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The $m = 2$ Amplituhedron and the Hypersimplex

Matteo Parisi[*1,2], Melissa Sherman-Bennett[†3], and Lauren Williams[‡4]

[1]*Center for Mathematical Sciences and Applications, Harvard University, Cambridge MA*
[2]*School of Natural Sciences, Institute for Advanced Study, Princeton NJ*
[3]*Department of Mathematics, University of Michigan, Ann Arbor MI*
[4]*Department of Mathematics, Harvard University, Cambridge MA*

Abstract. The hypersimplex $\Delta_{k+1,n}$ is the image of the positive Grassmannian $Gr^{\geq 0}_{k+1,n}$ under the moment map. It is a polytope of dimension $n-1$ in $\mathbb{R}^n$. Meanwhile, the amplituhedron $\mathcal{A}^Z_{n,k,2}$ is the image of $Gr^{\geq 0}_{k,n}$ under an amplituhedron map $\widetilde{Z}$ induced by a positive matrix Z. Introduced in the context of *scattering amplitudes*, it is not a polytope, and is a full dimensional subset of $Gr_{k,k+2}$. Nevertheless, there seem to be remarkable connections between these two objects, as conjectured by Lukowski–Parisi–Williams (LPW). We use ideas from oriented matroid theory, total positivity, and the geometry of the hypersimplex and positroid polytopes to obtain a deeper understanding of the amplituhedron. We show that the inequalities cutting out *positroid polytopes* — moment map images of positroid cells — translate into sign conditions cutting out *Grasstopes* — amplituhedron map images of positroid cells. Moreover, we subdivide the amplituhedron into *chambers*, just as the hypersimplex can be subdivided into simplices — with both chambers and simplices enumerated by the Eulerian numbers. We use these properties to prove the main conjecture of (LPW): a collection of positroid polytopes is a tiling of the hypersimplex if and only if the collection of T-dual Grasstopes is a tiling of the amplituhedron $\mathcal{A}^Z_{n,k,2}$ for all Z. We also prove Arkani-Hamed–Thomas–Trnka's conjectural sign-flip characterization of $\mathcal{A}^Z_{n,k,2}$.

Keywords: oriented matroid, Eulerian number, amplituhedron, matroid polytope, positroid, hypersimplex

1 Introduction

The *positive Grassmannian*[1] $\mathrm{Gr}^{\geq 0}_{k,n}$ is the subset of the real Grassmannian $\mathrm{Gr}_{k,n}$ where all Plücker coordinates are nonnegative [17, 19, 21]. This is a remarkable space with connections to cluster algebras, integrable systems, and high energy physics [1, 6, 11, 22], and

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[†]msherben@umich.edu. Supported by the NSF Graduate Research Fellowship, grant number DGE-1752814.

[‡]williams@math.harvard.edu. Supported by the NSF grants DMS-1854316 and DMS-1854512, as well as the Harvard–Radcliffe Institute for Advanced Study.

[1]More formally, the *totally nonnegative Grassmannian*.

it has a beautiful CW decomposition into *positroid cells* S_π, which are indexed by various combinatorial objects including *decorated permutations* π [19].

There are several interesting maps which one can apply to the positive Grassmannian $\mathrm{Gr}_{k,n}^{\geq 0}$ and its cells. The first map is the *moment map* μ, initially studied by Gelfand–Goresky–MacPherson–Serganova [7] in the context of the Grassmannian and its torus orbits, who showed that the image of the Grassmannian is the *hypersimplex* $\Delta_{k,n} \subset \mathbb{R}^n$, a polytope of dimension $n-1$. When one restricts μ to $\mathrm{Gr}_{k,n}^{\geq 0}$, the image is still the hypersimplex [29].

The second map is the *amplituhedron map*, introduced by Arkani-Hamed and Trnka [3] in the context of *scattering amplitudes* in $\mathcal{N} = 4$ SYM. In particular, any $n \times (k+m)$ matrix Z with maximal minors positive induces a map $\widetilde{Z}$ from $\mathrm{Gr}_{k,n}^{\geq 0}$ to the Grassmannian $\mathrm{Gr}_{k,k+m}$, whose image has full dimension mk and is called the *amplituhedron* $\mathcal{A}_{n,k,m}^Z$.

Given a surjective map $\phi\colon \mathrm{Gr}_{k,n}^{\geq 0} \to X$, it is natural to try to decompose X using images of positroid cells under ϕ. This leads to the following definition.[2]

Definition 1.1. Let $\phi\colon \mathrm{Gr}_{k,n}^{\geq 0} \to X$ be a continuous surjective map onto a cell complex or subset thereof, where $\dim X = d$. A *positroid tile* $\overline{\phi(S_\pi)}$ is (the closure of) the image of a d-dimensional positroid cell S_π on which ϕ is injective. A *positroid tiling* of X (with respect to ϕ) is a collection $\{\overline{\phi(S_\pi)}\}$ of positroid tiles such that pairs of distinct images $\phi(S_\pi)$ and $\phi(S_{\pi'})$ are disjoint and $\cup\overline{\phi(S_\pi)} = X$.

