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Stable Sets in Flag Spheres

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Abstract. We provide lower and upper bounds on the minimum size of a maximum stable set over graphs of flag spheres, as a function of the dimension of the sphere and the number of vertices. Further, we use stable sets to obtain an improved Lower Bound Theorem for the face numbers of flag spheres.

1 Introduction

Given a graph G, a set $X \subseteq V(G)$ is *stable* (or *independent*) if no edge of G has both ends in X. We denote by $\alpha(G)$ the size of a largest stable set in G; a stable set of size $\alpha(G)$ is called a *maximum stable set* of G. Stable sets are a basic concept in graph theory, but it is in general very difficult to understand what the structure of maximum stable sets is (this is related to the fact that the problem of computing $\alpha(G)$ is NP-complete). In this paper we study maximum stable sets in graphs whose clique complex is topologically a sphere of fixed dimension (these are called *graphs of flag spheres*). These graphs possess a beautiful recursive structure, since the neighborhood of every vertex is a graph of the same type but of lower dimension. They are also of great interest in topological combinatorics and beyond, *e.g.*, in the study of manifolds with nonpositive sectional curvature, via the Charney-Davis conjecture [2, 3].

Our main objective is the following natural invariant: the minimum size over maximum stable sets in n-vertex graphs of flag $(d-1)$-dimensional spheres, namely

$$\alpha(d,n) = \min(\alpha(G) : \; |V(G)| = n, \; cl(G) \text{ triangulates the } (d-1)\text{-dimensional sphere}).$$

(Here $cl(G)$ is the complex of cliques of G.) For fixed d we are interested in the growth of $\alpha(d,n)$ as $n \to \infty$.

Conjecture 1.1. *For every $d \geq 2$ and $n \geq 2d$, $\alpha(d,n) = \left\lceil \frac{n+d-3}{2(d-1)} \right\rceil$.*

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This conjecture holds for $d = 2$ (easy) and $d = 3$ (see Theorem 2.3, using the 4-color theorem (4CT) for the lower bound). For $d = 4$ we prove that the conjectured upper bound holds. For general $d \geq 4$ we show:

Theorem 1.2. *Let $d \geq 4$ and $n \geq 2d$. Then*

$$\frac{1}{4}n^{\frac{1}{d-2}} \leq \alpha(d,n) \leq \left\lceil \frac{\left\lceil \frac{n}{\lfloor d/4 \rfloor} \right\rceil + 1}{6} \right\rceil .$$

The lower bound slightly improves on the Ramsey bound ($\Omega(n^{\frac{1}{d}})$) by using the 4CT within the base case $d = 4$. The upper bound, which is roughly $\frac{2n}{3d}$ for large d, is obtained by taking the join of copies of the best flag 3-spheres constructed in Theorem 2.4 for the upper bound, and taking up to 3 extra suspensions to reach dimension $d - 1$. Indeed, a maximum stable set in the join is a maximum stable set in a component of the join – now, ignoring rounding, such component is a 3-sphere on a $4/d$ fraction of the n vertices, and a $1/6$ fraction of its vertices form a maximum stable set.

Our second result is an improved lower bound theorem on the number of edges for the class of flag spheres; the proof relies on the existence of a large stable set in such graphs. Deducing from this bound lower bounds on the number of higher dimensional k-faces appeared in the proof of [9, Proposition 3.2], following the MPW-reduction.

Theorem 1.3. *(i) Fix $\delta > 0$. There exists $d(\delta)$ such that for all $d \geq d(\delta)$ and n large enough, each n-vertex flag $(d-1)$-sphere has at least $(d + \frac{1-\delta}{2d+1})n$ edges.*

(ii) For all $d \geq 6$, and n large enough, each n-vertex flag $(d-1)$-sphere has at least $(d + \frac{0.987}{2d+1})n$ edges.

Note that the Lower Bound Theorem for simplicial spheres [1, 4] guarantees in (i) for simplicial spheres at least $(d - \delta)n$ edges and Gal's conjecture [3], which, if true, is tight, would imply at least $(2d - 3 - \delta)n$ edges (it does hold for $d \leq 5$). For $d \geq 6$ the lower bound in Theorem 1.3(ii) appears to be new. If Conjecture 1.1 holds then this lower bound would further improve to at least $(d + \frac{1}{2d-2})n$ edges, for all $d \geq 6$, for large enough n.

