert variety* X^{ss}_z are

$$\mathring{X}^{\mathrm{ss}}_z = \{A \in \mathrm{Mat}^{\mathrm{ss}}_n : \operatorname{rank} A_{[i][j]} = \operatorname{rank} z_{[i][j]} \text{ for } i, j \in [n]\},$$
$$X^{\mathrm{ss}}_z = \{A \in \mathrm{Mat}^{\mathrm{ss}}_n : \operatorname{rank} A_{[i][j]} \leq \operatorname{rank} z_{[i][j]} \text{ for } i, j \in [n]\}.$$

Here we identify $z \in \mathcal{I}^{\mathsf{FPF}}_n \subseteq S_n$ with its permutation matrix, so that $\operatorname{rank} z_{[i][j]}$ is the cardinality of $\{z(1), \ldots, z(i)\} \cap [j]$. These definitions would still make sense if z were an arbitrary permutation of $\mathbb{N}$, but then it could happen that $\mathring{X}^{\mathrm{ss}}_z = \varnothing$. We require $z \in \mathcal{I}^{\mathsf{FPF}}_n$ to exclude this degenerate case.

Skew-symmetric matrix Schubert cells arise as the orbits of a certain group action. Specifically, observe that the general linear group GL_n acts on $\mathrm{Mat}^{\mathrm{ss}}_n$ by $g : A \mapsto gAg^T$. Given any permutation w of $\mathbb{N}$, let $\mathrm{ss}_n(w)$ be the $n \times n$ matrix whose entry in position (i, j) is 1 if $w(j) = i < j = w(i)$, -1 if $w(j) = i > j = w(i)$, and 0 otherwise; see Example 3.3. The next theorem shows that Definition 3.1 is equivalent to the definition of X^{ss}_z given in the introduction.

Theorem 3.2 ([2]). Suppose $z \in \mathcal{I}^{\mathsf{FPF}}_n$. Then $\mathring{X}^{\mathrm{ss}}_z$ is the B_n-orbit of the skew-symmetric matrix $\mathrm{ss}_n(z)$. Moreover, X^{ss}_z is the Zariski closure of $\mathring{X}^{\mathrm{ss}}_z$ and is an irreducible variety.

3.1 Pfaffian generators for prime ideals

As in the introduction, we identify the coordinate ring $\mathbb{K}[\mathrm{Mat}^{\mathrm{ss}}_n]$ with $\mathbb{K}[u_{ij} : i, j \in [n], i > j]$ where u_{ij} represents the function $A \mapsto A_{ij}$. If A is a matrix and I and J are subsets of indices, then we write $A_{IJ} := [A_{ij}]_{(i,j) \in I \times J}$ for the corresponding $|I| \times |J|$ submatrix. We often apply this notation to the $n \times n$ skew-symmetric matrix of variables $\mathcal{U}^{\mathrm{ss}}$ with entries defined by

$$\mathcal{U}^{\mathrm{ss}}_{ij} = \begin{cases} -u_{ji} & \text{if } i < j \\ u_{ij} & \text{if } i > j \\ 0 & \text{if } i = j. \end{cases} \tag{3.1}$$

If A is a matrix then $\operatorname{rank} A \leq r$ if and only if all size $r+1$ minors of A vanish. Hence, X_z^{ss} is the zero locus of the ideal in $\mathbb{K}[u_{ji} : i,j \in [n], i > j]$ generated by all size $\operatorname{rank} z_{[i][j]} + 1$ minors of $\mathcal{U}^{\mathrm{ss}}_{[i][j]}$ for $(i,j) \in [n] \times [n]$. This ideal is often *not* prime, however.

