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A Type B Analog of the Whitehouse Representation

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Abstract. We give a Type B analog of Whitehouse's lifts of the Eulerian representations from S_n to S_{n+1} by introducing a family of B_n-representations that lift to B_{n+1}. As in Type A, we interpret these representations combinatorially via a family of orthogonal idempotents in the Mantaci–Reutenauer algebra, and topologically as the graded pieces of the cohomology of a certain $\mathbb{Z}_2$-orbit configuration space of $\mathbb{R}^3$. We show that the lifted B_{n+1}-representations also have a configuration space interpretation, and further parallel the Type A story by giving analogs of many of its notable properties, such as connections to equivariant cohomology and the Varchenko–Gelfand ring.

Keywords: configuration spaces, equivariant cohomology, Eulerian idempotents, symmetric group representations, hyperoctahedral group, Mantaci–Reutenauer algebra

1 Introduction

Let V be a representation of a finite group H; then V is said to have a *lift* to a group G containing H if there is a representation of G that restricts to V. The goal of this abstract is to (1) identify a family of representations of the hyperoctahedral group B_n that decompose the regular representation $\mathbb{Q}[B_n]$ and lift to B_{n+1}, and (2) interpret these representations combinatorially and topologically.

This work is inspired by the well-documented Type A story of a family of S_n-representations lifting to representations of S_{n+1} studied by Whitehouse [21], Early–Reiner [6], Mathieu [12], Getzler–Kapranov [9], Moseley–Proudfoot–Young [14], and others. These S_n-representations and their lifts arose from two distinct perspectives. The first is via a family of orthogonal idempotents $\{\mathfrak{e}_k\}_{0\leq k\leq n-1}$ known as the *Eulerian idempotents*. The $\mathfrak{e}_k$ are in *Solomon's descent algebra* $\Sigma[S_n]$, the subalgebra of $\mathbb{Q}[S_n]$ generated by sums of permutations $\sigma = (\sigma_1, \ldots, \sigma_n)$ with the same descent set

$$\mathrm{Des}(\sigma_1, \ldots, \sigma_n) := \{i \in [n-1] : \sigma_i > \sigma_{i+1}\}.$$

The Eulerian idempotents have been extensively researched in the world of algebraic combinatorics, and generate the *Eulerian representations* $\mathfrak{e}_k\,\mathbb{Q}[S_n]$, which lift to a family of S_{n+1}-representations called the *Whitehouse representations* [21], defined in Section 2.1.

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The second viewpoint comes from the study of $\mathrm{Conf}_n(\mathbb{R}^3)$, the configuration space comprised of n distinct ordered points in $\mathbb{R}^3$. Through this lens, one obtains a family of S_n-representations as the graded pieces of $H^* \mathrm{Conf}_n(\mathbb{R}^3)$, and lifted representations of S_{n+1} by considering the cohomology of a particular quotient of the configuration space of $\mathbb{S}^3$, the one-point compactification of $\mathbb{R}^3$. The cohomology of $\mathrm{Conf}_n(\mathbb{R}^3)$ is intrinsically linked to $H^* \mathrm{Conf}_n(\mathbb{R})$, a ring with an elegant combinatorial description via *Heaviside functions* due to Varchenko–Gelfand [19] (see Section 2.2).

Though not obvious, both viewpoints turn out to be equivalent and serve as a beautiful link between classical combinatorial objects and important topological ones.

Our goal here is to construct an analog to both perspectives for Type B. In our analogy, Solomon's descent algebra is replaced by the *Type B Mantaci–Reutenauer algebra* $\Sigma'[B_n]$, a combinatorially defined subalgebra of $\mathbb{Q}[B_n]$ generalizing $\Sigma[S_n]$ and containing the Type B Descent algebra $\Sigma[B_n]$. The role of the Eulerian idempotents will be played by a family of orthogonal idempotents $\{\mathfrak{g}_k\}_{0\leq k\leq n}$, obtained as a sum of certain orthogonal idempotents in $\Sigma'[B_n]$ introduced by Vazirani [20]. The Type B analog of the space $\mathrm{Conf}_n(\mathbb{R}^3)$ will be a $\mathbb{Z}_2$-*orbit configuration space* $\mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R}^3)$ (defined in (4.3)) first introduced by Feichtner–Ziegler in [7], and its lift will be a quotient of the $\mathbb{Z}_2$-orbit configuration space of $\mathbb{S}^3$ coming from the antipodal action on $\mathbb{S}^3$ (see (4.1)). In contrast to Type A, the strategy we adopt here is to begin with the "lifted" B_{n+1}-representations and use them to obtain representations of B_n which should lift.

Our main result is to give a full analogy to the Type A story by showing that the representations $\mathfrak{g}_k\,\mathbb{Q}[B_n]$ describe the graded pieces of $H^* \mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R}^3)$, and that these representations lift to B_{n+1}, where they also have a cohomological interpretation. Further, we fully flesh out the connection between $\mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R}^3)$ and $\mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R})$, and give a combinatorial description for $H^* \mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R})$ that parallels the one by Varchenko–Gelfand.

