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Wronskians, Total Positivity, and Real Schubert Calculus

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Abstract. A complete flag in $\mathbb{R}^n$ is a sequence of nested subspaces $V_1 \subset \cdots \subset V_{n-1}$ such that each V_k has dimension k. It is called *totally nonnegative* if all its Plücker coordinates are nonnegative. We may view each V_k as a subspace of polynomials in $\mathbb{R}[x]$ of degree at most $n-1$, by associating a vector $(a_1, \ldots, a_n)$ in $\mathbb{R}^n$ to the polynomial $a_1 + a_2 x + \cdots + a_n x^{n-1}$. We show that a complete flag is totally nonnegative if and only if each of its Wronskian polynomials $\mathrm{Wr}(V_k)$ is nonzero on the interval $(0, \infty)$. In the language of Chebyshev systems, this means that the flag forms a Markov system or *ECT*-system on $(0, \infty)$. This gives a new characterization and membership test for the totally nonnegative flag variety. Similarly, we show that a complete flag is totally positive if and only if each $\mathrm{Wr}(V_k)$ is nonzero on $[0, \infty]$. We use these results to show that a conjecture of Eremenko (2015) in real Schubert calculus is equivalent to the following conjecture: if V is a finite-dimensional subspace of polynomials such that all complex zeros of $\mathrm{Wr}(V)$ lie in the interval $(-\infty, 0)$, then all Plücker coordinates of V are real and positive. This conjecture is a totally positive strengthening of a result of Mukhin, Tarasov, and Varchenko (2009), and can be reformulated as saying that all complex solutions to a certain family of Schubert problems in the Grassmannian are real and totally positive. We also show that our conjecture is equivalent to a totally positive strengthening of the secant conjecture (2012).

Keywords: total positivity, Wronskian, Schubert calculus, flag variety, Chebyshev system

1 Introduction

Let $\mathrm{Fl}_n(\mathbb{R})$ denote the *complete flag variety*, consisting of all sequences $V = (V_1 \subset \cdots \subset V_{n-1})$ of nested subspaces of $\mathbb{R}^n$ such that each V_k has dimension k. Lusztig [16] introduced two remarkable subsets of $\mathrm{Fl}_n(\mathbb{R})$, called the *totally nonnegative part* $\mathrm{Fl}_n^{\geq 0}$ and the *totally positive part* $\mathrm{Fl}_n^{>0}$. They may be defined as the subsets where all Plücker coordinates are nonnegative or positive, respectively, up to rescaling. The totally nonnegative

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parts of flag varieties have been widely studied, with connections to representation theory [16], combinatorics and cluster algebras [7], high-energy physics [1], topology [9], and many other topics.

On the other hand, the *Wronskian* of a sequence of sufficiently differentiable functions $f_1(x), \ldots, f_k(x)$ (defined on $\mathbb{R}$ or $\mathbb{C}$) is

$$\mathsf{Wr}(f_1, \ldots, f_k) := \det\left(\frac{d^{i-1} f_j}{dx^{i-1}}\right)_{1 \le i,j \le k} = \det \begin{bmatrix} f_1 & \cdots & f_k \\ f_1' & \cdots & f_k' \\ \vdots & \ddots & \vdots \\ f_1^{(k-1)} & \cdots & f_k^{(k-1)} \end{bmatrix}. \tag{1.1}$$

The Wronskian is identically zero if $f_1, \ldots, f_k$ are linearly dependent; otherwise, it only depends on the linear span V of $f_1, \ldots, f_k$ up to multiplication by a nonzero scalar. In particular, it makes sense to write $\mathsf{Wr}(V)$, and its zeros are well-defined.

