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Peaks Are Preserved Under Run-Sorting (Extended Abstract)

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Abstract. We study a sorting procedure (run-sorting) on permutations, where runs are rearranged in lexicographic order. We describe a rather surprising bijection on permutations of length n, with the property that it sends the set of peak-values to the set of peak-values after run-sorting. We further show that the descent generating polynomials, $A_n(t)$ for run-sorted permutations, $\mathcal{RSP}(n)$ are real rooted, and satisfy an interlacing property similar to that satisfied by the Eulerian polynomials.

Keywords: run-sorted permutation, peak, real-rootedness, interlacing

1 Introduction

This is an extended abstract of [1]. For a fixed positive integer n, set $[n] := \{1, 2, \ldots, n\}$. We shall use the one-line notation, $\sigma(1)\sigma(2)\cdots\sigma(n)$ to represent a permutation $\sigma \in S_n$. A permutation σ can be decomposed into maximal increasing subsequences called *runs.* For instance, $\sigma = 1289\ 346\ 57$ has three runs namely $1289, 346$ and 57. If the runs of σ are lexicographically ordered, then we say that σ is *a run-sorted permutation* , which constitutes a subset of S_n, the set of all permutations of $\{1, 2, \ldots, n\}$. For example, $\sigma = 128934657$ is a run-sorted permutation over $[9]$ but $\sigma' = 85136472$ is not run-sorted. We let $\mathcal{RSP}(n)$ denote the set of run-sorted permutations of length n. We say that the word $\sigma \in S_n$ has k as a *descent* if $\sigma(k) > \sigma(k+1)$. The *descent set* of σ is denoted by $\mathrm{DES}(\sigma)$ and $\mathrm{des}(\sigma)$ is the cardinality of $\mathrm{DES}(\sigma)$. A *peak* of a permutation $\sigma \in S_n$, is an integer i, $1 < i < n$ such that $\sigma(i-1) < \sigma(i) > \sigma(i+1)$ and the corresponding $\sigma(i)$ is a *peak-value* of σ. Given a permutation σ, we let $\mathrm{runsort}(\sigma)$ denote the permutation obtained by rearranging the runs of σ lexicographically. Hence, if $\sigma \in S_n$, then $\mathrm{runsort}(\sigma) \in \mathcal{RSP}(n)$. We let $\mathrm{PKV}(\sigma)$ denote the set of peak-values of the permutation σ, and $\mathrm{SPV}(\sigma) := \mathrm{PKV}(\mathrm{runsort}(\sigma))$. We define the set of *descent bottoms* of a permutation σ as $\mathrm{DB}(\sigma) := \{\sigma_{i+1} : i \in [n-1] \text{ and } \sigma_i > \sigma_{i+1}\}$ and the set of *left-to-right minima* as $\mathrm{LRMin}(\sigma) := \{\sigma_i : \sigma_i = \min\{\sigma_1, \sigma_2, \ldots, \sigma_i\}\}$. The set of 132-*peak-values* of a

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permutation σ is defined as $\mathrm{PKV}^*(\sigma) := \{\sigma_i : 1 < i < n \text{ and } \sigma_{i-1} < \sigma_{i+1} < \sigma_i\}$. Note that $\mathrm{PKV}^*(\sigma) \subseteq \mathrm{PKV}(\sigma)$.

We present a recursive bijection on permutations, where we keep track of the peak values before and after run-sorting a permutation. In particular, we prove the following main result. We use the notation $\mathbf{x}_S := x_{s_1}x_{s_2}\cdots x_{s_k}$ whenever S is a finite set of positive integers. For example, $\mathbf{x}_{\mathrm{DB}(\pi)}$ is short for $\prod_{j\in\mathrm{DB}(\pi)} x_j$.

Theorem 1.1. *For $n \geq 1$, we have that*

$$\sum_{\pi\in S_n} \mathbf{x}_{\mathrm{DB}(\pi)}\mathbf{y}_{\mathrm{LRMin}(\pi)}\mathbf{z}_{\mathrm{PKV}^*(\pi)}\mathbf{w}_{\mathrm{PKV}(\pi)} = \sum_{\pi\in S_n} \mathbf{x}_{\mathrm{DB}(\pi)}\mathbf{y}_{\mathrm{LRMin}(\pi)}\mathbf{z}_{\mathrm{PKV}^*(\pi)}\mathbf{w}_{\mathrm{SPV}(\pi)}.$$

From this result, we see that this bijection after applying the runsort(σ) function not only preserves peak values, but also other statistics namely; descent bottoms, left-to-right minima and 132-peak-values.

