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Enumeration of Walks with Small Steps Avoiding a Quadrant

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Abstract. We address the enumeration of walks with weighted small steps avoiding a quadrant. In particular we give an exact, integral-expression solution for the generating function $\mathsf{C}(x, y; t)$ counting these walks by length and end-point. Moreover, we determine precisely when this generating function is algebraic, D-finite or D-algebraic with respect to x, showing that this complexity is the same as for walks in the quarter-plane with the same starting point, as long as the starting point (p, q) of the walks lies in the quarter plane then. Finally, we give an integral-free expression for the solution in the cases where (p, q) lies just outside the quarter plane, that is $p = 0$ or $q = 0$ with our convention, proving a conjecture of Raschel and Trotignon.

Keywords: lattice path, elliptic function, cone

1 Introduction

The systematic study of walks with small steps in the quarter plane was initiated by Bousquet-Mélou and Mishna in 2010 [5], and since then there has been great progress on the model [2, 14, 19, 18, 17, 1, 16, 8, 9]. The model is defined as follows: given a step set $S \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, determine the generating function

$$\mathsf{Q}(x, y; t) = \sum_{n=0}^{\infty} \sum_{i,j \geq 1} q(i, j; n) t^n x^i y^j,$$

where $q(i, j; n)$ is the number of walks of length n, starting at $(1, 1)$, and ending at (i, j) using steps in S and staying in the strictly positive quadrant.[1] A priori, there are 256 distinct step sets S, but after removing duplicates and cases that are equivalent to half-plane models, Bousquet-Mélou and Mishna identified 79 non-trivial and combinatorially distinct models. The study of these models is now in some sense complete as it is known for each S precisely where the generating function fits into the hierarchy

$$\text{Algebraic} \subset \text{D-finite} \subset \text{D-Algebraic}.$$

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[1]Note that most of the literature has considered the equivalent question of walks starting at $(0, 0)$ and staying in the non-strictly positive quadrant, for which the resulting generating function is $\frac{1}{xy} Q(x, y; t)$.

Recall that a generating function is called Algebraic with respect to a certain variable if it satisfies some non-trivial polynomial equation whose coefficients are polynomial in that variable, and it is called D-finite (resp. D-algebraic) if it satisfies a linear (resp. polynomial) differential equation with respect to that variable whose coefficients coefficients are polynomial in that variable. For a multivariate series to be algebraic (resp. D-finite, D-algebraic) it must be algebraic (resp. D-finite, D-algebraic) with respect to each variable.

Of the 79 models proposed by Mishna and Bousquet-Mélou, 4 models admit an algebraic generating function [5, 2], 19 further models admit a D-finite generating function [5, 14], 9 further models admit a D-algebraic generating function [1, 16] and the remaining 47 models admit a generating function which is not D-algebraic [8, 9]. Moreover, in the 74 cases known as *non-singular*, an exact integral expression is known for the generating function [19], while other exact expressions are known in the 5 singular cases [18, 17]. In recent years a number of articles have focused on the equivalent question for walks avoiding a quadrant [3, 20, 6, 11, 4], that is determining the generating function $\mathsf{C}(x, y; t)$ which counts walks starting at $(1, 1)$ whose intermediate points are required to lie in the three-quadrant cone

$$\mathcal{C} = \{(i, j) : i > 0 \text{ or } j > 0\}.$$

Between them these 5 articles classify 10 models into the complexity hierarchy (algebraic, D-finite, D-algebraic), while excursions have been enumerated for 4 further models [7, 12]. Remarkably the generating function has the same nature in each case as in the quarter plane, a fact that led Dreyfus and Trotignon to conjecture that the nature is the same for any of the 74 non-singular step-sets S [9]. We give exact integral expression solutions for $\mathsf{C}(x, y; t)$ in each of these 74 cases, analogous to those of Raschel in the quarter-plane [19]. We then prove that the nature of $\mathsf{C}(x, y; t)$ as a function of x (or y) is the same as that of $\mathsf{Q}(x, y; t)$, and we conjecture that these series also have the same nature as functions of t. In fact we do this in the more general setting of walks with *weighted* steps and starting at any point in the positive quadrant or on the positive x-axis.

Note that by our definition, steps directly between $(1, 0)$ and $(0, 1)$ are allowed, whereas they are forbidden in [6], for example. We expect the generating function for the model where these steps are forbidden to be closely related to the the model that we study, as found in [6, Section 8] for king walks, so we do not expect this to affect difference the nature of the generating function.