The remainder of the abstract proceeds as follows. Section 2 describes in detail the Type A motivation, including a "wish list" of properties for a Type B analog (Section 2.2.1); Sections 3 and 4 introduce the Type B representations and topology, respectively. Section 5 then gives the main results, where we realize the properties on our wishlist.

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2 Type A Motivation

2.1 The Eulerian and Whitehouse representations

The Eulerian idempotents $\{\mathfrak{e}_k\}_{0\leq k\leq n-1}$ were originally introduced by Reutenauer in [15], and have been extensively studied and generalized since then; see for instance [16]. They are obtained as a sum over a complete, primitive, orthogonal family of idempotents $\{\mathfrak{e}_\lambda\}_{\lambda\vdash n}$ in $\Sigma[S_n]$ constructed by Garsia–Reutenauer[1] in [8]:

$$\mathfrak{e}_{k-1} := \sum_{\substack{\lambda\vdash n\\ \ell(\lambda)=k}} \mathfrak{e}_\lambda. \tag{2.1}$$

Our focus will be on the family of S_n-representations generated by the $\mathfrak{e}_k$ and decomposing $\mathbb{Q}[S_n]$, called the *Eulerian representations*, $E_n^{(k)} := \mathfrak{e}_k\,\mathbb{Q}[S_n]$. The Eulerian representations have connections to many beloved objects such as the free Lie algebra [10], hyperplane arrangements [3] and configuration spaces (see Section 2.2).

For the purposes of this abstract, we are most interested in a property observed by Whitehouse in [21]: that each $E_n^{(k)}$ has a lift to S_{n+1}. View $S_n \leq S_{n+1}$ as the subgroup fixing the element $n+1$, let λ_{n+1} be the $n+1$ cycle $(12\ldots(n+1)) \in S_{n+1}$, and define

$$\Lambda_{n+1} := \frac{1}{n+1}\sum_{i=0}^{n}(\lambda_{n+1})^i.$$

Whitehouse shows the element $f_{n+1}^{(k)} := \Lambda_{n+1}e_n^{(k)}$ is an idempotent in $\mathbb{Q}[S_{n+1}]$, generating a family of representations $F_{n+1}^{(k)} := f_{n+1}^{(k)}\,\mathbb{Q}[S_{n+1}]$ which we will call the *Whitehouse representations*. She then proves that the $F_{n+1}^{(k)}$ are lifts of the $E_n^{(k)}$ [21, Proposition 1.4].

Example 1 ($n=3$). Denote by S^λ the irreducible symmetric group representation indexed by the partition λ. Then the S_3 Eulerian representations and their S_4 lifts are

$$\begin{aligned} E_3^{(0)} &= S^{(2,1)} & F_4^{(0)} &= S^{(2,2)}\\ E_3^{(1)} &= S^{(2,1)}\oplus S^{(1,1,1)} & F_4^{(1)} &= S^{(2,1,1)}\\ E_3^{(2)} &= S^{(3)} & F_4^{(2)} &= S^{(4)}. \end{aligned}$$

Each $F_4^{(k)}$ restricts to the representation $E_3^{(k)}$ via the symmetric group branching rules.

2.2 Configuration space cohomology

We will momentarily switch tracks here and focus on the topology of

$$\mathrm{Conf}_n(\mathbb{R}^d) := \{(x_1,\ldots,x_n)\in\mathbb{R}^{dn} : x_i \neq x_j\},$$

[1] The definition of the $\mathfrak{e}_\lambda$ is technical and therefore omitted.

a space with many fascinating and far-reaching mathematical connections. When $d = 2$, for example, $\mathrm{Conf}_n(\mathbb{R}^2)$ is the classifying space of the pure Artin braid group, and when $d = 1$, $\mathrm{Conf}_n(\mathbb{R})$ is the complement of the Braid arrangement. The symmetric group naturally acts on $\mathrm{Conf}_n(\mathbb{R}^d)$ by permuting coordinates, and this action induces a representation in cohomology.[2]

In the case that $d = 1$, the space $\mathrm{Conf}_n(\mathbb{R})$ is a disjoint union of $n!$ contractible pieces. Each piece is parametrized by a relative ordering of $x_1, \ldots, x_n$ in $\mathbb{R}$, and $H^* \mathrm{Conf}_n(\mathbb{R})$ is concentrated in degree 0, *i.e.* the space of linear functionals on $\mathrm{Conf}_n(\mathbb{R})$. Varchenko–Gelfand give a combinatorial set of generators for $H^0 \mathrm{Conf}_n(\mathbb{R})$ called *Heaviside functions*,

$$u_{ij}(x_1, \ldots, x_n) := \begin{cases} 1 & x_i < x_j \\ 0 & x_i > x_j \end{cases}$$

for $i \neq j \in [n] := \{1, \ldots, n\}$. The space of such Heaviside functions forms a $\mathbb{Z}$-algebra, where the u_{ij} are endowed with linear addition and component-wise multiplication:

$$u_{ij} \cdot u_{k\ell}(x_1, \ldots, x_n) = \begin{cases} 1 & x_i < x_j \text{ and } x_k < x_\ell \\ 0 & \text{otherwise.} \end{cases}$$