When ϕ is the moment map, the (closures of) the images of the positroid cells S_π are the *positroid polytopes* Γ_π [29], so a positroid tiling of the hypersimplex is a decomposition into positroid polytopes. When ϕ is the amplituhedron map $\widetilde{Z}$, the (closures of) the images of the positroid cells S_π are *Grasstopes* Z_π, which were first studied in [3] as the building blocks of conjectural positroid tilings of the amplituhedron. Note that neither the amplituhedron nor the Grasstopes are polytopes.

At first glance, the $(n-1)$-dimensional hypersimplex $\Delta_{k+1,n} \subset \mathbb{R}^n$ does not seem to have any relation to the $2k$-dimensional amplituhedron $\mathcal{A}_{n,k,2}^Z \subset \mathrm{Gr}_{k,k+2}$. Nevertheless, [16] showed that there are surprising parallels between them. In particular, [16] showed that *T-duality* gives a bijection between loopless cells S_π of $\mathrm{Gr}_{k+1,n}^{\geq 0}$ and coloopless cells $S_{\hat{\pi}}$ of $\mathrm{Gr}_{k,n}^{\geq 0}$, and conjectured that T-duality gives a bijection between positroid tilings of the hypersimplex $\Delta_{k+1,n}$ and positroid tilings of the amplituhedron $\mathcal{A}_{n,k,2}^Z$. They verified this conjecture for the recursively-defined tilings of the amplituhedron found by [5].

Here, we use *twistor coordinates* and the geometry of the hypersimplex and positroid polytopes to obtain a deeper understanding of the amplituhedron. We prove the conjecture of Lukowski–Parisi–Spradlin–Volovich [15] classifying the positroid tiles of $\mathcal{A}_{n,k,2}^Z$.

[2]There are many reasonable variations of this definition. One might want to relax the injectivity assumption, or to impose further restrictions on how boundaries of the images of cells should overlap.

We then give a new characterization of positroid tiles as regions where certain twistor coordinates have specified sign. We use this result to prove a conjecture of Arkani-Hamed–Thomas–Trnka [2]: $\mathcal{A}_{n,k,2}^{Z}$ can be characterized using sign flips of twistor coordinates.

Additionally, we draw striking parallels between $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}^{Z}$. We show that positroid tiles for $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}^{Z}$ are in bijection and establish a close connection between the inequalities cutting out corresponding positroid tiles. Further, we subdivide the amplituhedron into *sign chambers* according to the signs of twistor coordinates. We prove that the *realizable* chambers are exactly enumerated by the Eulerian numbers $E_{k,n-1}$, so the chamber decomposition of the amplituhedron is analogous to the subdivision of the hypersimplex into simplices enumerated by $E_{k,n-1}$. We then prove the main conjecture of [16]: a collection of positroid polytopes is a positroid tiling of $\Delta_{k+1,n}$ if and only if the collection of T-dual Grasstopes is a positroid tiling of $\mathcal{A}_{n,k,2}^{Z}$ for all Z.

2 Background

2.1 The Grassmannian and positive Grassmannian

Fix integers $0 < k \leq n$, let $[n]$ denote $\{1, \ldots, n\}$, and $\binom{[n]}{k}$ the set of all k-element subsets of $[n]$. The *(real) Grassmannian* $\mathrm{Gr}_{k,n}$ is the space of all k-dimensional subspaces of $\mathbb{R}^n$. An element of $\mathrm{Gr}_{k,n}$ can be viewed as a full rank $k \times n$ matrix, modulo left multiplication by invertible $k \times k$ matrices. We often abuse notation and identify a full-rank matrix with its rowspan. We embed $\mathrm{Gr}_{k,n}$ into $\mathbb{P}(\wedge^k \mathbb{R}^n)$ via the Plücker embedding: the subspace V represented by matrix C with rows $C_1, \ldots, C_k$ is mapped to the line spanned by $C_1 \wedge \cdots \wedge C_k$. We also frequently abuse notation and identify a vector with its span. The *Plücker coordinates* of V, denoted $p_I(V)$ for $I \in \binom{[n]}{k}$, are the homogeneous coordinates of $C_1 \wedge \cdots \wedge C_k$ in $\mathbb{P}(\wedge^k \mathbb{R}^n)$, using the standard basis on $\mathbb{R}^n$ and $\wedge^k \mathbb{R}^n$. Alternately, $p_I(V)$ is the maximal minor of C using column set I.

We will also use the notation $\langle C_1, \ldots, C_k \rangle$ for $C_1 \wedge \cdots \wedge C_k$.