The Wronskian appears in various contexts; we give three examples. First, when $k = 2$, we have

$$\mathsf{Wr}(f, g) = fg' - f'g = f^2\left(\frac{g}{f}\right)'.$$

Hence if f and g are polynomials with no common factors, then the zeros of $\mathsf{Wr}(f, g)$ are the critical points of the rational function $\frac{g}{f}$. Second, when $f_1, \ldots, f_k$ are linearly independent, the unique homogeneous linear differential operator $\mathcal{L}$ of order k with leading coefficient 1 and kernel spanned by $f_1, \ldots, f_k$ is given by

$$\mathcal{L}(g) = \frac{\mathsf{Wr}(f_1, \ldots, f_k, g)}{\mathsf{Wr}(f_1, \ldots, f_k)} = \frac{d^k g}{dx^k} + \cdots. \tag{1.2}$$

Third, when $f_1, \ldots, f_k$ are linearly independent and r is a scalar, by interpolation there exists a nonzero g in the linear span of $f_1, \ldots, f_k$ with a zero of order at least $k-1$ at r. For a generic r, the zero of g at r has order exactly $k-1$; it is precisely when r is a zero of $\mathsf{Wr}(f_1, \ldots, f_k)$ that there exists a g with a zero of order at least k at r.

We introduce a new connection between total positivity and Wronskians, and use it to show the hidden role that total positivity plays in certain conjectures in real Schubert calculus. We now explain our main results.

2 Complete flags and their Wronskians

Let $\mathbb{R}[x]_{\le n-1}$ denote the subspace of $\mathbb{R}[x]$ of polynomials of degree at most $n-1$. We identify $\mathbb{R}^n$ with $\mathbb{R}[x]_{\le n-1}$, as follows:

$$\mathbb{R}^n \leftrightarrow \mathbb{R}[x]_{\le n-1}, \quad (a_1, \ldots, a_n) \leftrightarrow a_1 + a_2 x + \cdots + a_n x^{n-1}. \tag{2.1}$$

In particular, we may view a complete flag $(V_1, \ldots, V_{n-1}) \in \mathrm{Fl}_n(\mathbb{R})$ as a sequence of nested subspaces inside $\mathbb{R}[x]_{\leq n-1}$. Our first main result characterizes the totally nonnegative and totally positive flag varieties $\mathrm{Fl}_n^{\geq 0}$ and $\mathrm{Fl}_n^{>0}$ in terms of their Wronskian polynomials:

Theorem 2.1. *Let $V = (V_1, \ldots, V_{n-1}) \in \mathrm{Fl}_n(\mathbb{R})$.*

(i) The flag V is totally nonnegative if and only if $\mathsf{Wr}(V_k)$ is nonzero on the interval $(0, \infty)$, for all $1 \leq k \leq n-1$.

(ii) The flag V is totally positive if and only if $\mathsf{Wr}(V_k)$ is nonzero on the interval $[0, \infty]$, for all $1 \leq k \leq n-1$. (Here, $\mathsf{Wr}(V_k)$ being nonzero at ∞ means that it has the expected degree $k(n-k)$.)

In the language of Chebyshev systems (see e.g. [11, 13]), the conclusions above say that V forms a *Markov system* (or *ECT-system*) on $(0, \infty)$ and $[0, \infty]$, respectively. An equivalent characterization is that for all $1 \leq k \leq n-1$, every nonzero polynomial $f \in V_k$ has at most $k-1$ zeros counted with multiplicity in $(0, \infty)$ and $[0, \infty]$, respectively.

Example 2.2. Let $n := 3$, and let $V = (V_1, V_2)$ denote a generic element of $\mathrm{Fl}_3(\mathbb{R})$, represented by the matrix

$$A := \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} \qquad (a, b, c \in \mathbb{R}).$$

That is, V_k (for $k = 1, 2$) is the span of the first k columns of A. Let us verify that Theorem 2.1(ii) holds for V. The Plücker coordinates of V are the left-justified (*i.e.* initial) minors of A:

$$\Delta_1(V) = 1,\ \Delta_2(V) = a,\ \Delta_3(V) = b \quad \text{and} \quad \Delta_{12}(V) = 1,\ \Delta_{13}(V) = c,\ \Delta_{23}(V) = ac - b.$$