This implies certain natural relations, for example that $u_{ij}^2 = u_{ij}$. Similarly, one can deduce that $1 - u_{ij} = u_{ji}$, so that $u_{ij} \cdot u_{jk} \cdot (1 - u_{ik}) = 0$, since it is impossible that $x_i < x_j < x_k$ but $x_i > x_k$. This is the essential idea behind Theorem 2.

Theorem 2 ([19]). *The ring $H^0 \mathrm{Conf}_n(\mathbb{R})$ has presentation $\mathbb{Z}[u_{ij}]/\mathcal{I}$, where $\mathcal{I}$ is generated by*

(i) $u_{ij}^2 = u_{ij}$, *(ii)* $u_{ij} = (1 - u_{ji})$, *(iii)* $u_{ij}u_{jk}(1 - u_{ik}) + (1 - u_{ij})(1 - u_{jk})u_{ik} = 0$.

Call the ring $\mathbb{Z}[u_{ij}]/\mathcal{I}$ the *Varchenko–Gelfand ring*. The presentation in Theorem 2 imposes an ascending filtration on the Varchenko–Gelfand ring obtained from the natural degree grading on $\mathbb{Z}[u_{ij}]/\mathcal{I}$: the m^{th} layer in the filtration is the span of monomials in the variables u_{ij} having degree at most m. We will see that the associated graded coming from this filtration, $\mathfrak{gr}(H^0 \mathrm{Conf}_n(\mathbb{R}))$, is closely related to $H^* \mathrm{Conf}_n(\mathbb{R}^d)$ for $d > 1$.

The space $\mathrm{Conf}_n(\mathbb{R})$ is relevant in part because it has a "hidden" S_{n+1}-action. To recover this action, let $U(1)$ be the circle group and consider $\mathrm{Conf}_{n+1}(U(1))$, the space of $n + 1$ distinct points in $U(1)$. The group $U(1)$ acts (left) diagonally, and the quotient by this action, $\mathcal{V}_{n+1}^1 := \mathrm{Conf}_{n+1}(U(1))/U(1)$ is S_n-equivariantly homeomorphic to $\mathrm{Conf}_n(\mathbb{R})$ via the map

$$f_A\colon\ \mathcal{V}_{n+1}^1 \xrightarrow{\cong} \mathrm{Conf}_n(\mathbb{R}), \tag{2.2}$$

$$(p_1, \ldots, p_{n+1}) \mapsto (\pi(p_{n+1}^{-1}p_1), \ldots, \pi(p_{n+1}^{-1}p_n)), \tag{2.3}$$

[2]When considering representations of $H^* \mathrm{Conf}_n(\mathbb{R}^d)$, we will assume our coefficients are in $\mathbb{Q}$. Otherwise, we will use coefficients in $\mathbb{Z}$, *e.g.*, for Theorems 2 and 4.

where π is the stereographic projection[3] from $U(1)$ to $\mathbb{R}$. The intuition here is that $\mathcal{V}^1_{n+1}$ has representatives $(p_1, \dots, p_n, 1)$ for $p_i \neq p_j \neq 1$ and like $\mathrm{Conf}_n(\mathbb{R})$, is comprised of $n!$ contractible pieces. Each disjoint piece of $\mathcal{V}^1_{n+1}$ is parametrized by a relative ordering of $p_1, \dots, p_{n+1}$ around the circle; these disjoint pieces (S_n-equivariantly) biject with the pieces of $\mathrm{Conf}_n(\mathbb{R})$. To move from $\mathcal{V}^1_{n+1}$ to $\mathrm{Conf}_n(\mathbb{R})$, read the ordering of $p_1, \dots, p_n$ around $U(1)$ counter-clockwise beginning after p_{n+1}.

The advantage of studying $\mathcal{V}^1_{n+1}$ is that it has an explicit S_{n+1}-action by coordinate permutation as well as a natural S_n-action given by permuting only $p_1, \dots, p_n$.