Definition 2.1 ([19, Section 3])**.** We say that $V \in \mathrm{Gr}_{k,n}$ is *totally nonnegative* (resp. *totally positive*) if it has a representative C so that $p_I(C) \geq 0$ (resp. $p_I(C) > 0$) for all $I \in \binom{[n]}{k}$. In an abuse of notation, we identify $V \in \mathrm{Gr}_{k,n}^{\geq 0}$ with its totally nonnegative representative C. The set of all totally nonnegative $V \in \mathrm{Gr}_{k,n}$ is the *totally nonnegative Grassmannian* $\mathrm{Gr}_{k,n}^{\geq 0}$. For $\mathcal{M} \subseteq \binom{[n]}{k}$, the *positroid cell* $S_{\mathcal{M}}$ is the set of $C \in \mathrm{Gr}_{k,n}^{\geq 0}$ such that $p_I(C) > 0$ if and only if $I \in \mathcal{M}$. We call $\mathcal{M}$ a *positroid* if $S_{\mathcal{M}}$ is nonempty.

Remark 2.2. Lusztig [17] earlier defined the positive and totally nonnegative part of a flag variety G/P, which Rietsch showed had a cell decomposition [21]. The two definitions, and the two cell complexes, agree [20, 28].

There are many ways to index the positroid cells of $\mathrm{Gr}_{k,n}^{\geq 0}$ [19], including *decorated permutations* π and *plabic graphs* G. Accordingly, we also use the notation S_π and S_G.

Definition 2.3. Let $\mathcal{M}$ be a positroid and choose $C \in S_{\mathcal{M}}$. The *decorated permutation* π of $\mathcal{M}$ is defined by $\pi_i := j$, where j is the label of the first column c_j of C such that c_i lies in the span of $c_{i+1}, c_{i+2}, \ldots, c_j$ (where indices are modulo n). If $\pi_i = i$, then i is a *loop* if c_i is the zero vector and is a *coloop* otherwise.

Definition 2.4. Let G be a *plabic graph*, *i.e.* a planar bipartite graph embedded in a disk, with black vertices $1, 2, \ldots, n$ on the boundary of the disk. An *almost perfect matching* M of G is a collection of edges which covers each internal vertex of G exactly once. The *boundary* of M, denoted ∂M, is the set of boundary vertices covered by M. The positroid associated to G is the collection $\mathcal{M} = \mathcal{M}(G) := \{\partial M : M \text{ an almost perfect matching of } G\}$.

See Figure 1 for examples of plabic graphs. We will always assume that plabic graphs are *reduced* [19, Definition 12.5], a technical condition meaning roughly that among all plabic graphs with the same positroid, we consider only those with the fewest possible faces.

2.2 The amplituhedron

In this section we discuss the *(tree) amplituhedron*, which was introduced by Arkani-Hamed and Trnka [3]. In what follows, we fix positive integers k, m, n with $k + m \leq n$. We let $\mathrm{Mat}_{n,p}^{>0}$ denote the set of $n \times p$ matrices whose maximal minors are positive.

Definition 2.5. Let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$. Then Z induces a map $\widetilde{Z}\colon \mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$ which maps $C \mapsto CZ$. The *(tree) amplituhedron* $\mathcal{A}_{n,k,m}^{Z}$ is the image $\widetilde{Z}(\mathrm{Gr}_{k,n}^{\geq 0})$ inside $\mathrm{Gr}_{k,k+m}$.

The fact that Z has positive maximal minors ensures that $\widetilde{Z}$ is well defined [3]. The amplituhedron $\mathcal{A}_{n,k,m}^{Z}$ has full dimension km inside $\mathrm{Gr}_{k,k+m}$. In special cases the amplituhedron recovers familiar objects: $\mathcal{A}_{k+m,k,m}^{Z}$ is isomorphic to $\mathrm{Gr}_{k,n}^{\geq 0}$, as in this case Z is a square matrix; $\mathcal{A}_{n,1,m}^{Z}$ is a *cyclic polytope* in projective space $\mathbb{P}^m$ [26]; and $\mathcal{A}_{n,k,1}^{Z}$ can be identified with the complex of bounded faces of a cyclic hyperplane arrangement [9].

We will consider the restriction of the $\widetilde{Z}$-map to positroid cells in $\mathrm{Gr}_{k,n}^{\geq 0}$. Recall the definition of positroid tile[3] from Definition 1.1, with ϕ taken to be the $\widetilde{Z}$ map.

Definition 2.6. Let $Z \in \mathrm{Mat}_{n,k+m}^{>0}$. Given a positroid cell S_π of $\mathrm{Gr}_{k,n}^{\geq 0}$, we define the *open Grasstope* $Z_\pi^\circ := \widetilde{Z}(S_\pi)$ and the *Grasstope*[4] $Z_\pi := \widetilde{Z}(\overline{S_\pi})$.