In Section 2, we give the detailed recursive bijection which keeps track of the peak values before applying the runsort function, as well as other statistics such as the descent bottoms, left-to-right minima and 132-peak-values. The main recursions are given in Lemma 2.1 and Theorem 2.3. It is from these recursions that we are able to arrive at Theorem 1.1. In Section 3, we compute the expected number of descents after run-sorting a uniformly random permutation in S_n. In Section 4, we prove that the descent generating polynomials, $A_n(t)$ for run-sorted permutations are real rooted, and give some consequences of this result. We also explore possible multivariate analogs of this statement. For example, in Theorem 4.9 we prove that the multi-variate Eulerian polynomials are same-phase stable, and ask if the multivariate $A_n(\mathbf{x})$ has the same property.

2 The peak-value distribution is preserved under run-sort

Let $\widehat{b}_{n,k}$ denote the number of permutations in S_n with exactly k peaks. In [8, p.24], it was discovered that the numbers $\widehat{b}_{n,k}$ satisfy the recursion

$$\widehat{b}_{n,k} = \begin{cases} 1 & \text{if } n = 1, k = 0, \\ (2k+2)\widehat{b}_{n-1,k} + (n-2(k-1)-2)\widehat{b}_{n-1,k-1} & \text{if } 0 \leq k < \frac{n}{2}, \\ 0 & \text{if } 2k \geq n \text{ or } k < 0. \end{cases} \tag{2.1}$$

and these numbers have been refined in different forms since then. Let us set

$$\widehat{B}_n(t) := \sum_{j\geq 0} \widehat{b}_{n,j} t^j.$$

In [11], it is proved that $\widehat{B}_n(t)$ satisfy the recursion

$$\widehat{B}_n(t) = (2 + t(n-2))\widehat{B}_{n-1}(t) + 2t(1-t)\widehat{B}'_{n-1}(t), \text{ for } n \geq 2 \tag{2.2}$$

and are real-rooted. We generalize this recursion further in in Proposition 2.2 by keeping track of the peak values. This is through a recursive process of constructing a permutation in S_n from a permutation in S_{n-1} by inserting n somewhere. For $\pi \in S_{n-1}$ and $a \in [n-1]$, we let $\mathrm{Stay}_a(\pi)$ denote the permutation obtained from π by inserting n immediately after a. We also let $\mathrm{Stay}_\emptyset(\pi)$ denote the permutation obtained from π by inserting n before π in one-line notation. Hence we have the following lemma.

Lemma 2.1. *For any $n \geq 1$, the bijection*

$$\mathcal{B}\colon \{\emptyset, 1, 2, \ldots, n-1\} \times S_{n-1} \to S_n$$

defined via $\mathcal{B}(a, \pi) := \mathrm{Stay}_a(\pi)$, has the following properties. For simplicity, we set $\pi' := \mathrm{Stay}_a(\pi)$ and we let k be the value immediately succeeding a in π (unless a is the last entry in π).

(1) $a = \emptyset$, so $\mathrm{PKV}(\pi') = \mathrm{PKV}(\pi)$.

(2) a is the last entry of π, so $\mathrm{PKV}(\pi') = \mathrm{PKV}(\pi)$.

(3) $a \in \mathrm{PKV}(\pi)$. Then
$$\mathrm{PKV}(\pi') = (\mathrm{PKV}(\pi) \setminus \{a\}) \cup \{n\}.$$

(4) $k \in \mathrm{PKV}(\pi)$. Then
$$\mathrm{PKV}(\pi') = (\mathrm{PKV}(\pi) \setminus \{k\}) \cup \{n\}.$$

(5) a is not the last entry of π, and neither a or k are in $\mathrm{PKV}(\pi)$. Then

$$\mathrm{PKV}(\pi') = \mathrm{PKV}(\pi) \cup \{n\}.$$

Proof. First note that the map $\mathcal{B}$ is indeed a bijection, as we can easily recover a from π'. Moreover, we have that π is recovered from π' by removing n. The other properties regarding the peaks follow via case-by-case analysis. □

As a corollary of Lemma 2.1, we can now easily deduce (2.1), and prove the following multivariate generalization of (2.2).