When we move to cohomology, the Heaviside functions u_{ij} also lift to *cyclic Heaviside functions* $v_{ijk} \in \mathcal{V}^1_{n+1}$, defined in [14] by Moseley–Proudfoot–Young as:

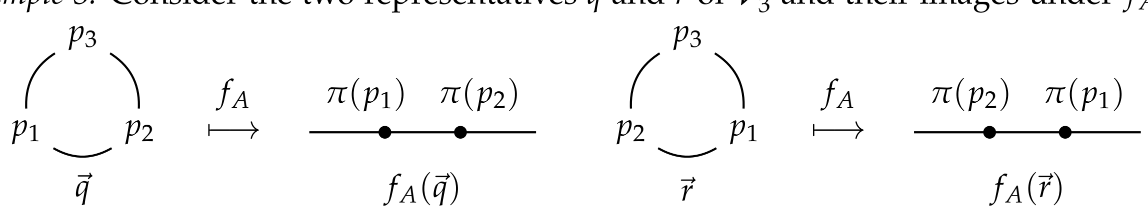
$$v_{ijk}(p_1, \dots, p_n) := \begin{cases} 1 & p_i < p_j < p_k \text{ in counter-clockwise order on } U(1) \\ 0 & \text{otherwise.} \end{cases}$$

The v_{ijk} again form a $\mathbb{Z}$-algebra and provide an elegant combinatorial description for the ring $H^0\,\mathcal{V}^1_{n+1}$. In fact the presentation can be recovered from the presentation in Theorem 2 via the induced isomorphism f_A^* sending u_{ij} to $v_{ij(n+1)}$, along with the additional relation due to Early–Reiner [6]:

$$v_{ijk} - v_{ij\ell} + v_{ik\ell} - v_{jk\ell} = 0;$$

see also [14]. As in the case of $H^0\,\mathrm{Conf}_n(\mathbb{R})$, the degree grading on $H^0\,\mathcal{V}^1_{n+1}$ from the v_{ijk} imposes an ascending filtration with associated graded $\mathfrak{gr}(H^0\,\mathcal{V}^1_{n+1})$.

Example 3. Consider the two representatives $\vec{q}$ and $\vec{r}$ of $\mathcal{V}^1_3$ and their images under f_A:



Note that $v_{123}(\vec{q}) = u_{12}(f_A(\vec{q})) = 1$, while $v_{123}(\vec{r}) = u_{12}(f_A(\vec{r})) = 0$. On the other hand $v_{213}(\vec{q}) = u_{21}(f_A(\vec{q})) = 0$ and $v_{213}(\vec{r}) = u_{21}(f_A(\vec{r})) = 1$.

When $d \geq 2$, the space $\mathrm{Conf}_n(\mathbb{R}^d)$ is no longer comprised of contractible, disjoint pieces but nonetheless has an elegant presentation due to F. Cohen.

Theorem 4 ([4])**.** *For $d \geq 2$, the ring $H^*\,\mathrm{Conf}_n(\mathbb{R}^d)$ has presentation $\mathbb{Z}\langle u_{ij}\rangle / \mathcal{J}$ for distinct $i, j, k, \ell \in [n]$, where $\mathcal{J}$ is generated by the relations $u_{ij}u_{k\ell} = (-1)^{d+1}u_{k\ell}u_{ij}$ and*

$$(i)\ \ u_{ij}^2 = 0, \qquad (ii)\ \ u_{ij} = (-1)^d u_{ji}, \qquad (iii)\ \ u_{ij}u_{jk} + u_{jk}u_{ki} + u_{ki}u_{ij} = 0.$$

The generator u_{ij} lies in $H^{d-1}\,\mathrm{Conf}_n(\mathbb{R}^d)$, which together with the relations in $\mathcal{J}$, implies that $H^*\,\mathrm{Conf}_n(\mathbb{R}^d)$ is concentrated in degrees $0, (d-1), 2(d-1), \dots, (n-1)(d-1)$.

[3]The point ∞ here is $1 \in U(1)$, and since $p_{n+1}^{-1}p_i \neq 1$, the map π is well-defined.

2.2.1 Property wish list for Type B

We are most concerned with the case that $d = 3$. In this situation, there are five notable properties of $H^* \operatorname{Conf}_n(\mathbb{R}^3)$ which will inspire our Type B work.

1. There is an isomorphism of S_n-representations[4] for $0 \leq k \leq n-1$:
$$E_n^{(n-1-k)} \cong_{S_n} H^{2k} \operatorname{Conf}_n(\mathbb{R}^3). \tag{2.4}$$
This was first deduced by comparing a result of Sundaram–Welker for subspace arrangements [18, Theorem 4.4(iii)] with descriptions of the characters of $E_n^{(k)}$ by Hanlon [10], and was later proved in the context of Coxeter groups in [3].

2. Equation (2.4) "lifts" to an isomorphism of S_{n+1} representations [6, Theorem]:
$$F_{n+1}^{(n-1-k)} \cong_{S_{n+1}} H^{2k}(\mathcal{V}_{n+1}^3), \tag{2.5}$$
where $\mathcal{V}_{n+1}^3 := \operatorname{Conf}_{n+1}(SU_2)/SU_2$. Recall that SU_2 is the group of 2×2 unitary matrices over $\mathbb{C}$ and is homeomorphic to $\mathbb{S}^3$; the quotient is by the diagonal action of SU_2 on $\operatorname{Conf}_{n+1}(SU_2)$. Intuitively, (2.5) comes from a S_n-equivariant homeomorphism found in Early-Reiner [6] and Moseley-Proudfoot-Young [14] analogous to (2.3). The notation $\mathcal{V}_{n+1}^3$ (resp. $\mathcal{V}_{n+1}^1$) indicates the relationship to $\mathbb{S}^3$ (resp. $\mathbb{S}^1$).

