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Equidistributions Around Special Kinds of Descents and Excedances Via Continued Fractions

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Abstract. We consider a sequence of four variable polynomials by refining Stieltjes' continued fraction for Eulerian polynomials. Using the combinatorial theory of Jacobi-type continued fractions and bijections we derive various combinatorial interpretations in terms of permutation statistics for these polynomials, which include special kinds of descents and excedances in a recent paper of Baril and Kirgizov. As a by-product, we derive several equidistribution results for permutation statistics, which enables us to confirm and strengthen a recent conjecture of Vajnovszki and also to obtain several companion permutation statistics for two bistatistics in a conjecture of Baril and Kirgizov.

Keywords: Eulerian polynomials, bijection, permutation statistic, equidistribution, cycle, continued fraction, descent, excedance, drop, derangement, gamma-positivity

1 Introduction

It is well-known [6, 20, 16] that the statistics "des" and "exc" are equidistributed over permutations of $[n] := \{1, \ldots, n\}$, their common generating function being the Eulerian polynomials $A_n(t)$, *i.e.*,

$$A_n(t) = \sum_{\sigma\in\mathfrak{S}_n} t^{\operatorname{des}\sigma} = \sum_{\sigma\in\mathfrak{S}_n} t^{\operatorname{exc}\sigma},$$

which satisfy the identity

$$\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{r=0}^{\infty} t^r (r+1)^n.$$

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Since MacMahon's pioneering work [14] various combinatorial variants and refinements of Eulerian polynomials have appeared, see [2, 8, 9, 13, 15, 19] for some recent papers.

In a recent paper [1] Baril and Kirgizov considered some special descents, excedances and cycles of permutations, that we recall in the following. For a permutation $\sigma := \sigma(1)\sigma(2)\cdots\sigma(n)$ of $1\ldots n$, an index $i \in [1, n-1]$ is called a

- *descent* (resp. *excedance*) if $\sigma(i) > \sigma(i+1)$ (resp. $\sigma(i) > i$);
- *descent of type 2* if i is a descent and $\sigma(j) < \sigma(i)$ for $j < i$;
- *pure excedance* if i is an excedance and $\sigma(j) \notin [i, \sigma(i)]$ for $j < i$;

and an index $i \in [2, n]$ is called a

- *drop* if $i > \sigma(i)$;
- *pure drop* if i is a drop and $\sigma(j) \notin [\sigma(i), i]$ for $j > i$.

Let des σ (resp. exc σ, drop σ, $\mathrm{des}_2\ \sigma$, pex σ and pdrop σ) denote the number of descents (resp. excedances, drops, descents of type 2, pure excedances and pure drops) of σ. Identifying σ with the bijection $i \mapsto \sigma(i)$ on $[n]$ we can decompose σ into disjoint cycles $(i, \sigma(i), \ldots, \sigma^{\ell}(i))$ with $\sigma^{\ell+1}(i) = i$ and $i \in [n]$. A cycle with $\ell = 1$ is called a *fixed point* of σ. Let cyc σ (resp. fix σ) denote the number of cycles (resp. fixed points) of σ. The number of non trivial cycles of σ [18, A136394] is defined by

$$\mathrm{pcyc}\ \sigma = \mathrm{cyc}\ \sigma - \mathrm{fix}\ \sigma. \tag{1.1}$$

For example, if $n = 8$ and $\sigma = 2\,3\,1\,4\,6\,8\,7\,5$, the descent indexes of type 2 are $\{2, 6\}$; the pure excedance indexes are $\{1, 5\}$ and the pure drop indexes are $\{3, 8\}$. Thus $\mathrm{des}_2\ \sigma = 2$, pex $\sigma = 2$, and pdrop $\sigma = 2$. Factorizing σ as product of disjoint cycles $(1\,2\,3)(4)(5\,6\,8)(7)$, we derive cyc $\sigma = 4$, fix $\sigma = 2$, and pcyc $\sigma = 2$.