2 Functional equations for walks avoiding a quadrant

We start with a step-set $S \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, a weight $w_s > 0$ for each $s \in S$ and a starting point (p, q) with $p > 0$, $q \geq 0$. We will determine the generating function

$\mathsf{C}(x,y;t)$ counting walks starting at (p,q), taking steps from S with all intermediate points lying in the three-quadrant cone $\mathcal{C}$ and with the weight of the walk being the product of the weights w_s of the steps. Note that the standard starting point is $(p,q) = (1,1)$ and in the unweighted case $w_s = 1$ for each $s \in S$.

The following lemma results from considering the final step of a walk counted by $\mathsf{C}(x,y;t)$:

Lemma 1. *Define the single step generating function* $\mathsf{P}(x,y)$ *by*

$$\mathsf{P}(x,y) = \sum_{(\alpha,\beta)\in S} w_{(\alpha,\beta)} x^\alpha y^\beta$$

Then there are series $A_H(\frac{1}{x};t) \in \frac{t}{x}\mathbb{Z}[\frac{1}{x}][[t]]$, $A_V(\frac{1}{y};t) \in \frac{t}{y}\mathbb{Z}[\frac{1}{y}][[t]]$ *and* $B(t) \in t\mathbb{Z}[[t]]$ *which satisfy*

$$\mathsf{C}(x,y;t) = x^p y^q + t\mathsf{P}(x,y)\mathsf{C}(x,y;t) - B(t) - A_H\left(\frac{1}{x};t\right) - A_V\left(\frac{1}{y};t\right). \tag{2.1}$$

Moreover, this equation together with the fact that $c(i,j;n) = 0$ *for* $i,j \leq 0$*, characterises the generating function*

$$\mathsf{C}(x,y;t) = \sum_{t\geq 0}\sum_{i,j\in\mathbb{Z}} c(i,j;n)x^i y^j t^n,$$

as well as the series A_H*,* A_V *and* B*.*

The series A_H, A_V and B in the lemma above can be understood combinatorially: They count walks starting at (p,q) and ending just outside $\mathcal{C}$ whose intermediate points all lie within $\mathcal{C}$. More precisely, $A_H(\frac{1}{x};t)$ counts those walks ending on the negative x-axis, $A_V(\frac{1}{y};t)$ counts those walks ending on the negative y-axis, and $B(t)$ counts those walks ending at $(0,0)$.

The unusual condition that the coefficients $c(i,j;n)$ of $\mathsf{C}(x,y;t)$ vanish for $i,j \leq 0$ makes this equation difficult to solve directly, so we partition $\mathcal{C}$ into three quadrants $\mathcal{Q}_{-1} = \{(i,j) : i > 0,\ j < 0\}$, $\mathcal{Q}_0 = \{(i,j) : i > 0,\ j \geq 0\}$ and $\mathcal{Q}_1 = \{(i,j) : i \leq 0,\ j > 0\}$, as shown in figure 1. A similar decomposition was used in [3, 6], but we have shifted the quadrants $\mathcal{Q}_{-1}$, $\mathcal{Q}_0$ down one space compared to those articles so that it is impossible to step directly between $\mathcal{Q}_{-1}$ and $\mathcal{Q}_1$ and so that our condition on the starting point (p,q) is now that $(p,q) \in \mathcal{Q}_0$.

Now, for $j = -1,0,1$, we define $\mathsf{Q}_j(x,y;t)$ to be the generating function counting walks in $\mathcal{C}$, starting at (p,q) and ending in $\mathcal{Q}_j$, so

$$\mathsf{C}(x,y;t) = \mathsf{Q}_{-1}(x,y;t) + \mathsf{Q}_0(x,y;t) + \mathsf{Q}_1(x,y;t),$$

and $\mathsf{Q}_{-1} \in \frac{x}{y}\mathbb{Z}\left[x,\frac{1}{y}\right][[t]]$, $\mathsf{Q}_0 \in x\mathbb{Z}[x,y][[t]]$ and $\mathsf{Q}_1 \in y\mathbb{Z}\left[\frac{1}{x},y\right][[t]]$. The following lemma rewrites (2.1) as three equations characterising $\mathsf{Q}_{-1}(x,y;t)$, $\mathsf{Q}_0(x,y;t)$ and $\mathsf{Q}_1(x,y;t)$.

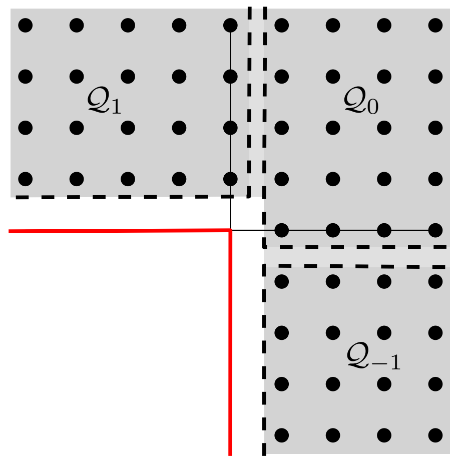

Figure 1: The three-quadrant cone $\mathcal{C}$ partitioned into three quadrants $\mathcal{Q}_1$, $\mathcal{Q}_0$ and $\mathcal{Q}_{-1}$.