3. There is a recursion relating the Eulerian and Whitehouse representations of S_n:
$$E_n^{(k)} = F_n^{(k-1)} \oplus \left(S^{(n-1,1)} \otimes F_n^{(k)} \right), \tag{2.6}$$
where $S^{(n-1,1)}$ is the reflection representation of S_n [6, Proposition 1].

4. The circle group $U(1)$ acts on $\mathbb{R}^3$ by rotation around the x-axis, thereby inducing an action on $\operatorname{Conf}_n(\mathbb{R}^3)$. The filtration induced from the $U(1)$-equivariant cohomology $H^*_{U(1)} \operatorname{Conf}_n(\mathbb{R}^3)$ implies a *graded* isomorphism of S_n-modules:
$$\mathfrak{gr}(H^0 \operatorname{Conf}_n(\mathbb{R})) \cong_{S_n} H^* \operatorname{Conf}_n(\mathbb{R}^3), \tag{2.7}$$
where $\mathfrak{gr}(H^0 \operatorname{Conf}_n(\mathbb{R}))$ coincides with the associated graded coming from the filtration by Heaviside functions [13].

5. Equation (2.7) also lifts to a graded S_{n+1}-module isomorphism [14]:
$$\mathfrak{gr}(H^0\, \mathcal{V}_{n+1}^1) \cong_{S_{n+1}} H^*(\mathcal{V}_{n+1}^3), \tag{2.8}$$
where again (2.8) comes from a $U(1)$-action on $\mathcal{V}_{n+1}^3$ and subsequent computation of $H^*_{U(1)}\, \mathcal{V}_{n+1}^3$. The grading on the left-hand-side also coincides with the associated graded coming from the filtration by cyclic Heaviside functions.

Our goal is to find a family of B_n-representations exhibiting analogs of these properties.

[4]In fact (2.4) holds for any $d \geq 3$ and odd by replacing $H^{2k} \operatorname{Conf}_n(\mathbb{R}^d)$ with $H^{(d-1)k} \operatorname{Conf}_n(\mathbb{R}^d)$.

3 The Mantaci–Reutenauer algebra and idempotents

We will begin our Type B story by introducing the family of B_n-representations arising in a generalization $\Sigma[S_n]$. Perhaps the most obvious generalization of the Type A descent algebra is the Type B descent algebra, with Coxeter length used to describe $\mathrm{Des}(\sigma)$. However, it turns out that the corresponding Eulerian representations of B_n (studied by the author in [3] for instance) do *not* lift to B_{n+1}!

Instead, we will work in the *Type B Mantaci–Reutenauer algebra* introduced in [11] and defined as follows. Consider $\sigma = (\sigma_1, \ldots, \sigma_n) \in B_n$ to be a signed permutation, meaning that $\sigma_i \in \{-n, \ldots, -1, 1 \cdots, n\}$. The *Mantaci–Reutenauer descents* of σ is the set

$$\mathrm{MRDes}(\sigma) := \left\{ i \in [n-1] : \begin{array}{l} |\sigma_i| > |\sigma_{i+1}| \text{ and } \sigma_i \text{ and } \sigma_{i+1} \text{ have the same sign or} \\ \sigma_i \text{ and } \sigma_{i+1} \text{ have opposite signs} \end{array} \right\}.$$

Note that $\mathrm{MRDes}(\sigma)$ partitions σ into $|\,\mathrm{MRDes}(\sigma)| + 1$ ordered blocks between each descent. Let $[n]^{\pm} := \{1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n}\}$. A *signed composition* of n is a sequence $(a_1, \ldots, a_\ell)$ where $a_i \in [n]^{\pm}$ and $|a_1| + \cdots + |a_\ell| = n$. (Here $|\overline{j}| = j$.) Denote by $\mathrm{sh}(\sigma)$ the signed composition of n obtained from $\mathrm{MRDes}(\sigma)$, where each block $\{\sigma_i, \ldots, \sigma_{i+m}\}$ contributes an $m+1$ to $\mathrm{sh}(\sigma)$ if each σ_i is positive and an $\overline{m+1}$ if σ_i is negative.

Example 5. If $\sigma = (3, 4, -1, -5, -2)$, then $\mathrm{MRDes}(\sigma) = \{2, 4\}$, which partitions σ into ordered blocks $(\{3, 4\}, \{-1, -5\}, \{-2\})$. Therefore $\mathrm{sh}(\sigma) = (2, \overline{2}, \overline{1})$.