Proposition 2.2. *We have that the multivariate polynomials $\widehat{B}_n(\mathbf{x})$ satisfy the recursion*

$$\widehat{B}_n = (2 + (n-2)x_n)\widehat{B}_{n-1} + 2x_n \sum_{j=3}^{n} (1 - x_j) \cdot \partial_{x_j} \widehat{B}_{n-1}. \tag{2.3}$$

2.1 A recursion which tracks peak-values after run-sort

In Lemma 2.1, we have recursively constructed permutations, while tracking the peak-values. We now want to track the peak-values after applying the run-sort function. Constructing this bijection is rather complicated, with several cases. As in Lemma 2.1, we still keep track of the position of insertion of n. Let $\mathrm{SPV}(\pi) := \mathrm{PKV}(\mathrm{runsort}(\pi))$ be the set of peak-values obtained after run-sorting π. In this subsection, we shall mainly consider the runs of $\pi \in S_{n-1}$ arranged in lexicographical order. The following result is now an analog of Lemma 2.1, and is the main result in this section.

Theorem 2.3. *For any $n \geq 1$, there is a bijection*

$$\mathcal{C}\colon \{\emptyset, 1, 2, \dots, n-1\} \times S_{n-1} \to S_n$$

which has the following properties. For simplicity, we set $\pi' := \mathcal{C}(a, \pi)$ and we let k be the value immediately succeeding a in $\mathrm{runsort}(\pi)$, unless a is the last entry in $\mathrm{runsort}(\pi)$:

(1) $a = \emptyset$, and $\mathrm{SPV}(\pi') = \mathrm{SPV}(\pi)$.

(2) a is the last entry of $\mathrm{runsort}(\pi)$, and $\mathrm{SPV}(\pi') = \mathrm{SPV}(\pi)$.

(3) $a \in \mathrm{SPV}(\pi)$. Then
$$\mathrm{SPV}(\pi') = (\mathrm{SPV}(\pi) \setminus \{a\}) \cup \{n\}.$$

(4) $k \in \mathrm{SPV}(\pi)$. Then
$$\mathrm{SPV}(\pi') = (\mathrm{SPV}(\pi) \setminus \{k\}) \cup \{n\}.$$

(5) a is not the last entry of $\mathrm{runsort}(\pi)$, and neither a or k are in $\mathrm{SPV}(\pi)$. Then
$$\mathrm{SPV}(\pi') = \mathrm{SPV}(\pi) \cup \{n\}.$$

A few words about the proof, as it is too intricate to describe in detail here. Cases 1, 2 and 3 are easy to handle, *i.e.*, by simply using Stay_a on the permutation. In the remaining cases, we (greedily) apply Stay_a whenever it has the desired property. Otherwise, we first need to perform some additional modification of the permutation, and carefully track in what situations the Stay_a map does not act as wanted. This leads to a handful of involutions, constructed in such a way that the total net effect of "bad things" cancel.

Since the recursions in Lemma 2.1 and Theorem 2.3 have the same structure, this allows us to construct an implicit bijection,

$$\eta\colon S_n \to S_n,$$

such that $\mathrm{PKV}(\sigma) = \mathrm{PKV}(\mathrm{runsort}(\eta(\sigma))$. With careful analysis of η (essentially, examining the five cases for each of $\mathcal{B}$ and $\mathcal{C}$), one can actually deduce that η preserves several other combinatorial statistics. This leads us to the following theorem which is the main result in this section.