Lemma 2. *Define the kernel* $K(x,y;t) = t\mathsf{P}(x,y) - 1$. *There are series* $V_1(y;t), V_2(y;t) \in \mathbb{Z}[y][[t]]$ *and* $H_1(x;t), H_2(x;t) \in \mathbb{Z}[x][[t]]$ *satisfying the three equations*

$$K(x,y;t)\mathsf{Q}_{-1}(x,y;t) = A_V\left(\frac{1}{y};t\right) + H_1(x;t) + \frac{1}{y}H_2(x;t) \tag{2.2}$$

$$K(x,y;t)\mathsf{Q}_0(x,y;t) = -xy + B(t) - V_1(y;t) - xV_2(y;t) - H_1(x;t) - \frac{1}{y}H_2(x;t) \tag{2.3}$$

$$K(x,y;t)\mathsf{Q}_1(x,y;t) = A_H\left(\frac{1}{x};t\right) + V_1(y;t) + xV_2(y;t). \tag{2.4}$$

Moreover, these three equations characterise the series $V_1(y;t)$, $V_2(y;t)$, $\mathsf{Q}_{-1}(x,y;t)$, $\mathsf{Q}_0(x,y;t)$, $\mathsf{Q}_1(x,y;t)$, $H_1(x;t)$, $H_2(x;t)$, $A_H\left(\frac{1}{x};t\right)$, $A_V\left(\frac{1}{y};t\right)$ *and* $B(t)$.

Combinatorially, the series $(V_1(0;t) - V_1(y;t))$ (resp. $xV_2(y;t)$, $(H_1(x;t) - H_1(0,t))$, $\frac{1}{y}(H_2(0;t) - H_2(x;t)))$ counts walks whose final step is from $\mathcal{Q}_0$ (resp. $\mathcal{Q}_1$, $\mathcal{Q}_{-1}$, $\mathcal{Q}_0$) to $\mathcal{Q}_1$ (resp. $\mathcal{Q}_0$, $\mathcal{Q}_0$, $\mathcal{Q}_{-1}$).

2.1 Parameterisation of the kernel curve

Following the method used in the quarter plane pioneered by Fayolle and Raschel [13, 19] we start by fixing $t \in \left(0, \frac{1}{\sum_{s\in S} w_s}\right)$ and then we consider the curve $\mathcal{W} = \{(x,y) : K(x,y;t) = 0\}$. From now on, we will make the following assumption:
Assumption: S is a non-singular step-set. That is, for any line ℓ through the origin, at least one element of S lies on each side of ℓ.
As explained in [6], if S did not have this property, the generating function $\mathsf{C}(x,y;t)$ would be algebraic.

Under this assumption, the curve $\mathcal{W}$ is known to have genus 1, so we will be able to parameterise it using elliptic functions $X(z)$ and $Y(z)$. More precisely the following lemma follows from [10, Proposition 2.1, Lemma 2.6].

Lemma 3. *There are meromorphic functions* $X(z), Y(z) : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ *and numbers* $\gamma, \tau \in i\mathbb{R}$ *with* $\Im(\pi\tau) > \Im(2\gamma) > 0$ *satisfying the following conditions*

- $K(X(z), Y(z)) = 0$
- $X(z) = X(z+\pi) = X(z+\pi\tau) = X(-\gamma - z)$
- $Y(z) = Y(z+\pi) = X(z+\pi\tau) = Y(\gamma - z)$
- $|X(-\frac{\gamma}{2})|, |Y(\frac{\gamma}{2})| < 1$
- *Counting with multiplicity, the functions* $X(z)$ *and* $Y(z)$ *each contain two poles and two roots in each fundamental domain* $\{z_c + r_1\pi + r_2\pi\tau : r_1, r_2 \in [0,1)\}$.

Moreover, $X(z)$ *and* $Y(z)$ *are differentially algebraic with respect to* z.

We intend to substitute $x \to X(z)$ and $y \to Y(z)$ into (2.2), (2.3) and (2.4), however we can only do this as long as the series in these equations converge, which occurs as long as $|x| \le 1 \le |y|$ for (2.2), $|x|, |y| \le 1$ for (2.3) and $|y| \le 1 \le |x|$ for (2.4). So to substitute $x \to X(z)$ and $y \to Y(z)$, we need to understand how $|X(z)|$ and $|Y(z)|$ compare to 1, for which we use the following lemma, which follows from [10, Lemma 2.9].