The *Mantaci–Reutenauer algebra* is the algebra $\Sigma'[B_n]$ generated by $x_\alpha \in \mathbb{Q}[B_n]$ where

$$x_\alpha := \sum_{\substack{\sigma \in B_n \\ \mathrm{sh}(\sigma) = \alpha}} \sigma.$$

Within $\Sigma'[B_n]$ is a family of complete, primitive and orthogonal idempotents[5] $\mathfrak{g}_{(\lambda^+, \lambda^-)}$ introduced by Vazirani in [20], where λ^+, λ^- are integer partitions with $|\lambda^+| + |\lambda^-| = n$.

The analog of the Eulerian idempotents will come from summing over these $\mathfrak{g}_{(\lambda^+, \lambda^-)}$:

$$\mathfrak{g}_k := \sum_{\substack{(\lambda^+, \lambda^-) \\ \ell(\lambda^+) = k}} \mathfrak{g}_{(\lambda^+, \lambda^-)}, \tag{3.1}$$

and the analog of the Eulerian representations is precisely $G_n^{(k)} := \mathfrak{g}_k \, \mathbb{Q}[B_n]$ for $0 \leq k \leq n$.

The above analogies are quite natural in the following sense.[6] Let $\tau\colon B_n \to S_n$ be the projection which forgets the signs of $\sigma \in B_n$. In [1] Aguiar–Bergeron–Nyman study the properties of τ and show that it extends to a surjective algebra homomorphism $\tau\colon \Sigma'[B_n] \to \Sigma[S_n]$. This homomorphism then relates the $\mathfrak{g}_k$ to the $\mathfrak{e}_k$:

Proposition 6. *We have* $\tau(\mathfrak{g}_0) = 0$ *and for* $1 \leq k \leq n$, $\tau(\mathfrak{g}_k) = \mathfrak{e}_{k-1}$.

[5] As in the case of the $\mathfrak{e}_\lambda$, the definition of these idempotents is technical and therefore omitted; see [20].

[6] The author is grateful to M. Aguiar for suggesting this line of inquiry.

4 Topology in Type B

In contrast to Type A, in Type B it is more natural to begin with the topology of the "hidden" action spaces analogous to $\mathcal{V}^1_{n+1}$ and $\mathcal{V}^3_{n+1}$. Recall that the antipodal map acts on SU_2 (*e.g.* $\mathbb{S}^3$), and $U(1)$ (*e.g.* $\mathbb{S}^1$) by -1. One then has two $\mathbb{Z}_2$-orbit configuration spaces

$$\mathrm{Conf}^{\mathbb{Z}_2}_{n+1}(U(1)) := \{(p_1,\dots,p_{n+1}) \in U(1)^n : p_i \neq \pm p_j\},$$
$$\mathrm{Conf}^{\mathbb{Z}_2}_{n+1}(SU_2) := \{(p_1,\dots,p_{n+1}) \in SU_2^n : p_i \neq \pm p_j\},$$

and corresponding quotients by the diagonal action of $U(1)$ and SU_2, respectively:

$$\mathcal{Y}^1_{n+1} := \mathrm{Conf}^{\mathbb{Z}_2}_{n+1}(U(1))/U(1), \qquad \mathcal{Y}^3_{n+1} := \mathrm{Conf}^{\mathbb{Z}_2}_{n+1}(SU_2)/SU_2. \tag{4.1}$$

4.1 Signed cyclic Heaviside functions and the $d = 1$ case

In direct analogy with Type A, the space $\mathcal{Y}^1_{n+1}$ is comprised of $2^n n!$ contractible pieces, each of which is parametrized by arrangements of $p_1, \cdots, p_{n+1}$ *and* $-p_1, \dots, -p_{n+1}$ on $U(1)$, where we require that each p_i be opposite its antipode $-p_i$. Given a point $\vec{p} = (p_1,\dots,p_{n+1}) \in \mathcal{Y}^1_{n+1}$, write $C(\vec{p}) = C(p_1,\dots,p_{n+1})$ as its arrangement *with* antipodes on $U(1)$ and $-p_i$ as $p_{\bar{i}}$. By convention $\bar{\bar{i}} = i$.

We define *signed cyclic Heaviside functions* y_{ijk} for distinct $i,j,k \in [n+1]^{\pm}$ as

$$y_{ijk}(\vec{p}) := \begin{cases} 1 & p_i < p_j < p_k \text{ counter-clockwise in } C(\vec{p}), \\ 0 & \text{otherwise.} \end{cases}$$

Once again, the y_{ijk} form a $\mathbb{Z}$-algebra with multiplication given by

$$y_{ijk} \cdot y_{qrs}(\vec{p}) := \begin{cases} 1 & p_i < p_j < p_k \text{ and } p_q < p_r < p_s \text{ counter-clockwise in } C(\vec{p}), \\ 0 & \text{otherwise.} \end{cases}$$

Analyzing the combinatorial properties of the y_{ijk} (and employing a standard Gröbner basis argument) allows one to determine a presentation for $H^0(\mathcal{Y}^1_{n+1})$.