Therefore V is totally positive if and only if

$$a,\ b,\ c,\ ac - b > 0. \tag{2.2}$$

On the other hand, let $f_k \in \mathbb{R}[x]$ (for $k = 1, 2$) be the polynomial corresponding to column k of A, so that

$$f_1(x) = 1 + ax + bx^2 \quad \text{and} \quad f_2(x) = x + cx^2.$$

Then

$$\mathsf{Wr}(V_1) = \mathsf{Wr}(f_1) = 1 + ax + bx^2 =: h_1(x),$$
$$\mathsf{Wr}(V_2) = \mathsf{Wr}(f_1, f_2) = f_1 f_2' - f_1' f_2 = 1 + 2cx + (ac - b)x^2 =: h_2(x).$$

We must show that (2.2) holds if and only if h_1 and h_2 are nonzero on $[0,\infty]$. If (2.2) holds, then h_1 and h_2 have positive coefficients, and hence they are positive on $[0,\infty]$. Conversely, suppose that h_1 and h_2 are nonzero on $[0,\infty]$. Since $h_1(0) = h_2(0) = 1$, they must be positive on $[0,\infty]$. Considering $x = \infty$, we get that h_1 and h_2 both have the expected degree 2, and $b, ac - b > 0$. It remains to show that $a, c > 0$. Since $ac > b > 0$, we see that a and c have the same sign. Proceed by contradiction and suppose that $a, c \leq 0$. Observe that h_1 is minimized at $x = -\frac{a}{2b}$, and h_2 is minimized at $x = -\frac{c}{ac-b}$. We obtain

$$h_1\Big(-\frac{a}{2b}\Big) = 1 - \frac{a^2}{4b} > 0 \quad \text{and} \quad h_2\Big(-\frac{c}{ac-b}\Big) = 1 - \frac{c^2}{ac-b} > 0.$$

Therefore $a^2 < 4b$ and $c^2 < ac - b$. We now reach a contradiction:

$$0 \leq (a-2c)^2 = a^2 - 4ac + 4c^2 < 4b - 4ac + 4(ac-b) = 0.$$

We outline the proof of part (ii) of Theorem 2.1, whence we obtain part (i) via a limiting argument. The forward direction is a consequence of the fact that the coefficients of $\mathrm{Wr}(V_k)$ are positively weighted sums of the Plücker coordinates of V_k. The reverse direction seems difficult to establish directly, as we have done in Example 2.2 when $n = 3$; rather, we argue as follows. First we note that $\mathrm{Fl}_n^{>0}$ is a connected component of the space of complete flags whose Wronskians are nonzero on $[0,\infty]$. Then we show that given $V \in \mathrm{Fl}_n(\mathbb{R})$ whose Wronskians are nonzero on $[0,\infty]$, there exists a path from V to $\mathrm{Fl}_n^{>0}$; specifically, V becomes totally positive upon replacing the variable x with $x + t$ for $t > 0$ sufficiently large. This is based on the fact that the operator $x \mapsto x + t$ can be written as $\exp(t\frac{d}{dx})$, and is therefore compatible with total positivity.

We do not know how to generalize the statement of Theorem 2.1 from $\mathrm{Fl}_n(\mathbb{R})$ to an arbitrary partial flag variety in $\mathbb{R}^n$. The argument in the previous paragraph breaks down because the boundary of the totally nonnegative part of a partial flag variety (other than $\mathrm{Fl}_n(\mathbb{R})$ and $\mathbb{P}^{n-1}(\mathbb{R})$) does not have a simple description in terms of Wronskians.