[3] In the literature, *e.g.* [15], these are sometimes called "generalized triangles".
[4] This is short for "Grassmann polytope", language introduced in [12].

By [12, Proposition 15.2], $\widetilde{Z}(\overline{S}_\pi) = \overline{\widetilde{Z}(S_\pi)}$, so the Grasstope Z_π is the closure of the open Grasstope Z_π°. If $k = 1$ and $m = 2$, the amplituhedron $\mathcal{A}_{n,1,2}^Z$ is a convex n-gon in $\mathbb{P}^2$. Its positroid tiles are exactly the triangles using 3 vertices of the polygon.

Images of positroid cells under the map $\widetilde{Z}$ have been studied since [3], which conjectured that the images of certain *BCFW* collections of $4k$-dimensional cells in $\mathrm{Gr}_{k,n}^{\geq 0}$ give a positroid tiling of the amplituhedron $\mathcal{A}_{n,k,4}^Z$.

Remark 2.7. It is believed that many combinatorial properties of $\mathcal{A}_{n,k,m}^Z$ are independent of Z, for example, whether or not Z_π is a positroid tile. We will see that this is true in Theorem 3.1 for $m = 2$. It is also believed that the collections of positroids $\{\mathcal{M}\}$ indexing a positroid tiling of $\mathcal{A}_{n,k,m}^Z$ should be independent of Z.

2.2.1 Twistor coordinates and the sign stratification of $\mathcal{A}_{n,k,m}^Z$

Our results on $\mathcal{A}_{n,k,2}^Z$ will use functions on $\mathrm{Gr}_{k,k+2}$ called *twistor coordinates* [2, 3].

Definition 2.8. Choose $Z \in \mathrm{Mat}_{n,k+m}^{>0}$ and denote its rows by $Z_1, \ldots, Z_n \in \mathbb{R}^{k+m}$. Given a matrix Y with rows $Y_1, \ldots, Y_k$ representing an element of $Gr_{k,k+m}$, and $i_1, \ldots, i_m$ a sequence of elements of $[n]$, the *twistor coordinate* is defined to be

$$\langle Y i_1 i_2 \cdots i_m \rangle := \langle Y_1, \ldots, Y_k, Z_{i_1}, \ldots, Z_{i_m} \rangle,$$

the determinant of the $(k+m) \times (k+m)$ matrix with rows $Y_1, \ldots, Y_k, Z_{i_1}, \ldots, Z_{i_m}$.

Note that the twistor coordinates of $Y \in \mathrm{Gr}_{k,k+m}$ are defined only up to a common scalar multiple. It follows from [9, Lemma 3.10, Proposition 3.12] that an element of $\mathrm{Gr}_{k,k+m}$ is uniquely determined by its twistor coordinates. In fact, [9] shows that $\mathrm{Gr}_{k,k+m}$ can be embedded into $\mathrm{Gr}_{m,n}$ so that the twistor coordinate $\langle Y i_1 \cdots i_m \rangle$ is the pullback of the Plücker coordinate $p_{i_1,\ldots,i_m}$ in $\mathrm{Gr}\, m, n$.

Since $Y \in \mathrm{Gr}_{k,k+m}$ is uniquely determined by its twistor coordinates, we can stratify $\mathcal{A}_{n,k,m}^Z \subset \mathrm{Gr}_{k,k+m}$ by the signs of the twistor coordinates. This was done in [9] when $m = 1$. Moreover, this sign stratification is closely related to the *oriented matroid stratification* of the Grassmannian $\mathrm{Gr}_{m,n}$ using the embedding $\mathrm{Gr}_{k,k+m} \hookrightarrow \mathrm{Gr}_{m,n}$ mentioned above.

Definition 2.9 (Amplituhedron chambers)**.** Fix positive $k < n$ and m such that $k + m \leq n$. Let $\sigma = (\sigma_{i_1,\ldots,i_m}) \in \{0, +, -\}^{\binom{n}{m}}$ be a nonzero sign vector, considered[5] modulo multiplication by ± 1. Set

$$\mathcal{A}_{n,k,m}^{Z,\sigma} := \{Y \in \mathcal{A}_{n,k,m}^Z \mid \mathrm{sign}\langle Y i_1 \cdots i_m \rangle = \sigma_{i_1,\ldots,i_m}\}.$$

[5] Plücker and twistor coordinates are defined only up to multiplication by a common scalar.