Outline: In Section 2 we construct low dimensional flag spheres whose maximum independent sets are small, proving Conjecture 1.1 for $d = 3$ and the upper bound there for $d = 4$, and deducing both bounds in Theorem 1.2. In Section 3 we prove Theorem 1.3 by combining stable sets with framework rigidity. In Section 4 we give some results and conjectures regarding the corresponding invariant for the other extreme:

$$\alpha_M(d,n) = \max(\alpha(G) : \ |V(G)| = n, \ cl(G) \text{ triangulates the } (d-1)\text{-dimensional sphere}).$$

2 The construction

We construct graphs, denoted $W_{d,k}$. First we analyze their α, and next we analyze their clique complex. Figure 1(middle) illustrates $W_{3,3}$.

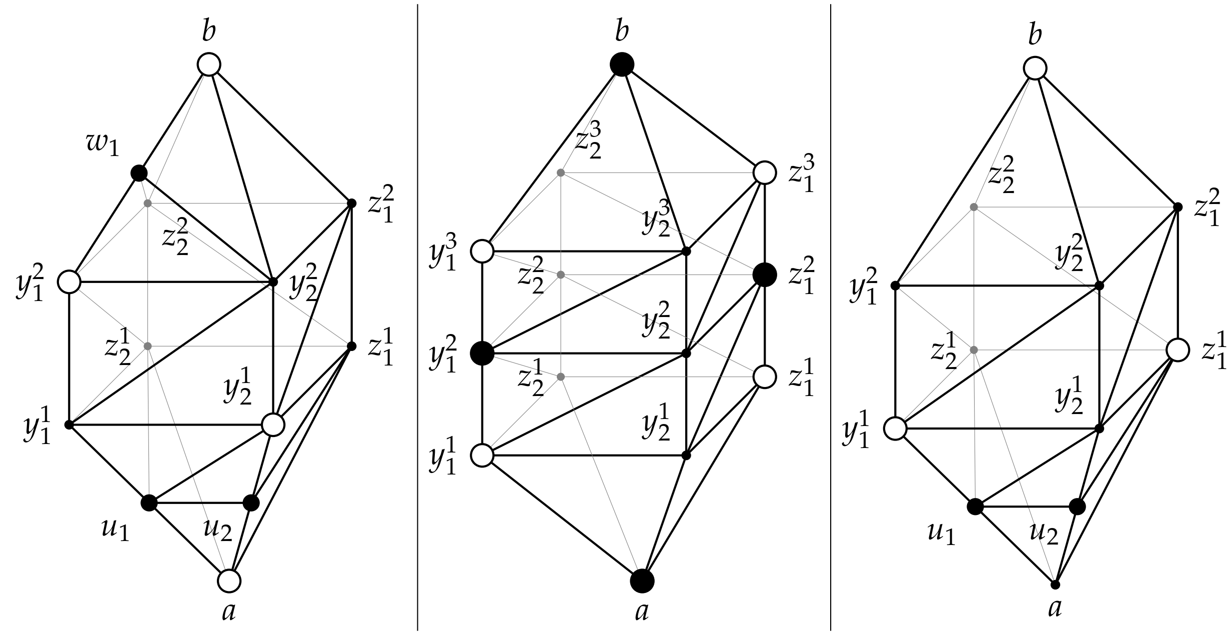

Figure 1: Middle: The graph $W_{3,3}$ is depicted. The bold black and bold white vertices indicate stable sets of size $\alpha(W_{3,3}) = 4$. The shaded edges indicate edges that are not visible from a front view of the depicted realization of the flag 2-sphere $cl(W_{3,3})$ in 3-space. **Right**: The graph $X(3,2,2)$ is depicted. The bold white vertices indicate a stable set of size $\alpha(X(3,2,2)) = 4$. **Left**: The graph $Y(3,2,1)$ is depicted. The bold white vertices indicate a stable set of size $\alpha(Y(3,2,1)) = 4$.

Fix an integer $d \geq 2$. For $k \geq 1$ let $W_{d,k}$ be the following graph. $V(W_{d,k}) = \{a, b\} \cup X_1 \cup \ldots \cup X_k$ where the sets $X_1, \ldots, X_k, \{a, b\}$ are pairwise disjoint and $|X_i| = 2d - 2$ for every $i \in \{1, \ldots, k\}$. Denote $X_i = \{y_1^i, \ldots, y_{d-1}^i, z_1^{i'}, \ldots, z_{d-1}^i\}$. Next we list the edges of $W_{d,k}$.