Example 3.3. Take $n = 6$ and let $z = (1,2)(3,5)(4,6) \in \mathcal{I}_6^{\mathsf{FPF}}$, so

$$\mathrm{ss}_n(z) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}.$$

As $\operatorname{rank} z_{[4][3]} = 2$, the ideal described above contains the four 3×3 minors in $\mathcal{U}^{\mathrm{ss}}_{[4][3]}$, one of which is

$$\det(\mathcal{U}^{\mathrm{ss}}_{\{1,2,4\},\{1,2,3\}}) = \det \begin{bmatrix} 0 & -u_{21} & -u_{31} \\ u_{21} & 0 & -u_{32} \\ u_{41} & u_{42} & u_{43} \end{bmatrix} = u_{21}(u_{32}u_{41} - u_{31}u_{42} + u_{21}u_{43}).$$

However, one can check that the single condition $\operatorname{rank} A_{[4][3]} \leq 2$ implies all the others defining X_z^{ss}, and that accordingly the ideal under consideration is generated by the four 3×3 minors in $\mathcal{U}^{\mathrm{ss}}_{[4][3]}$. In particular, it is generated by homogeneous degree 3 polynomials, so cannot contain either factor u_{21} or $u_{32}u_{41} - u_{31}u_{42} + u_{21}u_{43}$. (Since u_{21} is nonzero on the matrix $\mathrm{ss}(z) \in X_z^{\mathrm{ss}}$, the other factor must lie in the prime ideal of the irreducible variety X_z^{ss}. In fact, it generates it; see Example 3.5.)

In this section, we identify a different set of polynomials which turn out to generate the prime ideal $I(X_z^{\mathrm{ss}})$. The key idea in our construction is to replace minors of matrices by Pfaffians. Recall that the *Pfaffian* of a skew-symmetric $n \times n$ matrix A is

$$\mathrm{pf}(A) = \sum_{z \in \mathcal{I}_n^{\mathsf{FPF}}} (-1)^{\ell_{\mathsf{FPF}}(z)} \prod_{z(i) < i \in [n]} A_{z(i),i} \tag{3.2}$$

where once again $\ell_{\mathsf{FPF}}(z) = |D^{\mathrm{ss}}(z)|$. For example, we have $\mathrm{pf}(\mathcal{U}^{\mathrm{ss}}_{[2][2]}) = \mathrm{pf}\begin{bmatrix} 0 & -u_{21} \\ u_{21} & 0 \end{bmatrix} = -u_{21}$. If n is odd then the outer summation in (3.2) is empty so $\mathrm{pf}(A) = 0$. This is consistent with the well-known fact that $\mathrm{pf}(A)^2 = \det(A)$, which is zero if A is skew-symmetric of odd size.

Theorem 3.4. Given $z \in \mathcal{I}_n^{\mathsf{FPF}}$, the prime ideal $I(X_z^{\mathrm{ss}})$ in $\mathbb{K}[\mathrm{Mat}_n^{\mathrm{ss}}] = \mathbb{K}[u_{ij} : i,j \in [n], i > j]$ is generated by the Pfaffians $\mathrm{pf}(\mathcal{U}^{\mathrm{ss}}_{RR})$ for all nonempty sets $R \subseteq [n]$ of even size for which there exist indices $i, j \in [n]$ with $i \geq j$ such that $R \subseteq [i]$ and $|R \cap [j]| > \operatorname{rank} z_{[i][j]}$.

Example 3.5. Take $n = 6$ and let $z = (1,2)(3,5)(4,6) \in \mathcal{I}_6^{\mathsf{FPF}}$ as in Example 3.3. As mentioned in that example, X_z^{ss} is defined by the single rank condition $\operatorname{rank} A_{[4][3]} \leq 2$, and similarly one can deduce that it is only necessary to consider $(i,j) = (4,3)$ in

Theorem 3.4. That is, $I(X_z^{\mathrm{ss}})$ is generated by $\mathrm{pf}(\mathcal{U}_{RR}^{\mathrm{ss}})$ where $R \subseteq [4]$ has even size and $|R \cap [3]| > \mathrm{rank}\, z_{[4][3]} = 2$, i.e., by the single Pfaffian

$$\mathrm{pf}(\mathcal{U}^{\mathrm{ss}}_{\{1,2,3,4\},\{1,2,3,4\}}) = \mathrm{pf}\begin{bmatrix} 0 & -u_{21} & -u_{31} & -u_{41} \\ u_{21} & 0 & -u_{32} & -u_{42} \\ u_{31} & u_{32} & 0 & -u_{43} \\ u_{41} & u_{42} & u_{43} & 0 \end{bmatrix} = u_{32}u_{41} - u_{31}u_{42} + u_{21}u_{43}.$$