A *mesh pattern* of length k is a pair (τ, R), where τ is a permutation of length k and R is a subset of $[\![0,k]\!] \times [\![0,k]\!]$ with $[\![0,k]\!] = \{0, 1, \ldots, k\}$. Let (i, j) denote the box whose corners have coordinates $(i, j), (i, j+1), (i+1, j+1)$, and $(i+1, j)$. Note that a descent of type 2 can be viewed as an occurrence of the mesh pattern $(21, L_1)$ where $L_1 = \{1\} \times [0, 2] \cup \{(0, 2)\}$. By abuse of notation, we use des_2 to denote the mesh pattern corresponding to an occurrence of descent of type 2 in Figure 1. Similarly, we use pex (resp. pdrop) to denote an occurrence of pure excedance in Figure 1 although pex (resp. pdrop) is not a mesh pattern. See [1, 11] for further information about mesh patterns.

Recently Baril and Kirgizov [1] proved the equidistribution of the statistics "des_2", "pex" and "pcyc" over $\mathfrak{S}_n$ by bijections and conclude their paper with the following two conjectures on the equidistribution of two pairs of bistatistics.

Conjecture 1 (Baril and Kirgizov)**.** *The two bistatistics* $(\mathrm{des}_2, \mathrm{cyc})$ *and* $(\mathrm{pex}, \mathrm{cyc})$ *are equidistributed on* $\mathfrak{S}_n$.

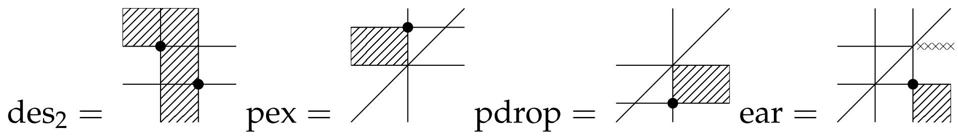

Figure 1: Illustration of the mesh patterns des_2 and pex and ear, where the cross line means that the value cannot be in the segment of the horizontal line

Conjecture 2 (Vajnovszki)**.** *The two bistatistics* $(\mathrm{des}_2, \mathrm{des})$ *and* $(\mathrm{pex}, \mathrm{exc})$ *are equidistributed on* $\mathfrak{S}_n$*.*

In this paper we shall take a different approach to their problems through the combinatorial theory of J-continued fractions developed by Flajolet and Viennot in the 1980s [7, 5], see [2, 4, 9, 19] for recent developments of this theory. Recall that a J-type continued fraction is a formal power series defined by

$$\sum_{n=0}^{\infty} a_n z^n = \cfrac{1}{1-\gamma_0 z - \cfrac{\beta_1 z^2}{1-\gamma_1 z - \cfrac{\beta_2 z^2}{\cdots}}},$$

where $(\gamma_n)_{n\geq 0}$ and $(\beta_n)_{n\geq 1}$ are two sequences in some commutative ring.

Define the polynomials $A_n(t,\lambda,y,w)$ by the J-fraction

$$\sum_{n\geq 0} z^n A_n(t,\lambda,y,w) \;=\; \cfrac{1}{1-wz-\cfrac{t\lambda y\, z^2}{1-(w+t+1)z-\cfrac{t(\lambda+1)(y+1)\,z^2}{\cdots}}} \tag{1.2}$$

with $\gamma_n = w + n(t+1)$ and $\beta_n = t(\lambda+n-1)(y+n-1)$.

It is known that $A_n(t,1,1,1)$ equals the Eulerian polynomial $A_n(t)$, see [9, 19]. Recently Sokal and the third author [19] have generalized the J-fraction for Eulerian polynomials in infinitely many intermediates, which are also generalizations of the polynomials $A_n(t,\lambda,y,w)$. The aim of this paper is to generalize the results in [1] by exploring the combinatorial interpretations of the polynomials $A_n(t,\lambda,y,w)$ in light of the aforementioned statistics. In particular, we confirm and strengthen Conjecture 2 (see Corollary 2) and obtain five equidistributed companions of the bistatistic $(\mathrm{pex}, \mathrm{cyc})$ in Conjecture 1 (see Theorem 3). This extended abstract is a summary of the recent paper [10].

2 Main results

For $\sigma \in \mathfrak{S}_n$, an index $i \in [n]$ is called (see [19]) a

- *cycle peak* (cpeak) if $\sigma^{-1}(i) < i > \sigma(i)$;
- *cycle valley* (cval) if $\sigma^{-1}(i) > i < \sigma(i)$;
- *cycle double rise* (cdrise) if $\sigma^{-1}(i) < i < \sigma(i)$;
- *cycle double fall* (cdfall) if $\sigma^{-1}(i) > i > \sigma(i)$;
- *fixed point* (fix) if $\sigma^{-1}(i) = i = \sigma(i)$.