Theorem 2.4. *For $n \geq 1$, we have that*

$$\sum_{\pi \in S_n} \mathbf{x}_{\mathrm{DB}(\pi)} \mathbf{y}_{\mathrm{LRMin}(\pi)} \mathbf{z}_{\mathrm{PKV}^*(\pi)} \mathbf{w}_{\mathrm{PKV}(\pi)} = \sum_{\pi \in S_n} \mathbf{x}_{\mathrm{DB}(\pi)} \mathbf{y}_{\mathrm{LRMin}(\pi)} \mathbf{z}_{\mathrm{PKV}^*(\pi)} \mathbf{w}_{\mathrm{SPV}(\pi)}.$$

Interestingly, the runsort(π) function seems to be attracting a lot of attention, with more researchers extending it to find more properties of permutations. For example, Coopman and Rubey, see [5] have recently proved that the number of inversions among permutations is equi-distributed with the number of occurrences of the vincular pattern $13-2$ after sorting the set of runs.

3 Probabilistic statements

It is a natural question for one to ask about the expected number of descents after applying the runsort on a uniformly random permutation in S_n. That hence leads us to the following theorem.

Theorem 3.1 (See also [11, p.110])**.** *Let $\sigma \in S_n$, with $n \geq 2$ be uniformly chosen. Then*

$$\mathbb{E}[\mathrm{des}(\mathrm{runsort}(\sigma))] = \mathbb{E}[\mathrm{peaks}(\mathrm{runsort}(\sigma))] = \mathbb{E}[\mathrm{peaks}(\sigma)] = (n-2)/3.$$

Proof. The first equality follows from the fact that every descent in a run-sorted permutation necessarily associated with a peak. The second identity follows immediately from Theorem 2.4, so it suffices to compute the expected number of peaks in a permutation. □

The runsort function led us to the question below.

Question 3.2. *Let $\sigma \in S_n$ be a uniformly chosen permutation, and let $\sigma' :=$ runsort(σ). Rescaling the permutation matrix with entries equal to 1 at (i, σ'_i), $i \in [n]$ gives a distinctive curve as seen in Figure 1, where a distinctive curve is seen. As $n \to \infty$ does this curve approach some limit curve?*

Fortunately, this question has recently been answered in the affirmative by Alon, Defant and Kravitz, see [2].

4 Realrootedness and interlacing roots

We now turn to proving real-rootedness and interlacing properties of some polynomials related to peaks and descents. We first need the following definitions.

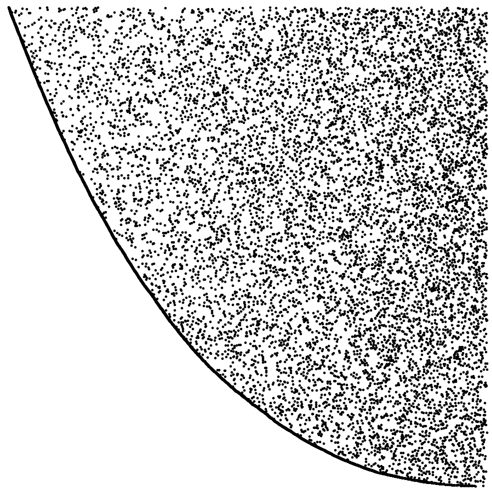

Figure 1: A random permutation matrix σ' after lexsort, for $n = 20000$. The entries equal to 1 are shaded black.

Definition 4.1 (see [10]). Let g be a polynomial of degree n with non-positive roots $g_1 \leq g_2 \leq \cdots \leq g_n$. If f is a degree $n-1$ polynomial with non-positive roots $f_1 \leq f_2 \leq \cdots \leq f_{n-1}$, we say that the roots of f *interlace* those of g, if

$$g_1 \leq f_1 \leq g_2 \leq f_2 \leq \cdots \leq f_{n-1} \leq g_n \leq 0.$$

Moreover, we say that f *alternates left of* g if $\deg(f) = \deg(g) = d$ and

$$f_1 \leq g_1 \leq f_2 \leq \cdots \leq f_d \leq g_d.$$

We say that f *interleaves* g if either f interlaces g or f alternates left of g. We write this as $f \ll g$.

Below in Theorem 4.4, we show that the polynomials

$$A_n(t) := \sum_{\sigma \in \mathcal{RSP}(n)} t^{\text{des}(\sigma)}$$

are real-rooted. Moreover, the roots of $A_{n-1}(t)$ *interlace* the roots of $A_n(t)$.