We call $\mathcal{A}^{Z,\sigma}_{n,k,m}$ an *(amplituhedron) sign stratum.* Clearly

$$\mathcal{A}^{Z}_{n,k,m} = \bigsqcup_{\sigma} \mathcal{A}^{Z,\sigma}_{n,k,m}.$$

If $\sigma \in \{+,-\}^{\binom{n}{m}}$, we call $\mathcal{A}^{Z,\sigma}_{n,k,m}$ an open *(amplituhedron) chamber*.[6]

For $m = 1$, all strata are nonempty [9, Definition 5.2], but this is not true for $m > 1$. Moreover, whether or not $\mathcal{A}^{\sigma}_{n,k,m}(Z)$ is empty depends on Z.

Definition 2.10. We say that a sign vector σ (or sign stratum $\mathcal{A}^{\sigma}_{n,k,m}$) is *realizable* for $\mathcal{A}^{\sigma}_{n,k,m}$ if $\mathcal{A}^{Z,\sigma}_{n,k,m}$ is nonempty for some $Z \in \mathrm{Mat}^{>0}_{n,k+m}$.

Previous work on the $m = 2$ amplituhedron. The original amplituhedron paper [3] gave a conjectural positroid tiling of $\mathcal{A}^{Z}_{n,k,2}$. A BCFW-style recursion for producing tilings of $\mathcal{A}^{Z}_{n,k,2}$ was conjectured in [10] and proved in [5]. A conjectural classification of $m = 2$ positroid tiles was given in [15]. A conjectural description of the boundaries of the $m = 2$ amplituhedron was given in [14]. In another direction, [2] gave a conjectural description of $\mathcal{A}^{Z}_{n,k,2}$ in terms of sign flips; [2] and independently [10] proved one direction.

2.3 The hypersimplex and positroid polytopes

In this section, we review the necessary background material on positroid polytopes. Throughout, for $x \in \mathbb{R}^n$ and $I \subset [n]$, we use the notation $x_I := \sum_{i \in I} x_i$.

Definition 2.11. Let $e_I := \sum_{i \in I} e_i \in \mathbb{R}^n$, where $\{e_1, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$. The $(k+1,n)$*-hypersimplex* is $\Delta_{k+1,n} := \mathrm{convex}(e_I : I \in \binom{[n]}{k})$.

The torus $T = \mathbb{R}^n \setminus \{0\}$ acts on $k \times n$ matrices by rescaling columns. This descends to an $(n-1)$-dimensional torus action on $Gr(k,n)$. We let TA denote the orbit of A under the action of T, and $\overline{TA}$ its closure.

The *moment map* from the Grassmannian $\mathrm{Gr}_{k+1,n}$ to $\mathbb{R}^n$ is defined as follows.

Definition 2.12 (The moment map)**.** Let A be a $(k+1) \times n$ matrix representing a point of $\mathrm{Gr}_{k+1,n}$. The *moment map* $\mu\colon \mathrm{Gr}_{k+1,n} \to \mathbb{R}^n$ is defined by

$$\mu(A) = \frac{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2 e_I}{\sum_{I \in \binom{[n]}{k+1}} |p_I(A)|^2}.$$

[6] We borrow the word "chamber" from the theory of hyperplane arrangements.

It is well-known that the image of the Grassmannian $\mathrm{Gr}_{k+1,n}$ under the moment map is the hypersimplex $\Delta_{k+1,n}$. If one restricts the moment map to $\mathrm{Gr}_{k+1,n}^{\geq 0}$ then the image is again the hypersimplex $\Delta_{k+1,n}$ [29, Proposition 7.10].

In general, it follows from classical work of Atiyah [4] and Guillemin–Sternberg [8] that the image $\mu(\overline{TA})$ is a convex polytope, whose vertices are the images of the torus-fixed points, *i.e.* the vertices are the points e_I such that $p_I(A) \neq 0$. This motivates the notion of *matroid polytope*. Recall that any full rank $(k+1) \times n$ matrix A gives rise to a matroid $\mathcal{M}(A) = ([n], \mathcal{B})$, where $\mathcal{B} = \{I \in \binom{[n]}{k+1} \mid p_I(A) \neq 0\}$.

Definition 2.13. Given a matroid $\mathcal{M} = ([n], \mathcal{B})$, the (basis) *matroid polytope* of $\mathcal{M}$ is $\Gamma_{\mathcal{M}} := \mathrm{convex}\{e_B : B \in \mathcal{B}\} \subset \mathbb{R}^n$.

Here, we are interested in *positroid polytopes*, that is, matroid polytopes $\Gamma_{\mathcal{M}}$ where $\mathcal{M}$ is a positroid. They arise as $\mu(\overline{TA})$ where A is a totally nonnegative matrix. Of more interest to us, they are also moment map images of positroid cells.