To verify Theorem 3.4 in this case, observe that the Pfaffian above is a square root of $\det(\mathcal{U}^{\mathrm{ss}}_{\{1,2,3,4\},\{1,2,3,4\}})$, so it vanishes on $A \in \mathrm{Mat}_6^{\mathrm{ss}}$ if and only if $\mathrm{rank}\, A_{[4][4]} \leq 3$. Because a skew-symmetric matrix has even rank, this is equivalent to $\mathrm{rank}\, A_{[4][3]} \leq 2$. Therefore the ideal $(u_{21}u_{43} - u_{31}u_{42} + u_{41}u_{32})$ does have zero locus X_z^{ss}, and it is prime because it is generated by an irreducible polynomial.

3.2 Gröbner bases

The generating set of $I(X_z^{\mathrm{ss}})$ given in Theorem 3.4 need *not* be a Gröbner basis.

Example 3.6. Let $n = 6$ and $z = (1,2)(3,6)(4,5)$. Using Theorem 3.8, one computes that $\mathrm{in}(I(X_z^{\mathrm{ss}})) = (u_{32}u_{41}, u_{32}u_{51}, u_{31}u_{42}u_{51})$. Theorem 3.4 implies that $I(X_z^{\mathrm{ss}})$ is generated by the two Pfaffians $g := \mathrm{pf}(\mathcal{U}^{\mathrm{ss}}_{\{1,2,3,4\},\{1,2,3,4\}}) = u_{32}u_{41} - u_{31}u_{42} + u_{21}u_{43}$ and $h := \mathrm{pf}(\mathcal{U}^{\mathrm{ss}}_{\{1,2,3,5\},\{1,2,3,5\}}) = u_{32}u_{51} - u_{31}u_{52} + u_{21}u_{53}$. Their initial terms are the first two generators of $\mathrm{in}(I(X_z^{\mathrm{ss}}))$, but the last monomial generator is evidently not in the ideal they generate, nor can it be the initial term of any Pfaffian of a submatrix of $\mathcal{U}^{\mathrm{ss}}$ (but it is the initial term of $u_{41}h - u_{51}g$).

Given sets $A = \{a_1 < \cdots < a_q\}$ and $B = \{b_1 > \cdots > b_q\}$, we define

$$f_{AB} := \mathrm{pf}\begin{bmatrix} \mathcal{U}^{\mathrm{ss}}_{BB} & \mathcal{U}^{\mathrm{ss}}_{B,A\ominus B} \\ \mathcal{U}^{\mathrm{ss}}_{A\ominus B,B} & 0 \end{bmatrix} \in \mathbb{K}[u_{ij} : i,j \in \mathbb{N}, i > j]$$

where $A \ominus B$ is the set of $a \in A$ for which no b exists with

$$(a,b), (b,a) \in \{(a_1,b_1), \ldots, (a_q,b_q)\}.$$

Theorem 3.7. If $z \in \mathcal{I}_n^{\mathsf{FPF}}$ then the elements f_{AB}, as (i,j) ranges over all pairs in $[n] \times [n]$ with $i \geq j$ and (A,B) ranges over all pairs in $\binom{[i]}{q} \times \binom{[j]}{q}$ for $q = \mathrm{rank}\, z_{[i][j]} + 1$, form a Gröbner basis for $I(X_z^{\mathrm{ss}})$ with respect to the reverse lexicographic term order.

For instance, if $z = (1,2)(3,6)(4,5)$ as in Example 3.6, then one of the pairs arising in Theorem 3.7 is $(A = \{1,4,5\}, B = \{1,2,3\})$. One computes that f_{AB} is then $u_{51}g - u_{41}h = -u_{31}u_{42}u_{51} + u_{21}u_{43}u_{51} + u_{31}u_{41}u_{52} - u_{21}u_{41}u_{53}$, whose initial term is the monomial generator which was missing in Example 3.6.