We let $f_{n,k}$ be the number of run-sorted permutations of $[n]$ having k runs. In [9], it was proved that the numbers $f_{n,k}$ satisfy the recurrence relation

$$f_{n,k} = k f_{n-1,k} + (n-2) f_{n-2,k-1} \text{ whenever } 1 \leq k < n. \tag{4.1}$$

Hence we have that

$$tA_n(t) = \sum_{\pi \in \mathcal{RSP}(n)} t^{\text{des}(\pi)+1} = \sum_{k \geq 1} t^k f_{n,k}. \tag{4.2}$$

From (4.2), let us set $R_n(t) := tA_n(t)$.

Lemma 4.2. *$R_n(t)$ satisfies the recurrence*

$$R_n(t) = tR'_{n-1}(t) + t(n-2)R_{n-2}(t), R_1(t) = R_2(t) = t. \tag{4.3}$$

Proof. By (4.1), we have that

$$R_n(t) = \sum_k t^k\left(kf_{n-1,k} + (n-2)f_{n-2,k-1}\right) = t\sum_k kt^{k-1}f_{n-1,k} + t(n-2)\sum_{k-1} t^{k-1}f_{n-2,k-1}.$$

This is now recognized as (4.3). □

From (4.3), we then prove Theorem 4.4 using the lemma below as a main tool.

Lemma 4.3 (See D.Wagner, [10, Sec. 3])**.** *Let $f, g, h \in \mathbb{R}[t]$ be real-rooted polynomials with only real, non-positive roots and positive leading coefficients. Then*

(i) if $f \ll h$ and $g \ll h$ then $f + g \ll h$.

(ii) if $h \ll f$ and $h \ll g$ then $h \ll f + g$.

(iii) $g \ll f$ if and only if $f \ll tg$.

Theorem 4.4. *The polynomials*

$$R_n(t) = \sum_{\pi \in \mathcal{RSP}(n)} t^{\mathrm{des}(\pi)+1}$$

satisfy $R_{n-1} \ll R_n$ for all $n \geq 1$. In particular, they are all real-rooted.

Proof. For $n = 1$, $R_0 \ll R_1$. By induction over n, we fix $n \geq 2$ and assume that we have $R_{n-2} \ll R_{n-1}$. It suffices to prove that $R_{n-1} \ll R_n$. By Rolle's theorem, we have that R'_{n-1} interlaces R_{n-1} *i.e.*, $R'_{n-1} \ll R_n$. Using Lemma 4.3, we have that $R_{n-1} \ll tR'_{n-1} + t(n-2)R_{n-2}$, and by using (4.3), we conclude that $R_{n-1} \ll R_n$, which in turn implies that $A_{n-1} \ll A_n$, hence real-rootedness. for all $n > 1$. □

Finally, we end with a recursion for a multivariate extension of $A_n(t)$.

Theorem 4.5. *For all integers $n \geq 1$, let*

$$A_n(\mathbf{x}) := \sum_{\pi \in \mathcal{RSP}(n)} \prod_{j \in \mathrm{DES}(\pi)} x_{n-j}.$$

Then

$$A_n(\mathbf{x}) = 1 + \sum_{i=1}^{n-2}\left(\binom{n-1}{i} - 1\right)x_i A_i(\mathbf{x}) \tag{4.4}$$

and

$$\sum_{n \geq 0} A_{n+1}(t)\frac{u^n}{n!} = \exp\left[u + t(e^u - u - 1)\right]. \tag{4.5}$$

Note the indexing of the descent set from the end.

In particular, we solve an open problem given in [9], where an explicit formula for function for the exponential generating function of the $A_n(t)$ was asked for.

Below, we illustrate $A_5(\mathbf{x})$ where $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$ for all $\pi \in \mathcal{RSP}(5)$, which keeps track of the descent set *by indexing from the end*. We find that $A_5(\mathbf{x}) = 1 + 3x_3 + 5x_2 + 3x_1 + 3x_3x_1$.