- a is complete to X_1 and b is complete to X_k and there are no other edges incident with a, b.
- For every i, the induced graph $W_{d,k}[X_i]$ is the 1-skeleton of the $(d-1)$-dimensional crosspolytope, a.k.a. the graph of the octahedral $(d-2)$-sphere, with non-edges $y_1^i z_1^i, \ldots, y_{d-1}^i z_{d-1}^i$.
- X_i is anticomplete to X_j if $|i - j| > 1$.

- For $i \in \{1, \ldots, k-1\}$ and $s, t \in \{1, \ldots, d-1\}$ let us say that the pair $(y^i_s z^i_s, y^{i+1}_t z^{i+1}_t)$ is *positive* if $y^i_s y^{i+1}_t$ and $z^i_s z^{i+1}_t$ are edges, and $y^i_s z^{i+1}_t$ and $z^i_s y^{i+1}_t$ are non-edges, and *negative* if $y^i_s y^{i+1}_t$ and $z^i_s z^{i+1}_t$ are non-edges, and $y^i_s z^{i+1}_t$ and $z^i_s y^{i+1}_t$ are edges. Then the pair $(y^i_s z^i_s, y^{i+1}_t z^{i+1}_t)$ is positive if $t \geq s$ and negative if $t < s$.
- All pairs of vertices of $W_{d,k}$ that are not mentioned above are non-edges.

Now we define certain edge subdivisions on $cl(W_{d,k})$. Consider a maximal simplex in the link of a (resp. b) in $cl(W_{d,k})$, say $y^1_1 y^1_2 y^1_3 \ldots y^1_{d-1}$ (resp. $y^k_1 y^k_2 y^k_3 \ldots y^k_{d-1}$). Given a simplicial complex Z and an edge xy of Z, we denote by $Z(xy)$ the complex obtained from Z by the stellar subdivision of Z at xy (also called *edge subdivision*), and by v_{xy} the new vertex resulting from such a subdivision. Make the following sequence of $2d-2$ edge subdivisions:

$X''(d,k,0) := cl(W_{d,k})$, and for $j \in \{1, \ldots, d-1\}$, having defined $X''(d,k,j-1)$ and u_{j-1} (for $j > 1$), let $X''(d,k,j) := X''(d,k,j-1)(ay^1_j)$ and $u_j := v_{ay^1_j}$. Let $X(d,k,j)$ be the graph that is the 1-skeleton of $X''(d,k,j)$ (thus $X(d,k,0) = W_{d,k}$). For example, Figure 1(right) illustrates $X(3,2,2)$.

Next let $Y''(d,k,0) := X''(d,k,d-1)$, and for $j \in \{1, \ldots, d-1\}$, having defined $Y''(d,k,j-1)$ and w_{j-1} (for $j > 1$), let $Y''(d,k,j) := Y''(d,k,j-1)(by^k_j)$ and $w_j := v_{by^k_j}$. Let $Y(d,k,j)$ be the graph that is the 1-skeleton of $Y''(d,k,j)$. For example, Figure 1(left) illustrates $Y(3,2,1)$.

Theorem 2.1. *For every $d \geq 2, k \geq 1, d-1 \geq j \geq 0$,*

$$\alpha(X(d,k,j)) = k+1 = \frac{|V(X(d,k,j))| - 2 - j}{2d-2} + 1.$$

For every $d \geq 3, k \geq 1, d-1 \geq j \geq 1$,

$$\alpha(Y(d,k,j)) = k+2 = \frac{|V(Y(d,k,j))| - 2 + (d-1-j)}{2d-2} + 1.$$

Proof sketch: Let G be one of the graphs $X(d,k,j)$ or $Y(d,k,j)$. Let U be the set of vertices of the form u_j in G, and let W be the set of vertices of the form w_j in G. Then $W \neq \emptyset$ only if $|U| = d-1$. Moreover $U \cup a$ and $W \cup b$ are both cliques in G. Denote by $N_G(v)$ the neighbors of v in G. Then, $X_1 \setminus N_G(a) \subseteq \{y^1_1, \ldots, y^1_{d-1}\}$, and for every j we have that $X_1 \setminus N_G(u_j) = \{y^1_1, \ldots, y^1_{j-1}, z^1_j\}$. In particular, $\alpha(G[X_1 \setminus N_G(v)]) \leq 1$ for every $v \in U \cup \{a\}$. Similarly, $\alpha(G[X_k \setminus N_G(v)]) \leq 1$ for every $v \in W \cup \{b\}$.