Proposition 2.14 ([29])**.** *Let $\mathcal{M}$ be a positroid. Then $\Gamma_{\mathcal{M}} = \mu(\overline{S_{\mathcal{M}}}) = \overline{\mu(S_{\mathcal{M}})}$.*

The interior $\Gamma_{\mathcal{M}}^{\circ}$ of a positroid polytope is the moment map image of $S_{\mathcal{M}}$; we call $\Gamma_{\mathcal{M}}^{\circ}$ the *open* positroid polytope. As in the amplituhedron case, we will focus on those positroid polytopes which are positroid tiles (*i.e.* $(n-1)$-dimensional positroid polytopes $\Gamma_{\mathcal{M}}$ where μ is injective on $S_{\mathcal{M}}$).

Theorem 2.15 ([16])**.** *A positroid polytope Γ_G is a tile for $\Delta_{k+1,n}$ if and only if G is a tree.*

2.4 T-duality

In [16], a curious parallel was conjectured between positroid tilings of $\mathcal{A}_{n,k,2}^{Z}$ and $\Delta_{k+1,n}$ involving *T-duality* [10].

Definition 2.16 (T-duality)**.** Let $\pi = a_1 a_2 \cdots a_n$ be a loopless decorated permutation (written in one-line notation). The *T-dual* decorated permutation is $\hat{\pi} = a_n a_1 a_2 \cdots a_{n-1}$, where any fixed points in $\hat{\pi}$ are declared to be loops.

T-duality induces a bijection from loopless positroid cells $S_\pi \in \mathrm{Gr}_{k+1,n}^{\geq 0}$ to coloopless positroid cells $S_{\hat{\pi}} \in \mathrm{Gr}_{k,n}^{\geq 0}$ [16, Lemma 5.2]. It thus also induces a bijection between loopless positroid polytopes $\Gamma_\pi \in \Delta_{k+1,n}$ and coloopless Grasstopes $Z_{\hat{\pi}} \in \mathcal{A}_{n,k,m}^{Z}$; we say Γ_π and $Z_{\hat{\pi}}$ are *T-dual*.

Conjecture 2.17. *A positroid polytope Γ_π is a positroid tile for $\Delta_{k+1,n}$ if and only if the T-dual Grasstope $Z_{\hat{\pi}}$ is a tile for $\mathcal{A}_{n,k,2}^{Z}$. Further, a collection $\{\Gamma_\pi\}$ of positroid polytopes is a positroid tiling of $\Delta_{k+1,n}$ if and only if the collection $\{Z_{\hat{\pi}}\}$ of T-dual Grasstopes is a positroid tiling of $\mathcal{A}_{n,k,2}^{Z}$ for all $Z \in \mathrm{Mat}_{n,k+2}^{>0}$.*

We prove this conjecture in Theorem 3.1 and Theorem 3.10.

3 Results

Our first main result is a characterization of the positroid tiles for $\mathcal{A}^Z_{n,k,2}$.[7] A *bicolored subdivision* of the n-gon P_n is a choice of noncrossing diagonals decomposing P_n into subpolygons. Each subpolygon is colored grey or white so that no two subpolygons of the same color share an edge; we view the middle diagram of Figure 1 as a decomposition of P_9 into a grey quadrilateral, a grey pentagon, and two white triangles. The *area* of a subdivision is the total number of grey triangles in any triangulation of the grey subpolygons.

Theorem 3.1. *Positroid tiles of $\Delta_{k+1,n}$ and $\mathcal{A}^Z_{n,k,2}$ are in bijection with bicolored subdivisions of P_n with area k. We read Γ_G and $Z_{\widehat{G}}$ off of the same bicolored subdivision σ of the n-gon by first choosing any triangulation τ of the grey polygons in σ. Then $G := G(\tau)$ is the dual plabic tree[8] of σ and $\widehat{G} := \widehat{G}(\tau)$ is obtained by placing a "tripod" in each grey triangle (see Figure 1).*

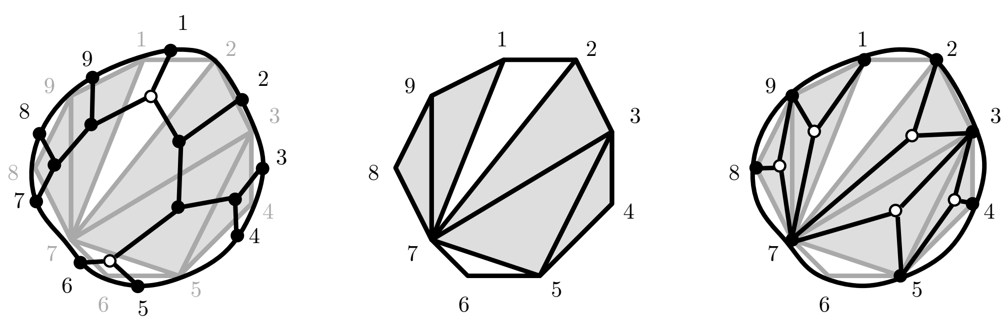

Figure 1: A triangulation of a bicolored subdivision σ with area 5, with the dual graph $G(\tau)$ to its left, and the T-dual graph $\widehat{G}(\tau)$ to its right.