Clearly every index i belongs to exactly one of these five types; we refer to this classification as the *cycle classification*. Next, an index $i \in [n]$ (or a value $\sigma(i)$) is called a

- *record* (rec) (or *left-to-right maximum*) if $\sigma(j) < \sigma(i)$ for all $j < i$ (the index 1 is always a record];
- *antirecord* (arec) (or *right-to-left minimum*) if $\sigma(j) > \sigma(i)$ for all $j > i$ (the index n is always an antirecord);
- *exclusive record* (erec) if it is a record and not also an antirecord;
- *exclusive antirecord* (earec) if it is an antirecord and not also a record.
- *exclusive antirecord cycle peak* (eareccpeak) if i is an exclusive antirecord and also a cycle peak.

The statistic eareccpeak was introduced in [19], in this paper we adopt the following concise notation instead

$$\mathrm{ear} := \mathrm{eareccpeak}\,. \tag{2.1}$$

An illustration of the pattern ear is given in Figure 1. Also, we shall denote the set of indexes of each type by capitalizing the first letter of type name. Hence $\mathrm{Cpeak}\,\sigma$ denotes the set of indexes of cycle peaks of σ. For example, if $\sigma = 2\,3\,1\,4\,7\,8\,6\,5 = (1\,2\,3)(4)(6\,8\,5\,7)$, then $\mathrm{Earec}\,\sigma = \{3, 8\}$ as $\sigma(3) = 1$ and $\sigma(8) = 5$ and $\mathrm{Cpeak}\,\sigma = \{3, 7, 8\}$, so $\mathrm{Ear}\,\sigma = \{3, 8\}$ and $\mathrm{ear}\,\sigma = 2$.

Our first result provides three interpretations for the polynomials $A_n(t, \lambda, y, w)$ in (1.2).

Theorem 1. *We have*

$$A_n(t,\lambda,y,w) = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{exc}\,\sigma}\lambda^{\mathrm{pex}\,\sigma}y^{\mathrm{ear}\,\sigma}w^{\mathrm{fix}\,\sigma} \tag{2.2a}$$

$$= \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{exc}\,\sigma}\lambda^{\mathrm{pcyc}\,\sigma}y^{\mathrm{ear}\,\sigma}w^{\mathrm{fix}\,\sigma} \tag{2.2b}$$

$$= \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{exc}\,\sigma}\lambda^{\mathrm{pcyc}\,\sigma}y^{\mathrm{pex}\,\sigma}w^{\mathrm{fix}\,\sigma}. \tag{2.2c}$$

By (1.2), the polynomial $A_n(t,\lambda,y,w)$ is invariant under $\lambda \leftrightarrow y$. Hence, the above theorem implies immediately the following result.

Corollary 1. *The six bistatistics* $(\mathrm{pex},\mathrm{ear})$, $(\mathrm{ear},\mathrm{pex})$, $(\mathrm{ear},\mathrm{pcyc})$, $(\mathrm{pcyc},\mathrm{ear})$, $(\mathrm{pex},\mathrm{pcyc})$ *and* $(\mathrm{pcyc},\mathrm{pex})$ *are equidistributed on* $\mathfrak{S}_n$.

Now we consider three specializations of $A_n(t,\lambda,y,w)$. First we let $B_n(t,\lambda,w) = A_n(t,\lambda,1,w) = A_n(t,1,\lambda,w)$, namely,

$$\sum_{n\geq 0} z^n B_n(t,\lambda,w) = \cfrac{1}{1-wz-\cfrac{t\lambda\, z^2}{1-(w+t+1)z-\cfrac{2t(\lambda+1)\,z^2}{\cdots}}} \tag{2.3}$$

with $\gamma_n = w + n(t+1)$ and $\beta_n = nt(\lambda+n-1)$.