Theorem 7. *The ring $H^0(\mathcal{Y}^1_{n+1})$ has presentation $\mathbb{Z}[y_{ijk}]/\mathcal{I}'$ for distinct $i,j,k \in [n+1]^{\pm}$, where $\mathcal{I}'$ is generated by the relations*

$$(i)\ \ y_{ijk}^2 = y_{ijk}, \qquad (ii)\ \ y_{ijk} = 1 - y_{jik}, \qquad (iii)\ \ y_{\bar{i}\,\bar{j}\,k} = y_{i\,j\,\bar{k}},$$

$$(iv)\ \ y_{ijk} - y_{ij\ell} + y_{ik\ell} - y_{jk\ell} = 0, \qquad (v)\ \ y_{ij\ell}y_{jk\ell}(1 - y_{ik\ell}) + (1 - y_{ij\ell})(1 - y_{jk\ell})y_{ik\ell} = 0.$$

Note that although the generators y_{ijk} are now indexed by $[n+1]^{\pm}$, the only new relation needed for $H^0\,\mathcal{Y}^1_{n+1}$ compared to $H^0\,\mathcal{V}^1_{n+1}$ is relation (iii). Like $H^0\,\mathcal{V}^1_{n+1}$, there is an ascending filtration on $H^0\,\mathcal{Y}^1_{n+1}$ by degree in the y_{ijk}, and the corresponding associated graded $\mathfrak{gr}(H^0\,\mathcal{Y}^1_{n+1})$ will again play an important role in understanding $H^*(\mathcal{Y}^3_{n+1})$.

In further parallel with Section 2.2, we would like to identify a genuine orbit configuration space of $\mathbb{R}$ (rather than a quotient) which is B_n-equivariantly homeomorphic to $\mathcal{Y}^1_{n+1}$. However, in this context we must be careful about how the antipodal map behaves under stereographic projection $\pi\colon \mathbb{S}^d \to \mathbb{R}^d$. In particular,

$$\pi(-p_i) = \frac{-\pi(p_i)}{|\pi(p_i)|^2} := \varphi(\pi(p_i)).$$

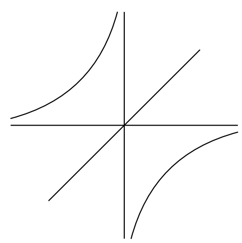
Hence, using the same map as in (2.3), we obtain a B_n-equivariant homeomorphism:

$$f_B\colon\ \mathcal{Y}^1_{n+1} \xrightarrow{\cong} \mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R}\setminus\{0\}), \tag{4.2}$$

where

$$\mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R}^d\setminus\{0\}) := \{(x_1,\dots,x_n) \in (\mathbb{R}^d\setminus\{0\})^n : x_i \neq x_j \neq \varphi(x_j)\}. \tag{4.3}$$

Example 8. The space $\mathrm{Conf}_2^{\langle\varphi\rangle}(\mathbb{R}\setminus\{0\})$ is the complement of the (non-linear!) spaces:



In cohomology, (4.2) induces an isomorphism of B_n-modules which identifies

$$y_{ij(n+1)} \longleftrightarrow \begin{cases} z_{ij} & i \neq \bar{j} \\ z_j & i = \bar{j}. \end{cases} \tag{4.4}$$

Theorem 7 then determines a presentation for $H^0\,\mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R}\setminus\{0\})$ in terms of z_{ij} and z_i for distinct $i, j \in [n]^{\pm}$; it too has an ascending filtration coming from the degree grading in the z_i and z_{ij}. The z_i variables can be interpreted as Heaviside-like functions where $z_i(\vec{x}) = 1$ if $x_i > 0$ and 0 otherwise. Unfortunately, unlike the u_{ij} in Type A, the z_{ij} have a decidedly more complicated description, which we omit for the sake of brevity.

4.2 The $d = 3$ case

In the case of $H^*(\mathcal{Y}^3_{n+1})$, there is also a simple presentation mirroring that of $H^*(\mathcal{V}^3_{n+1})$.

Theorem 9. *The ring $H^*(\mathcal{Y}^3_{n+1})$ has presentation $\mathbb{Z}[y_{ijk}]/\mathcal{J}'$ for distinct $i, j, k \in [n+1]^{\pm}$, where $\mathcal{J}'$ is generated by the relations*

(i) $y_{ijk}^2 = 0$, (ii) $y_{ijk} = -y_{jik}$, (iii) $y_{\bar{i}\,\bar{j}\,k} = y_{i\,j\,\bar{k}}$,

(iv) $y_{ijk} - y_{ij\ell} + y_{ik\ell} - y_{jk\ell} = 0$, (v) $y_{ij\ell}y_{jk\ell} - y_{ik\ell}y_{ij\ell} - y_{ik\ell}y_{jk\ell} = 0$.