1	12345
x_3	13245, 14235, 15234
x_2	12435, 12534, 13425, 13524, 14523
x_1	12354, 12453, 13452
x_3x_1	13254, 14253, 15243

4.1 Multivariate Eulerian polynomials

We noted that the $A_n(t)$ are similar to the Eulerian polynomials, and that the $A_n(t)$ are real-rooted. In this subsection, we consider the multivariate generalization $A_n(\mathbf{x})$, as well as the corresponding multivariate Eulerian polynomial $E_n(\mathbf{x})$. We manage to show that $E_n(\mathbf{x})$ satisfies a multivariate analog of real-rootedness, and we conjecture that $A_n(\mathbf{x})$ does too.

Definition 4.6. A multivariate polynomial $P \in \mathbb{C}[z_1, \dots, z_n]$ is called *stable* if it does not vanish on $\mathcal{H}^n$, where $\mathcal{H} \subset \mathbb{C}$ denote the upper half-plane $\{z \in \mathbb{C} : \mathrm{im}(z) > 0\}$.

The multivariate polynomials

$$\widetilde{E}_n(\mathbf{x}) := \sum_{\pi \in S_n} \prod_{\pi_j > \pi_{j+1}} x_{\pi_j}$$

are proved to be stable, see [6, Theorem 2.5], and [4]. It is worth noting that we can recover the classical Eulerian polynomials by setting $x_i \to t$ which in turn implies that the Eulerian polynomials are real-rooted.

We now describe a weaker notion of stability introduced in [7] below.

Definition 4.7. A polynomial $p(x_1, \dots, x_n) \in \mathbb{R}[x_1, \dots, x_n]$ is said to be *same-phase stable* if for every $\lambda \in \mathbb{R}^n_+$, we have that the univariate polynomial $p(\lambda_1 t, \lambda_2 t, \dots, \lambda_n t) \in \mathbb{R}[t]$ is real-rooted.

Definition 4.8. A sequence $\{f_1, f_2, \dots, f_n\}$ of polynomials with positive leading coefficients is said to be an *interlacing sequence* [1] if $f_i \ll f_j$ for all $1 \leq i < j \leq n$. We let $\mathcal{F}^+_n$ denote the set of all such interlacing sequences.

[1] Interleaving sequence would be a better name, but here we follow [3].

Let $E_n(\mathbf{x})$ be the *multivariate Eulerian polynomial*

$$E_n(\mathbf{x}) := \sum_{\pi \in S_n} \mathbf{x}_{\mathrm{DES}(\pi)}.$$

However, the $E_n(\mathbf{x})$ are not stable—we the following counter-example[2]. The polynomial

$$\begin{aligned} E_5(\mathbf{x}) = {} & 6x_2x_1 + 4x_2x_3x_1 + 16x_3x_1 + 9x_2x_4x_1 \\ & + x_2x_3x_4x_1 + 9x_3x_4x_1 + 11x_4x_1 \\ & + 4x_1 + 9x_2 + 11x_2x_3 + 9x_3 + 16x_2x_4 \\ & + 4x_2x_3x_4 + 6x_3x_4 + 4x_4 + 1 \end{aligned}$$

vanishes at

$$x_1 = -\frac{39}{16} + \frac{7i}{512},\ x_2 = -16 + i,\ x_3 = i,\ x_4 = \frac{-6523999 + 73341i}{5671874}.$$

However, we shall now show that the polynomials $E_n(\mathbf{x})$ are *same-phase stable* and satisfy a type of interlacing.

Theorem 4.9. *Let $\lambda_1, \lambda_2, \ldots$ be a fixed sequence of positive real numbers. Then for all $n \geq 1$,*

$$E_{n-1}(\lambda_1 t, \lambda_2 t, \ldots, \lambda_n t) \ll E_n(\lambda_1 t, \lambda_2 t, \ldots, \lambda_n t).$$

Proof. We refine the polynomial $E_n(\mathbf{x})$ by introducing

$$E_n^i(\mathbf{x}) := \sum_{\substack{\pi \in S_n \\ \pi(n) = i}} \mathbf{x}_{\mathrm{DES}(\pi)}.$$