Remark 1. By (2.2c) we recover the fix and cycle (p,q)-Eulerian polynomials [12, 13, 21]

$$A_n(x,p,1,pq) = B_n(x,p,pq) = \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{exc}\,\sigma}p^{\mathrm{cyc}\,\sigma}q^{\mathrm{fix}\,\sigma}. \tag{2.4}$$

To deal with descent statistics, we recall some linear statistics from [9]. For $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n) \in \mathfrak{S}_n$ with convention 0–∞, *i.e.*, $\sigma(0)=0$ and $\sigma(n+1)=n+1$, a value $\sigma(i)$ $(1\leq i\leq n)$ is called a

- *double ascent* (dasc) if $\sigma(i-1)<\sigma(i)$ and $\sigma(i)<\sigma(i+1)$;
- *double descent* (ddes) if $\sigma(i-1)>\sigma(i)$ and $\sigma(i)>\sigma(i+1)$;
- *peak* (peak) if $\sigma(i-1)<\sigma(i)$ and $\sigma(i)>\sigma(i+1)$;
- *valley* (valley) if $\sigma(i-1)>\sigma(i)$ and $\sigma(i)<\sigma(i+1)$.

A double ascent $\sigma(i)$ $(1\leq i\leq n)$ is called a *foremaximum* of σ if it is at the same time a record. Denote the number of foremaxima of σ by $\mathrm{fmax}\,\sigma$. For example, if $\sigma = 3\,4\,2\,1\,5\,8\,7\,6$, then $\mathrm{dasc}\,\sigma = \mathrm{ddes}\,\sigma = \mathrm{peak}\,\sigma = \mathrm{val}\,\sigma = 2$ and $\mathrm{fmax}\,\sigma = 2$ as the foremaxima of σ are $3,5$.

Theorem 2. *We have*

$$B_n(t,\lambda,w) = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{exc}\,\sigma}\lambda^{\mathrm{pcyc}\,\sigma}w^{\mathrm{fix}\,\sigma} \tag{2.5a}$$
$$= \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{exc}\,\sigma}\lambda^{\mathrm{ear}\,\sigma}w^{\mathrm{fix}\,\sigma} \tag{2.5b}$$
$$= \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{exc}\,\sigma}\lambda^{\mathrm{pex}\,\sigma}w^{\mathrm{fix}\,\sigma} \tag{2.5c}$$
$$= \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{des}\,\sigma}\lambda^{\mathrm{des}_2\,\sigma}w^{\mathrm{fmax}\,\sigma} \tag{2.5d}$$

and

$$\sum_{n\geq 0} B_n(t,\lambda,w)\frac{z^n}{n!} = e^{wz}\left(\frac{1-t}{e^{tz}-te^{z}}\right)^{\lambda}. \tag{2.5e}$$

The following corollary of Theorem 2 confirms and generalizes Conjecture 2.

Corollary 2. *The four bistatistics* (exc, pcyc), (exc, ear), (des, des_2) *and* (exc, pex) *are equidistributed over* $\mathfrak{S}_n$.

Remark 2. We will provide bijective proofs of Corollary 2 in [10].

Next let $C_n(y,\lambda) = A_n(1,\lambda,y,\lambda) = A_n(1,y,\lambda,\lambda)$. Using (1.1) we obtain the following result directly from Theorem 2.

Theorem 3. *We have*

$$C_n(y,\lambda) = \sum_{\sigma\in\mathfrak{S}_n} y^{\mathrm{pex}\,\sigma}\lambda^{\mathrm{ear}\,\sigma+\mathrm{fix}\,\sigma} = \sum_{\sigma\in\mathfrak{S}_n} y^{\mathrm{ear}\,\sigma}\lambda^{\mathrm{pex}\,\sigma+\mathrm{fix}\,\sigma} \tag{2.6a}$$
$$= \sum_{\sigma\in\mathfrak{S}_n} y^{\mathrm{pcyc}\,\sigma}\lambda^{\mathrm{ear}\,\sigma+\mathrm{fix}\,\sigma} = \sum_{\sigma\in\mathfrak{S}_n} y^{\mathrm{ear}\,\sigma}\lambda^{\mathrm{cyc}\,\sigma} \tag{2.6b}$$
$$= \sum_{\sigma\in\mathfrak{S}_n} y^{\mathrm{pcyc}\,\sigma}\lambda^{\mathrm{pex}\,\sigma+\mathrm{fix}\,\sigma} = \sum_{\sigma\in\mathfrak{S}_n} y^{\mathrm{pex}\,\sigma}\lambda^{\mathrm{cyc}\,\sigma}. \tag{2.6c}$$