A curious consequence of Theorem 2.1 is that $\mathrm{Fl}_n^{\geq 0}$ and $\mathrm{Fl}_n^{>0}$ can be described as the semialgebraic subsets of $\mathrm{Fl}_n(\mathbb{R})$ where the coefficients of each Wronskian polynomial are nonnegative and positive, respectively (up to rescaling). Note that this description involves $O(n^3)$ inequalities, whereas the usual description using Plücker coordinates involves $2^n - 2$ inequalities. Equivalently, we obtain a total nonnegativity test and a total positivity test for $\mathrm{Fl}_n(\mathbb{R})$ involving $O(n^3)$ functions. While total positivity tests for $\mathrm{Fl}_n(\mathbb{R})$ involving $O(n^2)$ functions are well-studied as part of the theory of cluster algebras (see [7, Section 1.3] and references therein), we do not know of any previous total nonnegativity tests for $\mathrm{Fl}_n(\mathbb{R})$ involving a subexponential number of fixed functions. In a separate paper, we will apply Theorem 2.1 to give an efficient total nonnegativity test for $\mathrm{GL}_n(\mathbb{R})$.

We point out that Saldanha, Shapiro, and Shapiro [19] have explored the connection between Wronskians of flags and total positivity. Also, Schechtman and Varchenko [20, Theorem 4.4] have interpreted parametrizations of totally positive complete flags in terms of Wronskian polynomials. While these parametrizations do not play a role in our arguments, it would be interesting to explore this connection further.

3 Conjectures in real Schubert calculus

Given a system of polynomial equations over the real numbers, one often wishes to know how many of the solutions over the complex numbers are real. In general, we can usually say little more than that the nonreal solutions come in complex-conjugate pairs. In some situations, a lower bound on the number of real solutions can be given. In very special situations, all complex solutions are guaranteed to be real. Shapiro and Shapiro discovered such a phenomenon in the Schubert calculus of Grassmannians.

Namely, for $0 \leq k \leq n$, let $\mathrm{Gr}_{k,n}(\mathbb{C})$ denote the *Grassmannian*, consisting of all k-dimensional subspaces V of $\mathbb{C}^n$. Let $W_1, \ldots, W_{k(n-k)}$ be sufficiently generic elements of $\mathrm{Gr}_{k,n}(\mathbb{C})$. Then the number of solutions $U \in \mathrm{Gr}_{n-k,n}(\mathbb{C})$ to the Schubert problem

$$U \cap W_l \neq \{0\} \quad \text{for } 1 \leq l \leq k(n-k) \tag{3.1}$$

is finite, and equals $d_{k,n} := \frac{1!2!\cdots(k-1)!}{(n-k)!(n-k+1)!\cdots(n-1)!}(k(n-k))!$. Define the *rational normal curve*

$$\gamma\colon \mathbb{C} \to \mathbb{C}^n, \quad x \mapsto \left(\tbinom{n-1}{i-1}x^{n-i}\right)_{i=1}^{n} = \left(x^{n-1}, (n-1)x^{n-2}, \tbinom{n-1}{2}x^{n-3}, \ldots, 1\right). \tag{3.2}$$

(Often one works instead with the curve $(1, x, x^2, \ldots, x^{n-1})$, which is equivalent; we use the alternative convention above in order to make some of the intermediate results simpler to state.) Shapiro and Shapiro conjectured (cf. [21]) that if each W_l is an osculating plane to γ at a real point, then all solutions U to the Schubert problem (3.1) are real (*i.e.* have a basis of real vectors). Their conjecture was verified when $k = 2$ by Eremenko and Gabrielov [3], and in general by Mukhin, Tarasov, and Varchenko [18], who also showed that the upper bound $d_{k,n}$ on the number of solutions is always obtained [17]. A different proof was recently given by Levinson and Purbhoo [15, Corollary 1.5].

Let $\mathbb{C}[x]_{\leq n-1}$ denote the subspace of $\mathbb{C}[x]$ of all polynomials of degree at most $n-1$, which we identify with $\mathbb{C}^n$ via (2.1) (with $\mathbb{R}$ replaced by $\mathbb{C}$). While not immediately apparent, a standard transformation (see e.g. [22, Section 10.1]) allows one to reformulate the Schubert problem above dually in terms of Wronskians. In particular, we have:

Theorem 3.1 (Mukhin, Tarasov, and Varchenko [17, Corollary 6.3])**.** *Let $0 \leq k \leq n$, and let $X \subseteq \mathbb{R}$ consist of $k(n-k)$ distinct points. Then there exist precisely $d_{k,n}$ elements $V \in \mathrm{Gr}_{k,n}(\mathbb{C})$ such that the zero set of $\mathrm{Wr}(V)$ is X. Moreover, each such V is real, i.e., it has a basis of real*

polynomials. In particular, if $V \in \mathrm{Gr}_{k,n}(\mathbb{C})$ such that all complex zeros of $\mathsf{Wr}(V)$ are real, then V is real.