Lemma 4. *The complex plane can be partitioned into simply connected regions* $\{\Omega_s\}_{s\in\mathbb{Z}}$ *(see Figure 2) satisfying*

$$\bigcup_{s\in\mathbb{Z}} \Omega_{4s} \cup \Omega_{4s+1} = \{z \in \mathbb{C} : |Y(z)| < 1\},$$
$$\bigcup_{s\in\mathbb{Z}} \Omega_{4s} \cup \Omega_{4s-1} = \{z \in \mathbb{C} : |X(z)| < 1\}$$

and for $s \in \mathbb{Z}$,

$$\pi + \Omega_s = \Omega_s,$$
$$s\pi\tau + \gamma - \Omega_{2s} \cup \Omega_{2s+1} = \Omega_{2s} \cup \Omega_{2s+1},$$
$$s\pi\tau - \gamma - \Omega_{2s} \cup \Omega_{2s-1} = \Omega_{2s} \cup \Omega_{2s-1}.$$

2.2 Analytic reformulation of functional equations

Using the results in the previous section, we can substitute $x = X(z)$ and $y = Y(z)$ into (2.2), (2.3) and (2.4) for z in the regions Ω_{-1}, Ω_0 and Ω_1, respectively, yielding (2.9), (2.10) and (2.11) in the following proposition:

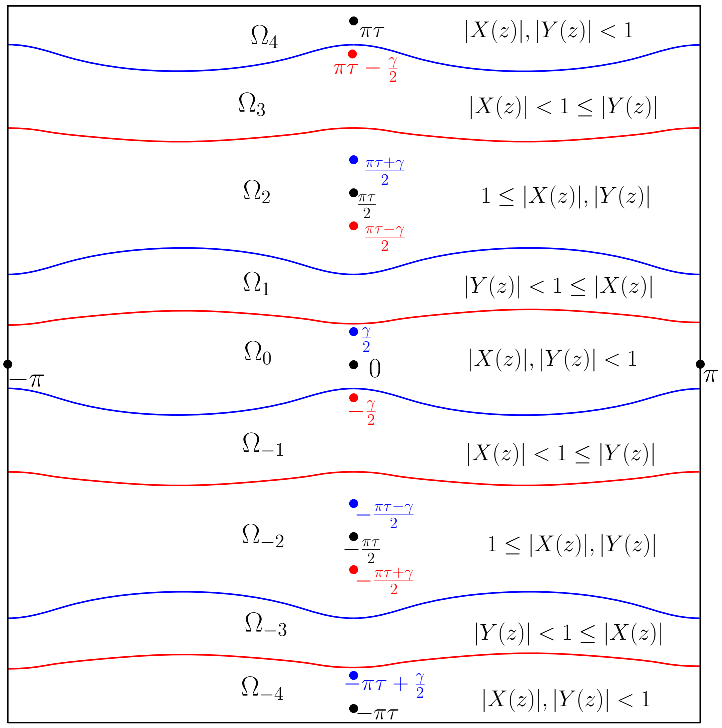

Figure 2: The complex plane partitioned into regions Ω_j. For z on the blue lines, $|Y(z)| = 1$, while on the red lines $|X(z)| = 1$.

Proposition 5. *The functions*

$$L_H(z) := H_1(X(z);t) + \frac{1}{Y(z)} H_2(X(z);t), \qquad \text{for } z \in \Omega_0 \cup \Omega_{-1}, \tag{2.5}$$

$$L_V(z) := V_1(Y(z);t) + X(z)V_2(Y(z);t), \qquad \text{for } z \in \Omega_0 \cup \Omega_1, \tag{2.6}$$

$$P_V(z) := A_V\left(\frac{1}{Y(z)};t\right), \qquad \text{for } z \in \Omega_{-1} \cup \Omega_{-2}, \tag{2.7}$$

$$P_H(z) := A_H\left(\frac{1}{X(z)};t\right), \qquad \text{for } z \in \Omega_1 \cup \Omega_2. \tag{2.8}$$

are well defined and satisfy the equations

$$0 = P_V(z) + L_H(z) \qquad \text{for } z \in \Omega_{-1}, \tag{2.9}$$

$$0 = -X(z)^p Y(z)^q + B(t) - L_V(z) - L_H(z) \qquad \text{for } z \in \Omega_0, \tag{2.10}$$

$$0 = P_H(z) + L_V(z) \qquad \text{for } z \in \Omega_1 \tag{2.11}$$

$$P_H(z) = P_H(\pi\tau - \gamma - z) = P_H(z + \pi) \tag{2.12}$$

$$P_V(z) = P_V(-\pi\tau + \gamma - z) = P_V(z + \pi) \qquad . \tag{2.13}$$

While these equations are a priori defined on different sets, they can be used to show that the functions extend meromorphically to all of $\mathbb{C}$, and so the equations hold on all of $\mathbb{C}$. Simply taking the sum of the three equations (2.9), (2.10) and (2.11) yields (2.14) in the theorem below.