3.3 Initial ideals

From Theorem 3.7 on Gröbner bases we can deduce an explicit formula for the initial ideal of $I(X_z^{\mathrm{ss}})$. Given sets $A = \{a_1 < \cdots < a_q\}, \{b_1 > \cdots > b_q\} \subseteq [n]$, let $u_{AB} = \prod_{i=1}^q u_{a_i b_i}$. Define $\phi : \mathbb{K}[u_{ij} : i,j \in [n]] \to \mathbb{K}[u_{ij} : 1 \leq j \leq i \leq n]$ to be the ring homomorphism with $u_{ii} \mapsto 0$ and $u_{ij}, u_{ji} \mapsto u_{ij}$ if $i > j$. Finally, let u_{AB}^{ss} be the squarefree radical of $\phi(u_{AB})$, where we take the radical of 0 to be 0.

Theorem 3.8. Given $z \in \mathcal{I}_n^{\mathsf{FPF}}$, the initial ideal of $I(X_z^{\mathrm{ss}})$ under reverse lexicographic term order is generated by all monomials of the form u_{AB}^{ss}, where $(A,B) \in \binom{[i]}{q} \times \binom{[j]}{q}$ for some $(i,j) \in [n] \times [n]$ with $i \geq j$ and $q = \operatorname{rank} z_{[i][j]} + 1$.

It is not necessarily the case that u_{AB} is the leading term in the polynomial f_{AB} defined in 3.2. However, this does hold if there are no indices $i < j$ with $b_i > a_j > a_i > b_j$, and given any (A,B), one can find (A', B') with this property and such that $u_{A'B'} = u_{AB}$.

Example 3.9. Take $n = 6$ and let $z = (1,2)(3,5)(4,6) \in \mathcal{I}_6^{\mathsf{FPF}}$ as in Examples 3.3 and 3.5. As mentioned in those examples, X_z^{ss} is defined by the single rank condition $\operatorname{rank} A_{[4][3]} \leq 2$, and similarly one can deduce that it is only necessary to consider $(i,j) = (4,3)$ and $q = 3 = \operatorname{rank} z_{[4][3]} + 1$ in Theorem 3.8. Thus, the ideal $\operatorname{in}(I(X_z^{\mathrm{ss}}))$ is generated by u_{AB}^{ss} where A ranges over the 3-element subsets of $\{1,2,3,4\}$ and $B = \{1,2,3\}$. The relevant monomials are listed below:

A	B	u_{AB}	$\phi(u_{AB})$	u_{AB}^{ss}
$\{1,2,3\}$	$\{3,2,1\}$	$u_{13}u_{22}u_{31}$	0	0
$\{1,2,4\}$	$\{3,2,1\}$	$u_{13}u_{22}u_{41}$	0	0
$\{1,3,4\}$	$\{3,2,1\}$	$u_{13}u_{32}u_{41}$	$u_{31}u_{32}u_{41}$	$u_{31}u_{32}u_{41}$
$\{2,3,4\}$	$\{3,2,1\}$	$u_{23}u_{32}u_{41}$	$u_{32}^2u_{41}$	$u_{32}u_{41}$

Theorem 3.8 now asserts that $\operatorname{in}(I(X_z^{\mathrm{ss}})) = (u_{32}u_{41})$ is the ideal generated by the rightmost column. Evidently this is indeed the initial ideal of $I(X_z^{\mathrm{ss}}) = (u_{32}u_{41} - u_{31}u_{42} + u_{21}u_{43})$ as computed in Example 3.5.