Using Theorem 3.1, we can enumerate the positroid tiles.

Proposition 3.2. *Positroid tiles of Δ_{k+1} and $\mathcal{A}^Z_{n,k,2}$ are in bijection with separable permutations on $[n-1]$ with k descents. They are enumerated by $R_{k,n-2}$ from [23, A175124].*

From a bicolored subdivision σ, we also obtain inequality descriptions of the corresponding T-dual positroid tiles. Interestingly, the inequality descriptions use exactly the same combinatorial information, the area statistic. Given two positive numbers $a, b \in [n]$, the *cyclic interval* $[a, b]$ is defined to be $\{a, a+1, \ldots, b-1, b\}$ if $a \leq b$ and $\{a, a+1, \ldots, n, 1, \ldots, b\}$ otherwise.

Theorem 3.3 (T-dual inequalities). *Let $\sigma, \tau, G(\tau), \widehat{G}(\tau)$ be as in Theorem 3.1. Let $h \to j$ be an arc in a grey triangle of τ, with $h < j$. Then*

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[7] The positroid tiles for $\Delta_{k+1,n}$ were characterized in [16] using *plabic trees*.

[8] technically, one should insert degree 2 white vertices to get a bipartite tree

1. $\mathrm{area}(h \to j) + 1 > x_{[h,j-1]} > \mathrm{area}(h \to j)$ *for* $x \in \Gamma^{\circ}_{G(\tau)}$,

2. $(-1)^{\mathrm{area}(h \to j)} \langle Yhj \rangle > 0$ *for* $Y \in Z^{\circ}_{\widehat{G}(\tau)}$,

where $\mathrm{area}(h \to j)$ *is the number of grey triangles to the left of* $h \to j$*. Letting* $h \to j$ *vary over all arcs of grey triangles of* τ*, these inequalities cut out the positroid tiles* $\Gamma^{\circ}_{G(\tau)}$ *and* $Z^{\circ}_{\widehat{G}(\tau)}$.

For example, for the triangulation in Figure 1, $\mathrm{area}(3 \to 7) = 2$, so the inequality $3 > x_3 + x_4 + x_5 + x_6 > 2$ holds for $x \in \Gamma^{\circ}_{G(\tau)}$ and the inequality $\langle Y37 \rangle > 0$ holds for $Y \in Z_{\widehat{G}(\tau)}$. Theorem 3.3 can be extended to give a correspondence between *facets* of T-dual positroid tiles lying in the interior of $\Delta_{k+1,n}$ or $\mathcal{A}^{Z}_{n,k,2}$, which correspond to arcs of σ separating grey and white subpolygons [18, Theorem 9.10].

Using Theorem 3.3, we also prove a description of $\mathcal{A}^{Z}_{n,k,2}$ conjectured by [2].

Theorem 3.4 (Sign-flip characterization of $\mathcal{A}_{n,k,2}$)**.** *Fix* $k < n$ *and* $Z \in \mathrm{Mat}^{>0}_{n,k+2}$*. Let*

$$\begin{aligned}\mathcal{F}^{\circ}_{n,k,2}(Z) :=\{Y \in Gr_{k,k+2} \mid \langle Yi\; i+1 \rangle > 0 \textit{ for } 1 \le i \le n-1, \textit{ and } (-1)^{k} \langle Y1\; n \rangle > 0,\\ \textit{and the sequence } \langle Y1\;2 \rangle, \langle Y1\;3 \rangle, \dots, \langle Y1\;n \rangle \textit{ has exactly } k \textit{ sign flips}\}.\end{aligned}$$

Then $\mathcal{A}^{Z}_{n,k,2} = \overline{\mathcal{F}^{\circ}_{n,k,2}(Z)}$.

We now turn to our results on tilings. In order to show the bijection between positroid tilings of $\Delta_{k+1,n}$ and $\mathcal{A}^{Z}_{n,k,2}$, we consider a refinement of all positroid tilings on both sides. This decomposes $\Delta_{k+1,n}$ and $\mathcal{A}^{Z}_{n,k,2}$ into pieces that are smaller than positroid tiles; we will give a correspondence between these smaller pieces.

On the hypersimplex side, we consider a triangulation of $\Delta_{k+1,n}$ [13, 24, 27], *i.e.* a decomposition into simplices, which refines all positroid tilings. The maximal simplices Δ_w of this triangulation are indexed by permutations w of $[n]$ with k descents and $w_n = n$. Such permutations are enumerated by the *Eulerian number* $E_{k,n-1}$ [25]. We will describe the simplices Δ_w using the following permutation statistic.