García-Puente, Hein, Hillar, Martín del Campo, Ruffo, Sottile, and Teitler [10] have conjectured a generalization of Theorem 3.1, known as the *secant conjecture*. It states that the solutions to the Schubert problem (3.1) all remain real if each osculating plane W_l to γ is replaced with a plane spanned by $\gamma(x_1), \ldots, \gamma(x_k)$ for any real points $x_1, \ldots, x_k$, such that the points chosen for each W_l lie in $k(n-k)$ disjoint intervals. Eremenko [5] showed that the secant conjecture is implied by Theorem 3.1 and the following conjecture. The case $k = 2$ of both conjectures follows from work of Eremenko, Gabrielov, Shapiro, and Vainshtein [4, Section 3] (cf. [5, p. 341]).

Conjecture 3.2 (Eremenko [5, 6])**.** *Let $V \in \mathrm{Gr}_{k,n}(\mathbb{R})$. Suppose that all complex zeros of $\mathsf{Wr}(V)$ are real, and let $I \subseteq \mathbb{R}$ be any interval on which $\mathsf{Wr}(V)$ is nonzero. Then every nonzero polynomial $f \in V$ has at most $k-1$ zeros in I.*

We now conjecture that certain of the Schubert problems considered above have all their solutions not only real, but totally nonnegative or totally positive. Namely, we say that $V \in \mathrm{Gr}_{k,n}(\mathbb{R})$ is *totally nonnegative* (respectively, *totally positive*) if all its Plücker coordinates are nonnegative (respectively, positive), up to rescaling. We also extend the definition of γ to $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ in a natural way. We make the following totally positive analogue of the secant conjecture:

Conjecture 3.3. *Let $0 \leq k \leq n$, and let $I_1, \ldots, I_{k(n-k)}$ be pairwise disjoint intervals of $\mathbb{P}^1(\mathbb{R})$. For $1 \leq l \leq k(n-k)$, let X_l be a multiset of k points contained in I_l, and let $W_l \in \mathrm{Gr}_{k,n}(\mathbb{R})$ be spanned by $\gamma(x), \gamma'(x), \ldots, \gamma^{(p-1)}(x)$ for all $x \in X_l$, where p denotes the multiplicity of x in X_l.*

(i) If $I_l \subseteq [0,\infty]$ for $1 \leq l \leq k(n-k)$, then the Schubert problem (3.1) has $d_{k,n}$ distinct solutions $U \in \mathrm{Gr}_{n-k,n}(\mathbb{C})$, which are all real and totally nonnegative.

(ii) If $I_l \subseteq (0,\infty)$ for $1 \leq l \leq k(n-k)$, then the Schubert problem (3.1) has $d_{k,n}$ distinct solutions $U \in \mathrm{Gr}_{n-k,n}(\mathbb{C})$, which are all real and totally positive.

A limiting argument implies that parts (i) and (ii) of Conjecture 3.3 for $\mathrm{Gr}_{n-k,n}(\mathbb{C})$ are equivalent. We also note that Conjecture 3.3 implies the secant conjecture. This is due to an action of $\mathrm{SL}_2(\mathbb{R})$ which allows us to assume, without loss of generality, that all the intervals I_l in the secant conjecture are contained in $(0,\infty)$.

A special case of Conjecture 3.3 is when each multiset X_l consists of a single point x_l of multiplicity k, so that W_l is the osculating plane to γ at x_l. Then Conjecture 3.3 has the following dual formulation, which is a totally positive analogue of Theorem 3.1:

Conjecture 3.4.[1] *Let $V \in \mathrm{Gr}_{k,n}(\mathbb{R})$.*

[1] Conjecture 3.4(i) was independently posed by Evgeny Mukhin and Vitaly Tarasov in 2017. I thank Chris Fraser for informing me of this.