Let S be a stable set of G. First we prove an upper bound on $|S|$. Clearly for every i we have that $\alpha(G[X_i]) = 2$. Moreover every vertex of X_{i+1} has a neighbor in every non-edge of $G[X_i]$, and every vertex of X_i has a neighbor in every non-edge of $G[X_{i+1}]$. Consequently, $|S \cap (X_i \cup X_{i+1})| \leq 2$.

Hence $|S \setminus (U \cup W \cup \{a,b\})| \leq k+1$. Suppose $|S \setminus (U \cup W \cup \{a,b\})| = k+1$. Then k is odd, and $|S \cap X_1| = |S \cap X_k| = 2$. It follows that $S \cap (U \cup W \cup \{a,b\}) = \emptyset$ and $|S| = k+1$.

Next suppose that $|S \setminus (U \cup W \cup \{a,b\})| = k$. Since $U \cup \{a\}$ and $W \cup \{b\}$ are both cliques, it follows that $|S| \leq k+2$, and so we may assume that $G = X(d,k,j)$ for some j (for otherwise $G = Y(d,k,j)$ and the upper bound on $\alpha(G)$ holds). With some extra work, of similar flavor, one shows that $|S| \leq k+1$ in this case, since $W = \emptyset$ in this case.

Next we show that if $G = X(d,k,j)$ for some $j \geq 0$ then $\alpha(G) = k+1$. Let $S' = \bigcup_{i \in 1,\ldots,k;\ i \text{ odd}} \{y_1^i, z_1^i\}$. If k is odd let $S = S'$. If k is even, let $S = S' \cup \{b\}$. In both cases $|S| = k+1$.

Finally we show that if $G = Y(d,k,j)$ for some $j \geq 1$ then $\alpha(G) = k+2$. Since $j \geq 1$, we have that a is anticomplete to $\{y_1^1, \ldots, y_{d-1}^1\}$ and $w_1 \in W$. Let

$$S = \{a, w_1\} \cup \bigcup_{\substack{i \in \{1,\ldots,k\} \\ k-i \text{ odd}}} \{y_1^i\} \cup \bigcup_{\substack{i \in \{1,\ldots,k\} \\ k-i \text{ even}}} \{z_1^i\}.$$

Then S a stable set of size $k+2$ in G.

So far we have proved that $\alpha(X(d,k,j)) = k+1$ for every $d \geq 2, k \geq 1$ and $j \geq 0$, and that $\alpha(Y(d,k,j)) = k+2$ for every $d \geq 3, k \geq 1$ and $j \geq 1$. The remaining equalities follow by a direct computation. □

Observe that $W_{d,1}$ is the 1-skeleton of the d-dimensional crosspolytope. Further,

Observation 2.2. *The clique complex of $W_{3,k}$ is a flag 2-sphere for every $k \geq 1$.*

Proof. For each i, $W_{3,k}[X_i]$ is a 4-cycle. Consider $W_{3,k}[X_i \cup X_{i+1}]$: adding to the two disjoint 4-cycles $W_{3,k}[X_i] \cup W_{3,k}[X_{i+1}]$ the edges $y_s^i y_s^{i+1}$ and $z_s^i z_s^{i+1}$ (for the positive pairs $(y_s^i z_s^i, y_s^{i+1} z_s^{i+1})$ with $s = 1,2$) makes a cylinder subdivided into 4 squares; adding the other edges for the positive pair with $s = 1, t = 2$ and for the negative pair with $s = 2, t = 1$ subdivides each of the four squares into two triangles. Thus, $W_{3,k}[X_1 \cup \ldots \cup X_k]$ is a triangulated cylinder, and adding a, b with their edges makes a flag 2-sphere. □

Next we show:

Theorem 2.3. *For every $n \geq 6$, $\alpha(3,n) = \lceil \frac{n}{4} \rceil$.*

Proof. Observe that $|V(X(3,k,j))| \equiv_4 2+j$, and $|V(Y(3,k,j))| \equiv_4 j$, and thus for every $n \geq 6$ there exist integers $k \geq 1$ and $j \geq 0$ and a graph $G \in \{X(3,k,j), Y(3,k,j)\}$ such that $|V(G)| = n$. Now by Theorem 2.1 for every $k \geq 1$ and $j \geq 0$ we have that $\alpha(X(3,k,j)) = \lceil \frac{|V(X(3,k,j))|}{4} \rceil$, and for every $k \geq 1$ and $j \geq 1$ we have that $\alpha(Y(3,k,j)) = \lceil \frac{|V(Y(3,k,j))|}{4} \rceil$. Finally, since $X''(3,k,j)$ and $Y''(3,k,j)$ are obtained from $cl(W_{3,k})$ by stellar edge subdivisions, it follows from Observation 2.2 that their clique complexes are flag

2-spheres. We have shown that for every $n \geq 6$, $\alpha(3,n) \leq \lceil \frac{n}{4} \rceil$. Since by the 4CT every n-vertex triangulation of the 2-dimensional sphere has a stable set of size $\lceil \frac{n}{4} \rceil$, $\alpha(3,n) \geq \lceil \frac{n}{4} \rceil$. $\square$

For $d = 4$, the graph $W_{4,k}$ induces a cell structure on the 3-sphere, consisting of tetrahedra with a vertex a or b and of triangular prisms consisting of a triangle on X_i and the corresponding triangle on X_{i+1} (the corresponding vertices differ only in the superscript). All these triangular prisms are triangulated by considering all tertrahedra defined by cliques of $W_{4,k}$ on this set of 6 vertices, except for the following two (for a fixed $1 \leq i \leq k-1$): $y_1^i, z_2^i, y_3^i; y_1^{i+1}, z_2^{i+1}, y_3^{i+1}$ and its "antipodal prism" $z_1^i, y_2^i, z_3^i; z_1^{i+1}, y_2^{i+1}, z_3^{i+1}$. We add the edge $y_1^i z_2^{i+1}$ to triangulate the first, and the edge $z_1^i y_2^{i+1}$ to triangulate the second (such added edge is "bent" inside the prism, the resulted triangulation of the prism is topological, not geometric); denote the resulting graph by $W'_{4,k}$. Let $X'(4,k,j)$ and $Y'(4,k,j)$ be the graphs obtained from $X(4,k,j)$ and $Y(4,k,j)$, respectively, by adding the same edges. See Figure 2 for an illustration of how the triangular prisms are triangulated.