Finally let $D_n(t,\lambda,y) = A_n(t,\lambda,y,0)$. From Theorem 1 we deduce

$$D_n(t,\lambda,y) = \sum_{\sigma\in\mathfrak{D}_n} t^{\mathrm{exc}\,\sigma}\lambda^{\mathrm{pex}\,\sigma}y^{\mathrm{ear}\,\sigma} \tag{2.7a}$$
$$= \sum_{\sigma\in\mathfrak{D}_n} t^{\mathrm{exc}\,\sigma}\lambda^{\mathrm{cyc}\,\sigma}y^{\mathrm{ear}\,\sigma} \tag{2.7b}$$
$$= \sum_{\sigma\in\mathfrak{D}_n} t^{\mathrm{exc}\,\sigma}\lambda^{\mathrm{cyc}\,\sigma}y^{\mathrm{pex}\,\sigma}, \tag{2.7c}$$

where $\mathfrak{D}_n$ is the set of derangements (that is, permutations without a fixed point.) in $\mathfrak{S}_n$.

Let $\mathfrak{D}_n^*$ the subset of $\mathfrak{D}_n$ consisting of derangements without cycle double rise. Furthermore, for $k \in [n]$ define the set

$$\mathfrak{D}_n^*(k) = \{\sigma \in \mathfrak{D}_n \mid \operatorname{exc}(\sigma) = k,\ \operatorname{cdrise}(\sigma) = 0\}. \tag{2.8}$$

We show that the polynomials $D_n(t, \lambda, y)$ have a nice γ-positive formula, see [3, 9] for further information.

Theorem 4. *We have*

$$D_n(t, \lambda, y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_{n,k}(\lambda, y) t^k (1+t)^{n-2k}, \tag{2.9}$$

where the gamma coefficient $\gamma_{n,k}(\lambda, y)$ has the following interpretations

$$\gamma_{n,k}(\lambda, y) = \sum_{\sigma \in \mathfrak{D}_n^*(k)} \lambda^{\operatorname{pex} \sigma} y^{\operatorname{ear} \sigma} \tag{2.10a}$$

$$= \sum_{\sigma \in \mathfrak{D}_n^*(k)} \lambda^{\operatorname{cyc} \sigma} y^{\operatorname{ear} \sigma} \tag{2.10b}$$

$$= \sum_{\sigma \in \mathfrak{D}_n^*(k)} \lambda^{\operatorname{cyc} \sigma} y^{\operatorname{pex} \sigma}. \tag{2.10c}$$

Remark 3. For $\sigma \in \mathfrak{D}_n^*(k)$, the mapping $\sigma \mapsto \sigma^{-1}$ is a bijection from $\mathfrak{D}_n^*(k)$ to $\mathfrak{D}_n^{**}(k)$ with

$$\mathfrak{D}_n^{**}(k) = \{\sigma \in \mathfrak{D}_n \mid \operatorname{drop}(\sigma) = k,\ \operatorname{cdfall}(\sigma) = 0\}. \tag{2.11}$$

Thus, when $y = 1$ both (2.10b) and (2.10c) reduce to [17, Theorem 11].

In [10], we construct two bijections on $\mathfrak{S}_n$ to prove the equality between (2.5c) and (2.5d), namely we have the following result.

Theorem 5. *There are bijections $\Phi_1 : \mathfrak{S}_n \to \mathfrak{S}_n$ and $\Phi_2 : \mathfrak{S}_n \to \mathfrak{S}_n$ such that*

$$(\operatorname{des}, \operatorname{des}_2)\, \sigma = (\operatorname{exc}, \operatorname{ear})\, \Phi_1(\sigma); \tag{2.12a}$$

$$(\operatorname{des}, \operatorname{des}_2, \operatorname{fmax})\, \sigma = (\operatorname{exc}, \operatorname{pex}, \operatorname{fix})\, \Phi_2(\sigma). \tag{2.12b}$$

Remark 4. Note that Φ_2 gives a bijective proof of Conjecture 2.

The rest of this paper is organized as follows: Theorem 1 is proved in Section 3 while the proofs of other theorems can be found in [10].