(i) If all complex zeros of $\mathsf{Wr}(V)$ *lie in the interval* $[-\infty, 0]$*, then* V *is totally nonnegative.*

(ii) If all complex zeros of $\mathsf{Wr}(V)$ *lie in the interval* $(-\infty, 0)$*, then* V *is totally positive.*

Above, $\mathsf{Wr}(V)$ *is considered to have a zero at* $-\infty$ *when its degree is less than* $k(n-k)$*.*

Note that it is equivalent to replace $\mathrm{Gr}_{k,n}(\mathbb{R})$ by $\mathrm{Gr}_{k,n}(\mathbb{C})$ in Conjecture 3.4, by Theorem 3.1. Conjecture 3.4 holds when $k = 1$; it states that if all complex zeros of a polynomial $f \in \mathbb{R}[x]$ are nonpositive (respectively, negative), then up to rescaling, all coefficients of f are nonnegative (respectively, positive). For an example, see Section 4, where we verify Conjecture 3.4(ii) for $\mathrm{Gr}_{2,4}(\mathbb{R})$. The case $n = 5$ of Conjecture 3.4 was proved by Fraser [8], and we have checked Conjecture 3.4 by computer for several instances with $n = 6$.

A limiting argument implies that parts (i) and (ii) of Conjecture 3.4 for $\mathrm{Gr}_{k,n}(\mathbb{R})$ are equivalent. Also, Conjecture 3.4 holds for $\mathrm{Gr}_{k,n}(\mathbb{R})$ if and only if it holds for $\mathrm{Gr}_{n-k,n}(\mathbb{R})$. This is due to the existence of a certain bilinear pairing on $\mathbb{R}^n$, such that the map $V \mapsto V^\perp$ preserves both the Wronskian and the collection of signs of Plücker coordinates. In particular, Conjecture 3.4 holds for $\mathrm{Gr}_{n-1,n}(\mathbb{R})$, since it holds for $\mathrm{Gr}_{1,n}(\mathbb{R})$.

Our second main result is that the three conjectures stated above are all equivalent to each other:

Theorem 3.5. *Let* $0 \leq k \leq n$*.*

(i) Conjecture 3.4 holds for $\mathrm{Gr}_{k,n}(\mathbb{R})$ *if and only if Conjecture 3.2 holds for both* $\mathrm{Gr}_{k,n}(\mathbb{R})$ *and* $\mathrm{Gr}_{n-k,n}(\mathbb{R})$*.*

(ii) Conjecture 3.4 holds for $\mathrm{Gr}_{k,n}(\mathbb{R})$ *if and only if Conjecture 3.3 holds for* $\mathrm{Gr}_{n-k,n}(\mathbb{C})$*.*

In particular, Conjecture 3.4 implies the secant conjecture. Another consequence of Theorem 3.5(ii) is that Conjecture 3.3 holds for $\mathrm{Gr}_{n-k,n}(\mathbb{C})$ if and only if it holds for $\mathrm{Gr}_{k,n}(\mathbb{C})$. This is in contrast to the secant conjecture, where the statements for $\mathrm{Gr}_{n-k,n}(\mathbb{C})$ and $\mathrm{Gr}_{k,n}(\mathbb{C})$ are not known to imply each other (see [10, Section 2.3]).

The proof of Theorem 3.5(i) uses Theorem 2.1, the $\mathrm{SL}_2(\mathbb{R})$-action, the bilinear pairing on $\mathbb{R}^n$, and classical results on Chebyshev and disconjugate systems of functions (see e.g. [2, 23]). Theorem 3.5(ii) then follows from the same argument used by Eremenko [5] to show that Conjecture 3.2 implies the secant conjecture. We point out that in reducing Conjecture 3.4(ii) to Conjecture 3.2 for a given subspace V, we call on Conjecture 3.2 applied to both V and $V^\perp$. In particular, although Conjecture 3.2 holds for $\mathrm{Gr}_{2,n}(\mathbb{R})$, it is open for $\mathrm{Gr}_{n-2,n}(\mathbb{R})$, and so we are not able to conclude that Conjecture 3.4 holds for $\mathrm{Gr}_{2,n}(\mathbb{R})$. Indeed, Conjecture 3.4 is open for $\mathrm{Gr}_{2,n}(\mathbb{R})$.