Definition 3.5. Let $w \in S_n$. We call a letter i in w a *cyclic descent*[9] if either $i \ge 2$ and $w^{-1}(i) < w^{-1}(i-1)$ or if $i = 1$ and $w^{-1}(1) < w^{-1}(n)$. We let $\mathrm{cDes_L}(w)$ denote the set of cyclic descents of w and let $D_{k+1,n}$ denote the set of permutations $w \in S_n$ with $k+1$ cyclic descents and $w_n = n$. Note that $|D_{k+1,n}| = E_{k,n-1}$.

For $w \in D_{k+1,n}$, let $w^{(a)}$ denote the cyclic rotation of w ending at a. We define $I_r := \mathrm{cDes_L}(w^{(r-1)})$. The *$w$-simplex* $\Delta_w \subseteq \Delta_{k+1,n}$ is the simplex with vertices $e_{I_1}, \dots, e_{I_n}$.

Example 3.6. Let $w = 324156$ in one-line notation. Then w has cyclic descents $\{1,2,3\} = I_1$. The rotation of w ending at 1 is 563241, which has cyclic descents $I_2 = \{2,3,5\}$. The rotation of w ending at 2 is 415632, which has cyclic descents $I_3 = \{1,3,4\}$.

[9] These are closely related to *left descents* of w, also called *recoils* in the literature.

Now, we turn to the amplituhedron side. We will consider the decomposition of $\mathcal{A}^Z_{n,k,2}$ into (closures of) amplituhedron chambers. This is a refinement of all positroid tilings because the tiles for $\mathcal{A}^Z_{n,k,2}$ are cut out by hypersurfaces $\langle Yij \rangle = 0$. We define one amplituhedron chamber for each $w \in D_{k+1,n}$.

Definition 3.7 (w-chambers)**.** Let $w \in D_{k+1,n}$ and let the vertices of Δ_w be $e_{I_1}, \dots, e_{I_n}$, as in Definition 3.5. Then the *open w-chamber* $\mathcal{A}^{Z,w}_{n,k,2}$ consists of $Y \in \mathrm{Gr}_{k,k+2}$ such that

$$\begin{aligned} \operatorname{sgn}\langle Yaj \rangle &= (-1)^{|I_a \cap [a,j-1]|-1} && \text{for } j > a \\ \operatorname{sgn}\langle Yaj \rangle &= (-1)^{|I_a \cap [a,j-1]|-k-1} && \text{for } j < a. \end{aligned}$$

The *(closed) w-chamber* is the closure $\widehat{\Delta}^Z_w := \overline{\mathcal{A}^{Z,w}_{n,k,2}}$.

Continuing Example 3.6, $\langle Y13 \rangle < 0$ on $\widehat{\Delta}^Z_w$, since $|\{1,2,3\} \cap [1,2]| - 1$ is odd.

The w-chambers are a distinguished subset of the amplituhedron chambers. We show that they are precisely the *realizable* chambers, in the sense of Definition 2.10.

Theorem 3.8. *An amplituhedron chamber is realizable if and only if it is an open w-chamber. In particular, the amplituhedron $\mathcal{A}^Z_{n,k,2}$ is the union of the w-chambers $\widehat{\Delta}^Z_w$, just as $\Delta_{k+1,n}$ is the union of the w-simplices Δ_w.*

Further, a w-simplex Δ_w and the corresponding w-chamber $\widehat{\Delta}^Z_w$ behave the same way with respect to containment in positroid tiles.

Theorem 3.9. *Let $\widehat{\Delta}^Z_w \neq \emptyset$ and let $Z_{\widehat{\pi}}$ be a positroid tile for $\mathcal{A}^Z_{n,k,2}$. Then $\widehat{\Delta}^Z_w$ is contained in $Z_{\widehat{\pi}}$ if and only if the w-simplex Δ_w is contained in the T-dual positroid tile Γ_π.*

Together, Theorem 3.8 and Theorem 3.9 give our main result on tilings.

Theorem 3.10 (Tilings of $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}$ are T-dual)**.** *The collection $\mathcal{C} = \{\Gamma_\pi\}$ is a positroid tiling of $\Delta_{k+1,n}$ if and only if for all $Z \in \mathrm{Mat}^{>0}_{n,k+2}$, the collection of T-dual Grasstopes $\widehat{\mathcal{C}} = \{Z_{\widehat{\pi}}\}$ is a positroid tiling of $\mathcal{A}^Z_{n,k,2}$.*

(k,n)	$(1,n)$	$(2,4)$	$(2,5)$	$(2,6)$	$(2,7)$	$(2,8)$	$(3,5)$	$(3,6)$	$(3,7)$
# pos. tilings	C_{n-2}	1	5	120	3073	6 443 460	1	14	3073

Table 1: Number of tilings of $\Delta_{k+1,n}$ and $\mathcal{A}^Z_{n,k,2}$ in bijection by Theorem 3.10 in the known cases. The *Catalan number* C_{n-2} is the number of triangulations of an n-gon.

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