We have that $E_{n-1}(\mathbf{x}) = E_n^n(\mathbf{x})$, since removing the last entry (which is n) in a permutation counted by $E_n^n(\mathbf{x})$, gives a bijection with elements counted by $E_{n-1}(\mathbf{x})$. To be more detailed, we set

$$v_n^i(t) := E_n^i(\lambda_1 t, \lambda_2 t, \ldots, \lambda_n t), \qquad v_n(t) := E_n(\lambda_1 t, \lambda_2 t, \ldots, \lambda_n t).$$

Next we prove that $\mathcal{V}_n := \{v_n^{n-i}\}_{i=1}^n$ is an interlacing sequence mimicking the approach used in [3, Example 7.8.8], and note that by conditioning on $\pi(n-1) = k$, we have

$$v_{n+1}^i(t) = \sum_{k \geq i} \lambda_n t \cdot v_n^k(t) + \sum_{k < i} v_n^k(t). \tag{4.6}$$

[2] We found this by simply using the Mathematica command `FindInstance`.

Equation (4.6) can be re-written as

$$\begin{bmatrix} v_{n+1}^{n+1} \\ v_{n+1}^{n} \\ v_{n+1}^{n-1} \\ \vdots \\ v_{n+1}^{3} \\ v_{n+1}^{2} \\ v_{n+1}^{1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ \lambda_n t & 1 & 1 & \dots & 1 & 1 \\ \lambda_n t & \lambda_n t & 1 & \dots & 1 & 1 \\ \lambda_n t & \lambda_n t & \lambda_n t & \dots & 1 & 1 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ \lambda_n t & \lambda_n t & \dots & \lambda_n t & \lambda_n t & 1 \\ \lambda_n t & \lambda_n t & \dots & \lambda_n t & \lambda_n t & \lambda_n t \end{bmatrix} \begin{bmatrix} v_n^n \\ v_n^{n-1} \\ \vdots \\ v_n^3 \\ v_n^2 \\ v_n^1 \end{bmatrix} \tag{4.7}$$

where we denote the big matrix by $G_n \in \mathbb{R}^{(n+1)\times n}$. Using [3, Theorem 7.8.5] allows us to easily verify that G_n maps $\mathcal{F}_n^+$ to $\mathcal{F}_{n+1}^+$. Hence by induction, it follows that $\mathcal{V}_n$ is an interlacing sequence for all n.

We now have that

$$v_{n-1}(t) = v_n^n(t) \text{ and } v_n(t) = v_n^1(t) + \cdots + v_n^n(t).$$

Since $v_{n-1}(t)$ interleaves all polynomials in $\{v_n^i(t)\}_{i=1}^n$, it must also interleave the sum, so $v_{n-1} \ll v_n(t)$, and we are done. □

Conjecture 4.10. *Let $A_n(\mathbf{x})$ be as in (4.4). Then $A_n(\mathbf{x})$ is same-phase stable, and for all $n \geq 1$, we have that*

$$A_{n-1}(\lambda_1 t, \lambda_2 t, \dots, \lambda_{n-1} t) \ll A_n(\lambda_1 t, \lambda_2 t, \dots, \lambda_n t),$$

whenever $\lambda_1, \lambda_2, \dots$ is a fixed sequence of positive real numbers.

As mentioned earlier, it has been shown that the $\widehat{B}_n(t)$ defined in (2.2) are real-rooted. Through computer experiments with mathematica, we were able to come up with the following conjecture.

Conjecture 4.11. *The polynomials $\widehat{B}_n(\mathbf{x}) = \sum_{\pi \in S_n} \mathbf{x}_{\mathrm{PKV}(\pi)}$, are all stable. Furthermore, if $\lambda_1, \lambda_2, \dots$ is a fixed sequence of positive real numbers, then*

$$\widehat{B}_{n-1}(\lambda_1 t, \lambda_2 t, \dots, \lambda_{n-1} t) \ll \widehat{B}_n(\lambda_1 t, \lambda_2 t, \dots, \lambda_n t) \text{ for all } n > 1.$$

We simply randomly generate positive real numbers λ_i, and verify (symbolically) that the polynomials are real-rooted. The interlacing property can then be checked numerically. We then performed several hundreds of such Monte-Carlo checks for each $n \leq 10$.

One possible approach to solve this conjecture is to use the multivariate recursion in Proposition 2.2, and use some sort of multivariate analog of Lemma 4.3.

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