Theorem 6. *The functions $P_H(z)$ and $P_V(z)$ extend to meromorphic functions on $\mathbb{C}$ which, along with the constant $B(t)$, are uniquely defined by the equation*

$$X(z)^p Y(z)^q = P_V(z) + B(t) + P_H(z), \tag{2.14}$$

along with (2.12), (2.13) *and the conditions*

- *$P_H(z)$ has no poles in $\Omega_0 \cup \Omega_1 \cup \Omega_2$,*
- *the poles of $X(z)$ for $z \in \Omega_1 \cup \Omega_2$ are roots of $P_H(z)$,*
- *$P_V(z)$ has no poles in $\Omega_0 \cup \Omega_{-1} \cup \Omega_{-2}$,*
- *the poles of $Y(z)$ for $z \in \Omega_{-1} \cup \Omega_{-2}$ are roots of $P_Y(z)$.*

Note that combining (2.14), (2.12), (2.13) yields

$$P_H(2\pi\tau - 2\gamma + z) - P_H(z) = W(z), \tag{2.15}$$

where $W(z)$ is an elliptic function with periods π and $\pi\tau$ given by

$$W(z) := (X(z-2\gamma)^p - X(z)^p)\, Y(z)^q.$$

3 Solving the functional equation

In the previous section we reduced the problem to finding the unique meromorphic functions $P_V, P_H \colon \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ and constant $B(t)$ characterised by Theorem 6 (for each t), as these determine $A_H(\frac{1}{x}, t)$ and $A_V(\frac{1}{y}, t)$ using 2.8 and 2.7, respectively, after which $\mathsf{C}(x, y; t)$ is determined by (2.1). An analogous result was found by Raschel for walks in the quarter plane [19], the main difference being that the transformations $z \to \pi\tau - \gamma - z$ and $z \to -\pi\tau + \gamma - z$ which fix $P_H(z)$ and $P_V(z)$ are $z \to \gamma - z$ and $z \to -\gamma - z$ in the quarter plane. Raschel used this equation to derive an integral-expression solution determining $\mathsf{Q}(x, y; t)$, and the equation has since been used to determine precisely when $\mathsf{Q}(x, y; t)$ is differentially algebraic [1, 8, 15] and to determine when it is algebraic or D-finite with respect to x or y [14, 16].

Due to this striking similarity we were able to use these methods to prove the same results for $\mathsf{C}(x, y; t)$, in particular showing that it is algebraic, D-finite or D-algebraic with respect to x (or y) in the same cases as $\mathsf{Q}(x, y; t)$. We note that Fayolle and Raschel also showed that the unweighted models that are algebraic or D-finite with respect to t have the same nature with respect to t, however these results relied on the precise ratios $\frac{\pi\tau}{\gamma}$ that could occur in these cases, so they do not apply so readily to our equation. Nonetheless, we expect that the same result holds for $\mathsf{C}(x, y; t)$.

3.1 Integral expression

In this section we give integral expressions analogous to those of Raschel [19] which determine $P_V(z)$ and $P_H(z)$ exactly. In order to write these expressions, we define an auxiliary elliptic function $\omega(z)$ which satisfies

$$\omega(z) = \omega(\pi\tau - \gamma - z) = \omega(-\pi\tau + \gamma - z) = \omega(z + \pi)$$

and shares the poles of $X(z)$ in $\Omega_1 \cup \Omega_2$, while $1/\omega(z)$ has the same poles as $Y(z)$ in $\Omega_{-1} \cup \Omega_{-2}$. Finally, $\omega(z)/X(z)$ converges to 1 at the poles of $X(z)$ in $\Omega_1 \cup \Omega_2$.

Theorem 7. *Let $z_0 \in \Omega_0$ and let $\mathcal{L}$ be a path from z_0 to $z_0 + \pi$ contained in the closure $\overline{\Omega_0}$ of Ω_0. Then $P_V(z)$, $P_H(z)$ and $B(t)$ are given by the integrals*

$$P_H(u) = \frac{1}{2\pi i t}\int_{\mathcal{L}} X(z)^p Y(z)^q \frac{\omega'(z)}{\omega(z) - \omega(u)} dz \qquad \text{for } u \in \Omega_1 \cup \Omega_2 \qquad (3.1)$$

$$P_V(u) = -\frac{1}{2\pi i t}\int_{\mathcal{L}} X(z)^p Y(z)^q \frac{\omega(u)}{\omega(z)} \frac{\omega'(z)}{\omega(z) - \omega(u)} dz \qquad \text{for } u \in \Omega_{-1} \cup \Omega_{-2} \qquad (3.2)$$

$$B(t) = -\frac{1}{2\pi i t}\int_{\mathcal{L}} X(z)^p Y(z)^q \frac{\omega'(z)}{\omega(z)} dz \qquad (3.3)$$

The proof of this theorem involves checking that these expressions satisfy the conditions in Theorem 6.