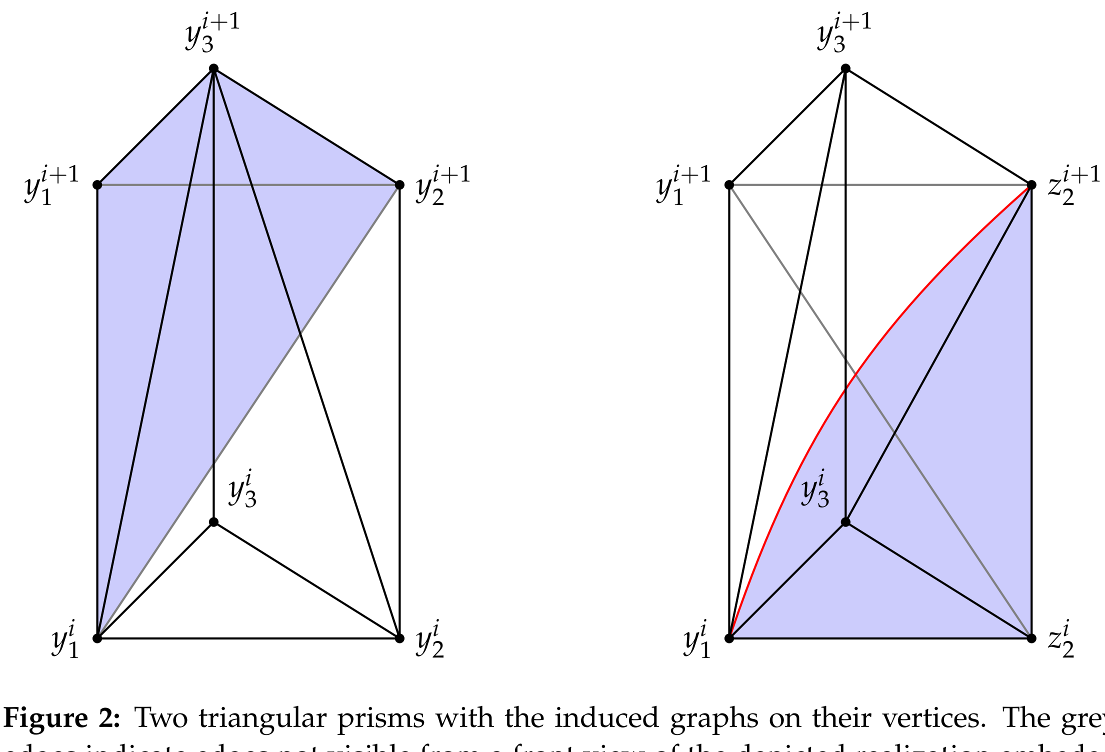

Figure 2: Two triangular prisms with the induced graphs on their vertices. The grey edges indicate edges not visible from a front view of the depicted realization embeded in 3-space. The red edge is bent inside the right prism. In purple are sample induced tetrahedra. Note that in each prism, its clique complex triangulates it.

Theorem 2.4. *The clique complex of $W'_{4,k}$ is a flag 3-sphere for every $k \geq 1$.*

Proof sketch: Recall the cell structure on the 3-sphere described above, by tetrahedra and triangular prisms, induced by $W_{4,k}$. First observe that for every triangular prism T on vertex set $V(T)$ and every added edge $uv = y_1^i z_2^{i+1}$ or $z_1^i y_2^{i+1}$ of $W'_{4,k}$ on vertices in $V(T)$, all cliques in $W'_{4,k}$ involving uv have their vertex sets contained in $V(T)$. Further, every clique of $W'_{4,k}[X_1 \cup \ldots \cup X_k]$ has its vertex set contained in $V(T)$ some triangular prism T. Hence, to show that $cl(W'_{4,k})$ is a flag 3-sphere it is enough to check that every induced subcomplex $cl(W'_{4,k}[V(T)])$ triangulates the prism T. Clearly the squares in each prism T are triangulated, as exactly one diagonal in each square is inserted (which diagonal depends on whether the corresponding pair is positive or negative). One verifies that each triangle on the boundary of T is contained in exactly one tetrahedron whose vertex set is contained in $V(T)$, and there is no 5-clique whose vertex set is contained in $V(T)$. Thus, to verify that $cl(W'_{4,k}[V(T)])$ triangulates the prism T it suffices to check for each tetrahedron A whose vertex set is contained in $V(T)$ that each triangle B in A and not in the boundary of T, satisfies that B is contained in exactly one more tetrahedron A' whose vertex set is contained in $V(T)$. One inspects that this is indeed the case. □

Next we show:

Theorem 2.5. *For all $n \geq 8$, $\alpha(4,n) \leq \lceil \frac{n+1}{6} \rceil$.*

Proof. Observe that $|V(X(4,k,j))| \equiv_6 2+j$, and $|V(Y(4,k,j))| \equiv_6 j-1$ (here $0 \leq j \leq 3$), and thus for every $n \geq 8$ there exist integers $k \geq 1$ and $j \geq 0$ and a graph $G \in \{X(4,k,j), Y(4,k,j)\}$ such that $|V(G)| = n$. Now by Theorem 2.1 for every $k \geq 1$ and $j \geq 0$ we have that $\alpha(X(4,k,j)) = \left\lceil \frac{|V(X(4,k,j))|+1}{6} \right\rceil$, and for every $k \geq 1$ and $j \geq 1$ we have that $\alpha(Y(4,k,j)) = \left\lceil \frac{|V(Y(4,k,j))+1|}{6} \right\rceil$. Since $X'(4,k,j)$ and $Y'(4,k,j)$ are obtained from $X(4,k,j)$ and $Y(4,k,j)$ by adding edges, we deduce that

$$\alpha(X'(4,k,j)) \leq \left\lceil \frac{|V(X(4,k,j))|+1}{6} \right\rceil = \left\lceil \frac{|V(X'(4,k,j))|+1}{6} \right\rceil$$

for every $k \geq 1$ and $j \geq 0$, and

$$\alpha(Y'(4,k,j)) \leq \left\lceil \frac{|V(Y(4,k,j))|+1}{6} \right\rceil = \left\lceil \frac{|V(Y'(4,k,j))|+1}{6} \right\rceil$$

and for every $k \geq 1$ and $j \geq 1$.