3 Proof of Theorem 1

We recall the two master J-fractions for permutations in [19]. First we associate to each permutation $\sigma \in \mathfrak{S}_n$ a pictorial representation by placing vertices $1, 2, \ldots, n$ along the horizontal axis and then draw an arc from i to $\sigma(i)$ above (resp. below) the horizontal axis in case $\sigma(i) > i$ (resp. $\sigma(i) < i$), if $\sigma(i) = i$ we do not draw any arc. Of course, the arrows on the arc are redundant, because the arrow on an arc above (resp. below) the axis always points to the right (resp. left). We then say that a quadruplet $i < j < k < l$ forms an

- *upper crossing* (ucross) if $k = \sigma(i)$ and $l = \sigma(j)$;
- *lower crossing* (lcross) if $i = \sigma(k)$ and $j = \sigma(l)$;
- *upper nesting* (unest) if $l = \sigma(i)$ and $k = \sigma(j)$;
- *lower nesting* (lnest) if $i = \sigma(l)$ and $j = \sigma(k)$.

See Figure 2 and Figure 3. We also need a refined version of the above statistics. We define

$$\mathrm{ucross}(j, \sigma) = \#\{i < j < k < l\colon\ k = \sigma(i) \text{ and } l = \sigma(j)\} \tag{3.1a}$$

$$\mathrm{unest}(j, \sigma) = \#\{i < j < k < l\colon\ k = \sigma(j) \text{ and } l = \sigma(i)\} \tag{3.1b}$$

$$\mathrm{lcross}(k, \sigma) = \#\{i < j < k < l\colon\ i = \sigma(k) \text{ and } j = \sigma(l)\} \tag{3.1c}$$

$$\mathrm{lnest}(k, \sigma) = \#\{i < j < k < l\colon\ i = \sigma(l) \text{ and } j = \sigma(k)\} \tag{3.1d}$$

We also consider the degenerate cases with $j = k$, by saying that a triplet $i < j < l$ forms an

- *upper pseudo-nesting* (upsnest) if $l = \sigma(i)$ and $j = \sigma(j)$;
- *lower pseudo-nesting* (lpsnest) if $i = \sigma(l)$ and $j = \sigma(j)$.

See Figure 4. Note that $\mathrm{upsnest}(\sigma) = \mathrm{lpsnest}(\sigma)$ for all σ (see [19]). We therefore write these two statistics simply as

$$\mathrm{lev}(\sigma) = \mathrm{upsnest}(\sigma) = \mathrm{lpsnest}(\sigma).$$

The refined level of a fixed point j ($\sigma(j) = j$) is defined by

$$\mathrm{lev}(j, \sigma) \;=\; \#\{i < j < l\colon\ l = \sigma(i)\} \;=\; \#\{i < j < l\colon\ i = \sigma(l)\}\,. \tag{3.2}$$

And we obviously have

$$\mathrm{ucross}(\sigma) \;=\; \sum_{j \in \mathrm{cval}} \mathrm{ucross}(j, \sigma) \tag{3.3}$$

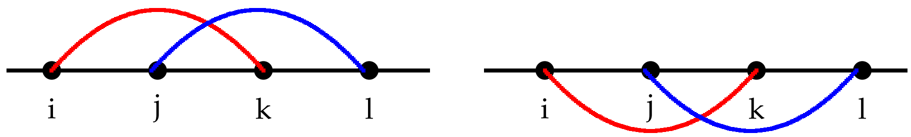

Figure 2: Upper crossing and lower crossing

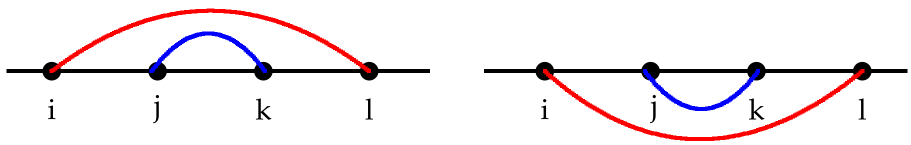

Figure 3: Upper nesting and lower nesting

and analogously for the other four statistics lcross, unest, lnest and lev.