3.2 Classification of $\mathsf{C}(x, y; t)$ into complexity hierarchy

In this section we very briefly describe the properties of the step set S, or equivalently $X(z)$ and $Y(z)$, which determine the nature of the generating function $\mathsf{C}(x, y; t)$.

In certain cases, called *finite group* cases, $\gamma = \frac{M}{N}\pi\tau$ for some positive $M, N \in \mathbb{Z}$ independent of t. Then applying (2.15) N times yields

$$P_H(2\pi\tau(N - M) + z) - P_H(z) = P_H((2\pi\tau - 2\gamma)N + z) - P_H(z) = E(z),$$

where $E(z)$ is called the orbit sum of the model. We can use this equation to solve for $P_H(z)$, yielding an expression for $\mathsf{C}(x, y; t)$ which is D-finite in x. in the cases where $E(z) = 0$, we can even prove that $\mathsf{C}(x, y; t)$ is algebraic in x.

In all other cases we have $\frac{\gamma}{\pi\tau} \notin \mathbb{Q}$ for generic t. The model is then said to decouple if there are rational functions R_1, R_2 satisfying $X(z)^p Y(z)^q = R_1(X(z)) + R_2(Y(z))$. These cases can be solved (with integral-free expressions) as, using (2.14), the function

$$f(z) := R_1(X(z)) - P_H(z) = -R_2(Y(z)) + B(t) + P_v(z)$$

is an elliptic function which can be determined exactly. In these cases every function used is D-algebraic in all of its variables.

Finally in the infinite group cases which do not decouple, the generating function can be shown to be non-D-algebraic in x using Galois theory of q-difference equations, as in [8, 15] for $\mathsf{Q}(x, y; t)$.

3.3 Special case: walks starting on x-axis

We will now study the special cases where the walk starts at some point $(p, 0)$ for $p > 0$. Trotignon and Raschel conjectured that with this starting point all finite group models admit algebraic generating functions [20]. Indeed, if $q = 0$ it is easy to see that the orbit sum $E(z)$ defined is section 3.2 is equal to 0, so this follows from our more general results. Moreover, if $q = 0$, the model trivially decouples, so even in the infinite group case the generating function $\mathsf{C}(x, y; t)$ is D-algebraic.

The following lemma follows directly from Theorem 6, where $\omega(z)$ is defined as in Section 3.1.

Lemma 8. *If the starting point of the walks is $(p, 0)$ for some $p \geq 1$, then there is a degree p polynomial H satisfying*

$$P_V(z) = H(\omega(z)) - H(0), \tag{3.4}$$

$$B(t) = H(0), \tag{3.5}$$

$$P_H(z) = X(z)^p - H(\omega(z)). \tag{3.6}$$

Moreover, this polynomial is uniquely determined by the fact that the right hand side of (3.6) has a root at $z = \delta$.

In fact, we can convert these directly to formulae for A_H, A_V and B using series $W_1\left(\frac{1}{x}; t\right) \in x\mathbb{Z}[\frac{1}{x}][[t]]$ and $W_2\left(\frac{1}{y}; t\right) \in \frac{1}{y}\mathbb{Z}[\frac{1}{y}][[t]]$ satisfying $W_1\left(\frac{1}{X(z)}; t\right) = \omega(z)$ for $z \in \Omega_{-1} \cup \Omega_{-2}$ and $W_2\left(\frac{1}{Y(z)}; t\right) = \omega(z)$ for $z \in \Omega_1 \cup \Omega_2$. Note that these depend on the step-set but not the starting point (p, q) of the walk. For general p, we can rewrite Lemma 8 as the following theorem:

Theorem 9. *If the starting point of the walks is $(p, 0)$ for some $p \geq 1$, then there is a degree p polynomial H satisfying*

$$A_V\left(\frac{1}{y}\right) = H\left(W_2\left(\frac{1}{y}; t\right)\right) - H(0), \tag{3.7}$$

$$B(t) = H(0), \tag{3.8}$$

$$A_H\left(\frac{1}{x}\right) = x^p - H\left(W_1\left(\frac{1}{x}; t\right)\right). \tag{3.9}$$

Moreover, this polynomial is uniquely determined by the fact that the right hand side of (3.9) is a series in $\frac{1}{x}\mathbb{Z}[\frac{1}{x}][[t]]$.