Finally, since $cl(X'(4,k,j))$ and $cl(Y'(4,k,j))$ are obtained from $cl(W'(4,k))$ by stellar edge subdivisions, it follows from Theorem 2.4 that their clique complexes are flag 3-spheres. This completes the proof. □

Remark 2.6. In fact, $\alpha(X'(4,k,j)) = \left\lceil \frac{|V(X(4,k,j))|+1}{6} \right\rceil$ for every $k \geq 1$ and $j \geq 0$, and $\alpha(Y'(4,k,j)) = \left\lceil \frac{|V(Y(4,k,j))|+1}{6} \right\rceil$ for every $k \geq 1$ and $j \geq 1$.

Indeed, for $d = 4$ the sets S constructed in the proof of Theorem 2.1 are also independent in $X'(4,k,j)$ and $Y'(4,k,j)$ resp.

Finally we prove the lower bound of Theorem 1.2.

Theorem 2.7. *Let $d \geq 4$. Then for all $n \geq 2d$,*

$$\alpha(d,n) \geq \frac{1}{4} n^{\frac{1}{d-2}}$$

Proof sketch: The proof is by induction on d. Let Δ be a $(d-1)$-flag sphere. Recall Δ has at least $2d$ vertices [8], say it has n vertices.

For the base case let $d = 4$. Then the link of v in Δ, denoted $lk_v(\Delta)$, is a planar triangulation for every vertex v of Δ, and therefore, by the 4CT, $lk_v(\Delta)$ contains a stable set of size $\left\lceil \frac{|V(lk_v(\Delta))|}{4} \right\rceil$. Thus if for some vertex v of Δ we have that $|V(lk_v(\Delta))| \geq n^{\frac{1}{2}}$, then the theorem holds. If $|V(lk_v(\Delta))| < n^{\frac{1}{2}}$ for every v, then a stable set of size $\frac{n}{n^{\frac{1}{2}}} = n^{\frac{1}{2}} > \frac{1}{4}n^{\frac{1}{2}}$ can be obtained greedily. This finishes the case when $d = 4$.

Now we turn to general d. In this case $lk_v(\Delta)$ is a $(d-2)$-flag sphere for every vertex v of Δ, and therefore, inductively, $lk_v(\Delta)$ contains a stable set of size $\frac{1}{4}|V(lk_v(\Delta))|^{\frac{1}{d-3}}$. Thus if for some vertex v of Δ we have that $|V(lk_v(\Delta))| \geq n^{\frac{d-3}{d-2}}$, then the theorem holds. If $|V(lk_v(\Delta))| < n^{\frac{d-3}{d-2}}$ for every v, then a stable set of size $\frac{n}{n^{\frac{d-3}{d-2}}} = n^{\frac{1}{d-2}} > \frac{1}{4}n^{\frac{1}{d-2}}$ can be obtained greedily. This completes the proof. $\square$

3 Lower bounds on f_1

The goal of this section is to prove Theorem 1.3.

Proof. Let $\Delta = cl(G)$ be a flag $(d-1)$-sphere on $n = f_0(\Delta)$ vertices and $f_1 = f_1(\Delta)$ edges. Let $\epsilon > 0$, and assume $f_1 < (d+\epsilon)n$. We look for the largest $\epsilon = \epsilon(d)$ for which we reach a contradiction (when d is chosen large enough, and then n is chosen large enough with respect to d).

By an easy restatement of Turán's theorem from [7] there is a stable set I of G with $|I| \geq \frac{n}{2(d+\epsilon)+1}$.

We may assume $d \geq 4$. Then, we use the following well known facts: (i) G is generically d-rigid, hence its space of stresses (a.k.a. affine 2-stresses [6]) has dimension $g_2(\Delta) := f_1 - dn + \binom{d+1}{2}$, see Kalai [4]. (ii) For every vertex link, its graph is generically $(d-1)$-rigid and is not stacked (by flagness) hence, by the Cone Lemma, see, *e.g.*, [10, Corollary 1.5], for every vertex $v \in \Delta$ there exists a stress supported in the closed star of v (namely in the induced graph of G on v and its neighbors) such that some edge containing v has a nonzero weight.

Now, as I is independent, the stresses mentioned above for $v \in I$ are linearly independent (each has a unique edge with a nonzero weight) and hence

$$f_1 - dn + \binom{d+1}{2} \geq |I| \geq \frac{n}{2(d+\epsilon)+1},$$

Thus, for n large enough with respect to d, we can ignore the $\binom{d+1}{2}$ term and get: $\epsilon n > \frac{n}{2(d+\epsilon)+1}$, namely $\epsilon > \frac{1}{2(d+\epsilon)+1}$.