4 Stanley–Reisner ideals and K-polynomials

4.1 Pipe dreams and Stanley–Reisner ideals

In this subsection, we derive an alternate expression for $\operatorname{in}(I(X_z^{\mathrm{ss}}))$ as an intersection of prime monomial ideals indexed by the *involution pipe dreams* corresponding to z, as introduced in [6]. Define the *pipe dream reading word* of a finite set $D \subset \mathbb{N} \times \mathbb{N}$ to be the word $\operatorname{word}(D)$ whose letters list the numbers $i + j - 1$ as (i,j) ranges over all elements of D in the order that makes $(i, -j)$ increase lexicographically (i.e., which reads the rows in order, but going right to left). For example, the pipe dream reading word of $D = \{(1,4),(1,3),(2,6),(5,5),(5,4),(5,3)\}$ is **437987**.

Definition 4.1. A *reduced fpf-involution word* for $z \in \mathcal{I}_n^{\mathsf{FPF}}$ is a sequence of positive integers $i_1 i_2 \cdots i_l$ of shortest possible length such that

$$z = s_{i_l} \cdots s_{i_2} s_{i_1} (1,2)(3,4) \ldots (n-1,n) s_{i_1} s_{i_2} \cdots s_{i_l},$$

where $s_i \in S_n$ is the transposition $(i, i+1)$.

A set $D \subseteq \triangleright_n := \{(i,j) \in [n] \times [n] : i > j\}$ is a *reduced (fpf-involution) pipe dream* of z if the pipe dream reading word of D is a reduced fpf-involution word of z. Let $\mathcal{FP}(z)$ denote the set of reduced fpf-involution pipe dreams of z.

Bergeron and Billey associated a set of *reduced pipe dreams* to any permutation [1], and our fpf-involution pipe dreams are closely related: $D \in \mathcal{FP}(z)$ if and only if D is the intersection of $\triangleright_n$ with a pipe dream of z which is symmetric about its main diagonal [6, Theorem 3.12]

Given a set $D \subseteq \triangleright_n$, we write $(u_{ij} : (i,j) \in D)$ to denote the ideal in the coordinate ring $\mathbb{K}[\mathrm{Mat}_n^{\mathrm{ss}}] = \mathbb{K}[u_{ij} : i,j \in [n], i > j]$ generated by u_{ij} for all $(i,j) \in D$.

Theorem 4.2. Let $z \in \mathcal{I}_n^{\mathsf{FPF}}$. Then $\mathrm{in}(I(X_z^{\mathrm{ss}})) = \bigcap_{D \in \mathcal{FP}(z)} (u_{ij} : (i,j) \in D) \subseteq \mathbb{K}[\mathrm{Mat}_n^{\mathrm{ss}}]$.

Example 4.3. Let $z = (1,2)(3,5)(4,6)$. Then $z = s_4(1,2)(3,4)(5,6)s_4$, the only reduced fpf-involution word of z is **4**, and the two elements of $\mathcal{FP}(z)$ are $\{(4,1)\}$ and $\{(3,2)\}$. Theorem 4.2 then says that $\mathrm{in}(I(X_z^{\mathrm{ss}})) = (u_{41}) \cap (u_{32}) = (u_{41}u_{32})$, in agreement with Example 3.9.

The *Stanley–Reisner ideal* of a simplicial complex Δ with vertex set $[N]$ is the ideal in $\mathbb{K}[x_1, \ldots, x_N]$ generated by the elements $\prod_{v \in E} x_v$ for all $E \subseteq [N]$ with $E \notin \Delta$. Any ideal generated by squarefree monomials is a Stanley–Reisner ideal, and $\mathrm{in}(I(X_z^{\mathrm{ss}}))$ has this property by Theorem 3.8. The next theorem identifies it as the Stanley–Reisner ideal of an explicit complex; it is not hard to deduce from Theorem 4.2.

Definition 4.4. The *fpf-subword complex* associated to $z \in \mathcal{I}_n^{\mathsf{FPF}}$ and a subset $Q \subseteq \triangleright_n$ is the simplicial complex with vertices Q and faces

$$\Sigma(z, Q) = \{S \subseteq Q : Q \setminus S \text{ contains an fpf-involution pipe dream for } z\}.$$

Theorem 4.5. If $z \in \mathcal{I}_n^{\mathsf{FPF}}$ then $\mathrm{in}(I(X_z^{\mathrm{ss}}))$ is the Stanley–Reisner ideal of $\Sigma(z, \triangleright_n)$. Moreover, the complex $\Sigma(z, Q)$ is vertex-decomposable (hence shellable) for any $Q \subseteq \triangleright_n$.