In the $p=1$ case, we have $W_2\left(\frac{1}{y};t\right) = A_V\left(\frac{1}{y}\right)$ and $W_1\left(\frac{1}{x};t\right) = x - B(t) - A_H\left(\frac{1}{x}\right)$, which can be used as alternative definitions for W_1 and W_2.

3.4 Special case: simple walks

We now describe the case of simple walks, that is, unweighted walks with step-set $S = \{(0,1),(1,0),(0,-1),(-1,0)\}$. In this case $X(z)$ and $Y(z)$ can be written in terms of the Jacobi theta function

$$\vartheta(z,\tau) := \sum_{n=0}^{\infty} e^{i\pi\tau n(n+1)}\left(e^{(2n+1)iz} - e^{-(2n+1)iz}\right).$$

as

$$X(z) = e^{-i\gamma}\frac{\vartheta(z,\tau)\vartheta(z+\gamma,\tau)}{\vartheta(z-\gamma,\tau)\vartheta(z+2\gamma,\tau)} \qquad \text{and} \qquad Y(z) = e^{-i\gamma}\frac{\vartheta(z,\tau)\vartheta(z-\gamma,\tau)}{\vartheta(z+\gamma,\tau)\vartheta(z-2\gamma,\tau)},$$

where $\gamma = \frac{\pi\tau}{4}$ and is related to t by

$$e^{-i\gamma}\frac{\vartheta\left(\frac{\gamma}{2},\tau\right)^2}{\vartheta\left(\frac{3\gamma}{2},\tau\right)^2} = \frac{1+2t-\sqrt{1+4t}}{2t}.$$

Moreover, the function $\omega(z)$ defined in Section 3.1 is given by

$$\omega(z) = e^{-3i\gamma}\frac{\vartheta(2\gamma,\tau)\vartheta'(0,\frac{3\tau}{2})\vartheta(\gamma,\frac{3\tau}{2})}{\vartheta'(0,\tau)\vartheta(2\gamma,\frac{3\tau}{2})\vartheta(3\gamma,\frac{3\tau}{2})}\cdot\frac{\vartheta(z+\gamma,\frac{3\tau}{2})\vartheta(z+2\gamma,\frac{3\tau}{2})}{\vartheta(z-\gamma,\frac{3\tau}{2})\vartheta(z+4\gamma,\frac{3\tau}{2})},$$

which has π and $\frac{3\pi\tau}{2}$ as periods. Since $X(z)$ and $\omega(z)$ share the periods π and $3\pi\tau$, they are related by a polynomial equation. One such equation is

$$\frac{1}{2t} - Y(z) - \frac{1}{Y(z)} = X(z) + \frac{1}{X(z)} - \frac{1}{2t} = \frac{\omega(z)+c_1}{\omega(z)-c_1}\left(\omega(z) + \frac{c_1^2}{\omega(z)} + c_2\right),$$

where c_1 and c_2 are given by

$$c_1 = -e^{-i\gamma}\frac{\vartheta(2\gamma,\tau)\vartheta'(0,\frac{3\tau}{2})\vartheta(\gamma,\frac{3\tau}{2})}{\vartheta'(0,\tau)\vartheta(2\gamma,\frac{3\tau}{2})\vartheta(3\gamma,\frac{3\tau}{2})},$$

$$c_2 = \frac{1+4t}{2t}\cdot\frac{1+c_3}{1-c_3} + c_1c_3 + \frac{c_1}{c_3}, \qquad \text{where} \qquad c_3 = -e^{i\gamma}\frac{\vartheta(\frac{5\gamma}{2},\frac{3\pi\tau}{2})}{\vartheta(\frac{\gamma}{2},\frac{3\pi\tau}{2})}.$$

So the series $W_1\left(\frac{1}{x};t\right) \in x\mathbb{Z}[\frac{1}{x}]$ and $W_2\left(\frac{1}{y};t\right) \in \frac{1}{y}\mathbb{Z}[\frac{1}{y}]$ defined in Section 3.3 satisfy

$$-\frac{1}{2t} + x + \frac{1}{x} = \frac{W_1\left(\frac{1}{x};t\right) + c_1}{W_1\left(\frac{1}{x};t\right) - c_1}\left(W_1\left(\frac{1}{x};t\right) + \frac{c_1^2}{W_1\left(\frac{1}{x};t\right)} + c_2\right),$$

$$\frac{1}{2t} - y - \frac{1}{y} = \frac{W_2\left(\frac{1}{y};t\right) + c_1}{W_2\left(\frac{1}{y};t\right) - c_1}\left(W_2\left(\frac{1}{y};t\right) + \frac{c_1^2}{W_2\left(\frac{1}{y};t\right)} + c_2\right).$$

Then for any starting point $(p, 0)$, the generating functions A_H, A_V, B and hence $\mathsf{C}(x, y; t)$ can be determined by Theorem 9. Moreover, c_1, c_2 and t can be shown to be modular functions of τ, so they are all algebraically related. Hence in these cases the generating function $\mathsf{C}(x, y; t)$ is algebraic in t as well as the other variables.