Solving the quadric for ϵ we get a contradiction if $\epsilon < \frac{-(2d+1)+\sqrt{(2d+1)^2+8}}{4}$.

Hence for arbitrarily small $\delta > 0$, if d is large enough we reach a contradiction for $\epsilon = \frac{1-\delta}{2d+1}$, proving part (i). For part (ii), note that $\sqrt{x^2+8} - x > \frac{3.95}{x}$ for $x \geq 13 = 2 \cdot 6 + 1$, thus for all $d \geq 6$ (and large enough n) we will reach a contradiction if $\epsilon \leq \frac{3.95}{4(2d+1)} = \frac{0.987}{2d+1}$. □

Note that if Conjecture 1.1 holds then plugging the larger value for $|I|$ yields $f_1 \geq (d + \frac{1}{2d-2})n$ for all $d \geq 6$ and large enough n.

Conjecture 3.1. *For all $d \geq 5$, the graph of every flag $(d-1)$-sphere is $(d+1)$-rigid.*

If true, this conjecture would imply $f_1 \geq (d+1)f_0 - \binom{d+2}{2}$ for flag spheres of dimension $d-1 \geq 4$. A standard use of the Cone and Gluing Lemmas, see Kalai [4], reduces Conjecture 3.1 to the case $d = 5$. For $d < 5$ its assertion is false.

4 $\alpha_M(d,n)$

Fix $d \geq 4$ and let $n \to \infty$. Then there exist simplicial $(d-1)$-spheres on n vertices where the proportion of vertices in an independent set is arbitrarily close to 1. To see this, start with the boundary complex Δ of a cyclic d-polytope with $m > d$ vertices, and note that Δ is a neighborly $(d-1)$-sphere, *i.e.* all $\binom{m}{\lfloor \frac{d}{2} \rfloor}$ subsets consisting of $\left\lfloor \frac{d}{2} \right\rfloor$ vertices are faces in Δ. It is easy to check that Δ has $\Theta(m^{\lfloor \frac{d}{2} \rfloor})$ facets. Perform stellar subdivisions on all facets. Then the set I of the newly added vertices is stable and of size $\Theta(m^{\lfloor \frac{d}{2} \rfloor})$, while only the original m vertices are not in I.

In contrast, for flag spheres we conjecture that the proportion of vertices in an independent set can not exceed 1/2.

Conjecture 4.1. *For all $d \geq 2$, $\alpha_M(d,n) = \left\lfloor \frac{n-2(d-2)}{2} \right\rfloor$.*

This conjecture clearly holds for $d = 2$ and we prove it for $d = 3$. The lower bound holds for all $d \geq 2$ by the following construction: consider the $(d-2)$-fold suspension over the $(n-2(d-2))$-gon. A maximum stable set is obtained by taking every second vertex along the $(n-2(d-2))$-gon.

Theorem 4.2. *For all $n \geq 6$, $\alpha_M(3,n) = \lfloor \frac{n-2}{2} \rfloor$.*

Proof. The construction above proves the lower bound $\alpha_M(3,n) \geq \lfloor \frac{n-2}{2} \rfloor$. To show $\alpha_M(3,n) \leq \lfloor \frac{n-2}{2} \rfloor$, let I be a maximum stable set in the graph $G = (V,E)$ of a flag 2-sphere on n vertices (it forces $n \geq 6$). Let $G' = (V,B)$ be the subgraph of G whose edges are those with exactly one vertex in I. Then G' is bipartite and planar. Further, G' has at least two vertices in I (as each vertex in G has a non-neighbor) and at least two (in fact 4) vertices in the complement of I (as each vertex in I has degree at least 4 by flagness). Thus, G' has at most $2n-4$ edges (this is known, see, *e.g.*, [5, Lemmas 4.2, 4.3] for a proof). On the other hand,

$$|B| = \sum_{v \in I} \deg(v) \geq 4|I|,$$

as each vertex in G has degree at least 4, and for all $v \in I$ the degree is preserved when passing to G'. Thus $4|I| \leq 2n-4$, hence $|I| \leq \lfloor \frac{n-2}{2} \rfloor$. $\square$

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