4.2 K-polynomials

We now describe how the results of the preceding subsection lead to a new proof of a combinatorial formula for the torus-equivariant K-theory class of X^{ss}_z from [15]. Our approach is modeled after Knutson and Miller's study of subword complexes and the resulting combinatorial formulas for Grothendieck polynomials [9].

Suppose $R = \mathbb{K}[u_1, u_2, \dots, u_N]$ is a polynomial ring that is graded by a (multiplicative) free abelian group G in the sense that each variable u_i is assigned a degree $\deg(u_i) \in G$, a monomial $u_1^{a_1} \cdots u_N^{a_N}$ has degree $\deg(u_1)^{a_1} \cdots \deg(u_N)^{a_N}$, and a polynomial is homogeneous if its terms all have the same degree. If $I \subseteq R$ is a homogeneous ideal (that is, generated by homogeneous elements), then one can also speak of degrees and homogeneous elements in R/I. The *K-polynomial* of I is then the following formal $\mathbb{Z}$-linear combination of elements of G:

$$\mathcal{K}(I) = \prod_{i=1}^{N}(1 - \deg(u_i)) \sum_{g \in G} \dim_{\mathbb{K}}((R/I)_g) g,$$

where $(R/I)_g$ is the subspace of degree g homogeneous elements in R/I. See [16, §8] for a more careful definition, including conditions guaranteeing that this formal generating function is well-defined.

Example 4.6. Suppose $R = k[x_1, x_2]$, $G = \mathbb{Z}t_1 \oplus \mathbb{Z}t_2$, $\deg x_i = t_i$, and $I = (x_1^2)$. A homogenous basis for R/I is $\{x_1^i x_2^j + I : 0 \leq i \leq 1, 0 \leq j\}$, so $\mathcal{H}(R/I) = \sum_{j=0}^{\infty}(1+t_1)t_2^j = \frac{1+t_1}{1-t_2}$ and $\mathcal{K}(I) = \frac{1+t_1}{1-t_2}(1-t_1)(1-t_2) = 1 - t_1^2$.

Let G be the multiplicative abelian group freely generated by $a_1, a_2, \dots, a_n$. We give the coordinate ring $\mathbb{K}[\text{Mat}^{\text{ss}}_n] = \mathbb{K}[u_{ij} : n \geq i > j \geq 1]$ a G-grading by setting $\deg(u_{ij}) = a_i a_j$. These degrees arise as torus weights induced from the action of the torus $T_n \subseteq B_n$ of diagonal matrices on Mat^{ss}_n. Since a skew-symmetric matrix Schubert variety X^{ss}_z is B_n-stable, it is T_n-stable, which implies that $I(X^{\text{ss}}_z)$ is homogeneous under this grading.

Definition 4.7. The *symplectic Grothendieck polynomial* associated to $z \in \mathcal{I}^{\mathsf{FPF}}_n$ is the K-polynomial $\mathfrak{G}^{\mathsf{Sp}}_z := \mathcal{K}(I(X^{\text{ss}}_z))$.

Although *a priori* one only knows that $\mathfrak{G}^{\mathsf{Sp}}_z \in \mathbb{Z}[[a_1, a_2, \dots]]$, it turns out that this formal generating function always has only finitely many nonzero terms. The functions $\mathfrak{G}^{\mathsf{Sp}}_z$ are the same as the polynomials $\Upsilon^K_{\pi,(\text{GL}_{2n},\text{Sp}_{2n})}$ given in [18] as representatives for the ordinary K-theory classes of the $\text{Sp}_n(\mathbb{C})$-orbit closures on $\text{GL}_n(\mathbb{C})/B_n$. Symplectic Grothendieck polynomials also appear in [13, 14] (after making the change of variables $a_i \mapsto 1 - x_i$) as representatives for the T-equivariant K-theory classes of the varieties X^{ss}_z. The latter polynomials (in $\mathbb{Z}[x_1, x_2, \dots]$) have well-defined "stable limits" converging to symmetric functions that expand positively in terms of Ikeda and Naruse's K-theoretic Schur P-functions (with parameter $\beta = -1$); see [11, Thm. 1.9] and [13, Thm. 1.6].