The proofs of the results stated above, and further discussion, appear in the full version of this paper [12].

4 Example: $\mathrm{Gr}_{2,4}(\mathbb{R})$

We verify Conjecture 3.4(ii) for $\mathrm{Gr}_{2,4}(\mathbb{R})$. Let $V \in \mathrm{Gr}_{2,4}(\mathbb{R})$ be represented by the matrix

$$\begin{bmatrix} 1 & 0 \\ a & 1 \\ b & c \\ 0 & d \end{bmatrix}, \quad \text{where } a, b, c, d \in \mathbb{R}. \tag{4.1}$$

That is, V is spanned by $f_1, f_2 \in \mathbb{R}[x]_{\leq 3}$, where

$$f_1(x) = 1 + ax + bx^2 \quad \text{and} \quad f_2(x) = x + cx^2 + dx^3.$$

The Plücker coordinates of V are the 2×2 minors of the matrix (4.1):

$$\Delta_{12} = 1, \quad \Delta_{13} = c, \quad \Delta_{14} = d, \quad \Delta_{23} = ac - b, \quad \Delta_{24} = ad, \quad \Delta_{34} = bd.$$

The Wronskian of V is

$$\begin{aligned} \mathsf{Wr}(V) = \mathsf{Wr}(f_1, f_2) = f_1 f_2' - f_1' f_2 &= 1 + 2cx + (3d + ac - b)x^2 + 2adx^3 + bdx^4 \\ &= \Delta_{12} + 2\Delta_{13}x + (3\Delta_{14} + \Delta_{23})x^2 + 2\Delta_{24}x^3 + \Delta_{34}x^4. \end{aligned}$$

Now suppose that all zeros of $\mathsf{Wr}(V)$ lie in the interval $(-\infty, 0)$. We may write

$$\mathsf{Wr}(V) = (1 + r_1 x)(1 + r_2 x)(1 + r_3 x)(1 + r_4 x) = 1 + e_1 x + e_2 x^2 + e_3 x^3 + e_4 x^4,$$

where $r_1, r_2, r_3, r_4 > 0$, and e_i (for $1 \leq i \leq 4$) is the ith elementary symmetric polynomial in r_1, r_2, r_3, r_4. We wish to show that V is totally positive, *i.e.*, all its Plücker coordinates are positive. Using the Plücker relation $\Delta_{13}\Delta_{24} = \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}$, we find that the Plücker coordinates of V are

$$\Delta_{12} = 1, \quad \Delta_{13} = \frac{e_1}{2}, \quad \Delta_{14} = \frac{e_2 \pm \sqrt{\kappa}}{6}, \quad \Delta_{23} = \frac{e_2 \mp \sqrt{\kappa}}{2}, \quad \Delta_{24} = \frac{e_3}{2}, \quad \Delta_{34} = e_4,$$

where

$$\kappa := e_2^2 - 3e_1e_3 + 12e_4 = \frac{(r_1 - r_2)^2(r_3 - r_4)^2 + (r_1 - r_3)^2(r_2 - r_4)^2 + (r_1 - r_4)^2(r_2 - r_3)^2}{2}.$$

(The fact that $\kappa \geq 0$ is equivalent to Theorem 3.1 for $\mathrm{Gr}_{2,4}(\mathbb{C})$; cf. [21, Example 2.2].) Therefore V is totally positive if and only if $e_2^2 > \kappa$. This is equivalent to $e_1e_3 > 4e_4$, which then follows by expanding both sides in r_1, r_2, r_3, r_4.

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