We introduce five infinite families of indeterminates $\mathbf{a} = (\mathsf{a}_{\ell,\ell'})_{\ell,\ell'\geq 0}$, $\mathbf{b} = (\mathsf{b}_{\ell,\ell'})_{\ell,\ell'\geq 0}$, $\mathbf{c} = (\mathsf{c}_{\ell,\ell'})_{\ell,\ell'\geq 0}$, $\mathbf{d} = (\mathsf{d}_{\ell,\ell'})_{\ell,\ell'\geq 0}$, $\mathbf{e} = (\mathsf{e}_{\ell})_{\ell\geq 0}$ and define the polynomial $Q_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e})$ by

$$\begin{aligned} Q_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e}) \;=\; & \\ & \sum_{\sigma\in\mathfrak{S}_n} \prod_{i\in\mathrm{Cval}} \mathsf{a}_{\mathrm{ucross}(i,\sigma),\,\mathrm{unest}(i,\sigma)} \prod_{i\in\mathrm{Cpeak}} \mathsf{b}_{\mathrm{lcross}(i,\sigma),\,\mathrm{lnest}(i,\sigma)} \times \\ & \quad \prod_{i\in\mathrm{Cdfall}} \mathsf{c}_{\mathrm{lcross}(i,\sigma),\,\mathrm{lnest}(i,\sigma)} \prod_{i\in\mathrm{Cdrise}} \mathsf{d}_{\mathrm{ucross}(i,\sigma),\,\mathrm{unest}(i,\sigma)} \prod_{i\in\mathrm{Fix}} \mathsf{e}_{\mathrm{lev}(i,\sigma)}\;. \end{aligned} \tag{3.4}$$

The following is the first master J-fraction for permutations in [19, Theorem 2.9].

Theorem 6. *[19] The ordinary generating function of the polynomials $Q_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e})$ has the J-type continued fraction*

$$\sum_{n=0}^{\infty} Q_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e})\, z^n \;=\;$$

$$\cfrac{1}{1-\mathsf{e}_0 z-\cfrac{\mathsf{a}_{00}\mathsf{b}_{00}z^2}{1-(\mathsf{c}_{00}+\mathsf{d}_{00}+\mathsf{e}_1)z-\cfrac{(\mathsf{a}_{01}+\mathsf{a}_{10})(\mathsf{b}_{01}+\mathsf{b}_{10})z^2}{1-(\mathsf{c}_{01}+\mathsf{c}_{10}+\mathsf{d}_{01}+\mathsf{d}_{10}+\mathsf{e}_2)z-\cfrac{(\mathsf{a}_{02}+\mathsf{a}_{11}+\mathsf{a}_{20})(\mathsf{b}_{02}+\mathsf{b}_{11}+\mathsf{b}_{20})z^2}{1-\cdots}}}} \tag{3.5}$$

with coefficients

$$\gamma_n = \mathsf{c}^{\star}_{n-1} + \mathsf{d}^{\star}_{n-1} + \mathsf{e}_n, \tag{3.6a}$$

$$\beta_n = \mathsf{a}^{\star}_{n-1}\,\mathsf{b}^{\star}_{n-1}, \tag{3.6b}$$

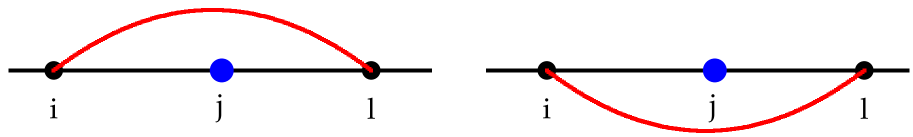

Figure 4: Upper pseudo-nesting and lower pseudo-nesting of a fixed point

where

$$\mathsf{a}^\star_{n-1} \stackrel{\text{def}}{=} \sum_{\ell=0}^{n-1} \mathsf{a}_{\ell,n-1-\ell} \tag{3.7}$$

and likewise for $\mathsf{b}, \mathsf{c}, \mathsf{d}$.

We again define five infinite families of indeterminates: $\mathbf{a} = (\mathsf{a}_\ell)_{\ell\ge 0}$, $\mathbf{b} = (\mathsf{b}_{\ell,\ell'})_{\ell,\ell'\ge 0}$, $\mathbf{c} = (\mathsf{c}_{\ell,\ell'})_{\ell,\ell'\ge 0}$, $\mathbf{d} = (\mathsf{d}_{\ell,\ell'})_{\ell,\ell'\ge 0}$, $\mathbf{e} = (\mathsf{e}_\ell)_{\ell\ge 0}$; note that $\mathbf{a}$ now has one index rather than two. We then define the polynomial $\widehat{Q}_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\lambda)$ by