Example 4.8. Let D be a subset of $\triangle_n = \{(i,j) \in [n] \times [n] : i > j\}$. As in the proof of Theorem 4.2, write $\mathsf{Ideal}(D) = (u_{ij} : (i,j) \in D) \subset \mathbb{K}[u_{ij} : n \geq i > j \geq 1] =: R$. The set of monomials $\prod_{(i,j)\in D^c} u_{ij}^{m_{ij}}$ where $D^c := \triangle_n \setminus D$ descends to a basis for $R/\mathsf{Ideal}(D)$. Accordingly,

$$\mathcal{K}(\mathsf{Ideal}(D)) = \prod_{n\geq i>j\geq 1}(1-a_ia_j) \prod_{(i,j)\in D^c} \sum_{m_{ij}\geq 0} (a_ia_j)^{m_{ij}} = \prod_{(i,j)\in D}(1-a_ia_j).$$

If $z \in \mathcal{I}_n^{\mathsf{FPF}}$ is *fpf-dominant* in the sense that $\mathcal{FP}(z)$ has a unique element D, then $I(X_z^{\mathsf{ss}}) = \mathsf{Ideal}(D)$ by Theorem 4.2, and $\mathfrak{G}_z^{\mathsf{Sp}} = \prod_{(i,j)\in D}(1-a_ia_j)$, which recovers the skew-symmetric half of [13, Thm. 3.8].

If $z \in \{1\} \sqcup \mathcal{I}_n^{\mathsf{FPF}}$ and $s = s_i = (i, i+1) \in S_n$, then we define

$$z * s = \begin{cases} 1 & \text{if } (i,i+1) \text{ is a cycle of } z \\ z & \text{if } z = 1 \text{ or } z(i) > z(i+1) \\ szs & \text{otherwise.} \end{cases}$$

Note that if $z \in \mathcal{I}_n^{\mathsf{FPF}}$ then either $z * s \in \mathcal{I}_n^{\mathsf{FPF}}$ or $z * s = 1$, but we always have $1 * s = 1$. The operation $*$ extends to a right action of the 0-*Hecke monoid* of S_n but not to a group action. If $i_1 i_2 \cdots i_l$ is a word, define $\delta_{\mathsf{FPF}}(i_1 i_2 \cdots i_l) = (\cdots((1_{\mathsf{FPF}} * s_{i_1}) * s_{i_2}) \cdots) * s_{i_l}$.

Definition 4.9. An *extended (fpf-involution) pipe dream* for $z \in \mathcal{I}_n^{\mathsf{FPF}}$ is a subset $D \subseteq \triangle$ whose pipe dream reading word $i_1 i_2 \cdots i_l$ satisfies $z = \delta_{\mathsf{FPF}}(i_1 i_2 \cdots i_l)$. Let $\mathcal{FP}^+(z)$ be the set of all extended pipe dreams for z.

The set $\mathcal{FP}^+(z)$ is called $\mathsf{InvDreams}(z)$ in [15]. The next theorem can be deduced from Theorem 4.5 and a more detailed analysis of the fpf subword complex $\Sigma(z, \triangle_n)$ using the techniques of Knutson and Miller in [9, §4].

Theorem 4.10 ([15, Thm. 4.5])**.** Let $z \in \mathcal{I}_n^{\mathsf{FPF}}$. Then

$$\mathfrak{G}_z^{\mathsf{Sp}} = \sum_{D \in \mathcal{FP}^+(z)} (-1)^{|D| - \ell_{\mathsf{FPF}}(z)} \prod_{(i,j)\in D}(1-a_ia_j).$$

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