The generators y_{ijk} are of cohomological degree 2, and so Theorem 9 implies that $H^*(\mathcal{Y}^3_{n+1})$ is concentrated in degrees $0, 2, \ldots, 2n$.

We would like to recover from Theorem 7 a presentation[7] for the cohomology of $\mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R}^3 \setminus \{0\})$. As in the $d = 1$ case, there is a B_n-equivariant homeomorphism between $\mathcal{Y}^3_{n+1}$ and $\mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R}^3 \setminus \{0\})$ analogous to (2.3). This again induces a B_n-module isomorphism in cohomology identifying the generator $y_{ij(n+1)}$ with z_{ij} or z_i as in (4.4).

From this identification, one can readily use Theorem 9 to obtain a presentation for $H^* \mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R}^3 \setminus \{0\})$ with respect to z_{ij} and z_i for $i, j \in [n]^{\pm}$.

5 Main results: Type B wishlist realized

We now present an analog of the properties described in Section 2.2.1 for Type B.

Theorem 10.

1. *There is an isomorphism of B_n-representations for $0 \leq k \leq n$:*
$$G_n^{(n-k)} \cong_{B_n} H^{2k} \mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R}^3 \setminus \{0\}), \tag{5.1}$$
and thus the total representation of $H^* \mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R}^3 \setminus \{0\})$ is $\mathbb{Q}[B_n]$.

2. *The representation in (5.1) lifts to B_{n+1}, where it is described by $H^{2k}\, \mathcal{Y}^3_{n+1}$;*

3. *For $0 \leq k \leq n$, there is an isomorphism of B_n-representations:*
$$H^{2k} \mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R}^3 \setminus \{0\}) \cong_{B_n} H^{2(k-1)}(\mathcal{Y}_n) \oplus \left(V \otimes H^{2k}(\mathcal{Y}_n)\right),$$
where $V = \chi^{((n-1,1),0)} \oplus \chi^{((n-1),(1))}$; this notation refers to the fact that irreducible representations of B_n are indexed by partitions (λ^+, λ^-) where $|\lambda^+| + |\lambda^-| = n$; see [17].

4. *The circle group $U(1)$ acts on $\mathbb{R}^3$ by rotation around the x-axis, inducing an action on $\mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R}^3 \setminus \{0\})$. The filtration induced from the $U(1)$-equivariant cohomology implies a* graded *isomorphism of B_n-modules:*
$$\mathfrak{gr}(H^0 \mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R} \setminus \{0\})) \cong_{B_n} H^* \mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R}^3 \setminus \{0\}), \tag{5.2}$$
where $\mathfrak{gr}(H^0 \mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R} \setminus \{0\}))$ coincides with the associated graded coming from the filtration by degree in the variables z_i and z_{ij} for distinct $i, j \in [n]^{\pm}$.

[7] This question was first studied in [7]. However, the presentation given has an error (Lemma 7) which is corrected using the lifted presentation in Theorem 9 and identification of generators in (4.4).

5. *Equation (5.2) also lifts to a graded B_{n+1}-module isomorphism*

$$\mathfrak{gr}(H^0\,\mathcal{Y}^1_{n+1}) \cong_{S_{n+1}} H^*(\mathcal{Y}^3_{n+1}), \tag{5.3}$$

where again (5.3) comes from a $U(1)$-action on $\mathcal{Y}^3_{n+1}$ and subsequent computation of $H^*_{U(1)}\,\mathcal{Y}^3_{n+1}$. Once more $\mathfrak{gr}(H^0\,\mathcal{Y}^1_{n+1})$ coincides with the filtration by degree in the signed cyclic Heaviside functions y_{ijk} for distinct $i,j,k \in [n]^{\pm}$.

Proof idea.

1. The isomorphism (5.1) comes from a combination of character computations of $\mathfrak{g}_{(\lambda^+,\lambda^-)}\,\mathbb{Q}[B_n]$ in [5], adapting techniques in [2, Theorem 9.1], and analyzing a finer (descending) filtration of the ring $H^*\,\mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R}^3\setminus\{0\})$ by degree in the variable z_i for $i \in [n]^{\pm}$.

2. The lift follows by using the B_n-equivariant homeomorphism $\mathrm{Conf}_n^{\langle\varphi\rangle}(\mathbb{R}^3\setminus\{0\}) \cong \mathcal{Y}^3_{n+1}$.

3. The recursion comes from studying the B_n-action on the cohomology induced by the spectral sequence $SU_2\setminus\{\pm p_1, \pm p_2, \ldots, \pm p_n\} \longrightarrow \mathcal{Y}^3_{n+1} \longrightarrow \mathcal{Y}^3_n$.

4. The techniques used to prove (5.2) are adapted from [13, Lemma 4.2].

5. The techniques used to prove (5.3) are adapted from [14, Remark 2.9].

□

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