$$\begin{aligned}\widehat{Q}_n(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e},\lambda) \;=\; & \\ & \sum_{\sigma\in\mathfrak{S}_n} \lambda^{\mathrm{cyc}(\sigma)} \prod_{i\in\mathrm{Cval}} \mathsf{a}_{\mathrm{ucross}(i,\sigma)+\mathrm{unest}(i,\sigma)} \prod_{i\in\mathrm{Cpeak}} \mathsf{b}_{\mathrm{lcross}(i,\sigma),\,\mathrm{lnest}(i,\sigma)} \times \\ & \prod_{i\in\mathrm{Cdfall}} \mathsf{c}_{\mathrm{lcross}(i,\sigma),\,\mathrm{lnest}(i,\sigma)} \prod_{i\in\mathrm{Cdrise}} \mathsf{d}_{\mathrm{ucross}(i,\sigma)+\mathrm{unest}(i,\sigma),\,\mathrm{unest}(\sigma^{-1}(i),\sigma)} \prod_{i\in\mathrm{Fix}} \mathsf{e}_{\mathrm{lev}(i,\sigma)}\,. \end{aligned} \tag{3.8}$$

The following is the second master J-fraction for permutations in [19, Theorem 2.14].

Theorem 7. *[19] The ordinary generating function of the polynomials $\widehat{Q}_n(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{e},\lambda)$ has the J-type continued fraction*

$$\sum_{n=0}^{\infty} \widehat{Q}_n(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{e},\lambda)\, z^n \;=\; \cfrac{1}{1-\lambda\mathsf{e}_0 z - \cfrac{\lambda\mathsf{a}_0\mathsf{b}_{00} z^2}{1-(\mathsf{c}_{00}+\mathsf{d}_{00}+\lambda\mathsf{e}_1)z - \cfrac{(\lambda+1)\mathsf{a}_1(\mathsf{b}_{01}+\mathsf{b}_{10})z^2}{1-(\mathsf{c}_{01}+\mathsf{c}_{10}+\mathsf{d}_{10}+\mathsf{d}_{11}+\lambda\mathsf{e}_2)z - \cfrac{(\lambda+2)\mathsf{a}_2(\mathsf{b}_{02}+\mathsf{b}_{11}+\mathsf{b}_{20})z^2}{1-\cdots}}}} \tag{3.9}$$

with coefficients

$$\gamma_n = \sum_{\ell=0}^{n-1} \mathsf{c}_{\ell,n-1-\ell} \;+\; \sum_{\ell=0}^{n-1} \mathsf{d}_{n-1,\ell} \;+\; \lambda\mathsf{e}_n, \tag{3.10a}$$

$$\beta_n = (\lambda+n-1)\,\mathsf{a}_{n-1} \sum_{\ell=0}^{n-1} \mathsf{b}_{\ell,n-1-\ell}, \tag{3.10b}$$

We derive the following dual version of Theorem 7 from (3.8) by constructing a bijection, see [10].

Proposition 1 (Dual form of Theorem 7). *We have*

$$\widehat{Q}_n(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{e},\lambda) \;=\; \sum_{\sigma\in\mathfrak{S}_n} \lambda^{\mathrm{cyc}(\sigma)} \prod_{i\in\mathrm{Cval}} \mathsf{b}_{\mathrm{ucross}(i,\sigma),\,\mathrm{unest}(i,\sigma)} \prod_{i\in\mathrm{Cpeak}} \mathsf{a}_{\mathrm{lcross}(i,\sigma)+lnest(i,\sigma)} \times \prod_{i\in\mathrm{Cdfall}} \mathsf{d}_{\mathrm{lcross}(i,\sigma)+lnest(i,\sigma),\,lnest(\sigma^{-1}(i),\sigma)} \prod_{i\in\mathrm{Cdrise}} \mathsf{c}_{\mathrm{ucross}(i,\sigma),\,\mathrm{unest}(i,\sigma)} \prod_{i\in\mathrm{Fix}} \mathsf{e}_{\mathrm{lev}(i,\sigma)}\,. \tag{3.11}$$

We derive Theorem 1 from Theorem 6, Theorem 7 and Proposition 1. Please see [10] for more details.

Remark 5. As the polynomials Q_n and $\widehat{Q}_n$ are originally defined using cyclic statistics of permutations, it is then suggested in [19] to seek for interpretations using linear statistics for these master polynomials. In [10], we give two such interpretations for the polynomials Q_n and as an application, we give a group action proof for a gamma-expansion formula Equation (2.10a) (see Theorem 4). And we conclude the paper [10] with some open questions.

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