4 Nature of $\mathsf{C}(x, y; t)$ with respect to t

The main remaining problem is to prove that the generating function $\mathsf{C}(x, y; t)$ has the same nature (algebraic, D-finite, D-algebraic) as a function of t as it does as a function of x and y. Even for $\mathsf{Q}(x, y; t)$, which counts walks confined to a quadrant, this has not been proven for weighted models, so it is not surprising that we have so far been unable to prove it for $\mathsf{C}(x, y; t)$. However, we expect that if this is proven for $\mathsf{Q}(x, y; t)$, the result for $\mathsf{C}(x, y; t)$ will follow using the same method applied to Theorem 6.

References

[1] O. Bernardi, M. Bousquet-Mélou, and K. Raschel. "Counting quadrant walks via Tutte's invariant method". *Combin. Theory* **1** (2021), #3.

[2] A. Bostan and M. Kauers. "The complete generating function for Gessel walks is algebraic". *Proc. Amer. Math. Soc.* **138**.9 (2010), pp. 3063–3078.

[3] M. Bousquet-Mélou. "Square lattice walks avoiding a quadrant". *J. Combin. Theory Ser. A* **144** (2016), pp. 37–79.

[4] M. Bousquet-Mélou. "Enumeration of three-quadrant walks via invariants: some diagonally symmetric models". 2021. arXiv:2112.05776.

[5] M. Bousquet-Mélou and M. Mishna. "Walks with small steps in the quarter plane". *Contemp. Math* **520** (2010), pp. 1–40.

[6] M. Bousquet-Mélou and M. Wallner. "More models of walks avoiding a quadrant". 2021. arXiv:2110.07633.

[7] T. Budd. “Winding of simple walks on the square lattice”. *J. Combin. Theory Ser. A* **172** (2020), pp. 105191, 59.

[8] T. Dreyfus, C. Hardouin, J. Roques, and M. Singer. “On the nature of the generating series of walks in the quarter plane”. *Invent. Math.* **213**.1 (2018), pp. 139–203.

[9] T. Dreyfus, C. Hardouin, J. Roques, and M. F. Singer. “Walks in the quarter plane: Genus zero case”. *J. Combin. Theory Ser. A* **174** (2020), p. 105251.

[10] T. Dreyfus and K. Raschel. “Differential transcendence & algebraicity criteria for the series counting weighted quadrant walks”. *Publ. Math. Besançon-Algèbr. Théor. Nr.* 1 (2019), pp. 41–80.

[11] T. Dreyfus and A. Trotignon. “On the nature of four models of symmetric walks avoiding a quadrant”. *Ann. Combin.* (2021), pp. 1–28.

[12] A. Elvey Price. “Counting lattice walks by winding angle”. *Proceedings of the 32nd Conference on Formal Power Series and Algebraic Combinatorics*. 2020.

[13] G. Fayolle, R. Iasnogorodski, and V. Malyshev. *Random walks in the quarter-plane*. Vol. 40. Applications of Mathematics (New York). Springer-Verlag, Berlin, 1999.

[14] G. Fayolle and K. Raschel. “On the Holonomy or Algebraicity of Generating Functions Counting Lattice Walks in the Quarter-Plane”. *Markov Process. Related Fields* **16**.3 (2010), pp. 485–496.

[15] C. Hardouin and M. F. Singer. “On Differentially Algebraic Generating Series for Walks in the Quarter Plane”. *Selecta Math. (N.S.)* **27**.5 (2021), pp. 1–49.

[16] I. Kurkova and K. Raschel. “On the functions counting walks with small steps in the quarter plane”. *Publ. Math. Inst. Hautes Études Sci.* **116**.1 (2012), pp. 69–114.

[17] S. Melczer and M. Mishna. “Singularity analysis via the iterated kernel method”. *Combin., Probab. Comput.* **23**.5 (2014), pp. 861–888.

[18] M. Mishna and A. Rechnitzer. “Two non-holonomic lattice walks in the quarter plane”. *Theoret. Comput. Sci.* **410**.38-40 (2009), pp. 3616–3630.

[19] K. Raschel. “Counting walks in a quadrant: a unified approach via boundary value problems”. *J. Eur. Math. Soc. (JEMS)* **14**.3 (2012), pp. 749–777.

[20] K. Raschel and A. Trotignon. “On Walks Avoiding a Quadrant”. *Electron. J. Combin.* **26**.3 (2019), Paper 3, 31.