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0-Hecke–Clifford Modules From Diagrams

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Abstract. We construct modules of 0-Hecke algebras and 0-Hecke–Clifford algebras from fillings of diagrams of boxes in the plane. We apply this general construction method to solve a problem posed in (N. Jing and Y. Li, 2015), to define a new basis of the peak algebra analogous to quasiymmetric Schur functions, and to manifest a new connection between quasiymmetric Schur functions and Schur Q-functions.

Keywords: 0-Hecke algebra, 0-Hecke–Clifford algebra, quasisymmetric Schur functions, Schur Q-functions, quasisymmetric Schur Q-functions

1 Introduction

1.1 Overview

The Hecke algebra is a deformation of the group algebra of the symmetric group by a parameter q, where the specialization $q = 0$ is known as the 0-Hecke algebra. The Grothendieck group of the 0-Hecke algebra is isomorphic [5] to the Hopf algebra QSym of quasisymmetric functions, an important and widely-studied algebra that contains the symmetric functions. Under this isomorphism, called the *quasisymmetric characteristic*, the images of the simple 0-Hecke modules are precisely the *fundamental quasisymmetric functions*, a basis of QSym introduced in [6] as enumerating functions of P-partitions.

Recently, several authors have constructed modules of 0-Hecke algebras whose quasisymmetric characteristics are interesting bases of QSym. Such modules were constructed for the dual immaculate functions in [3], for the quasisymmetric Schur functions in [17], for the extended Schur functions in [14], and for the Young row-strict quasisymmetric Schur functions in [1]. There has also been recent interest in the structure of such modules, *e.g.*, in [9] all four of these families of 0-Hecke modules are interpreted in terms of 0-Hecke modules defined on intervals in the weak Bruhat order.

The Hecke–Clifford (super)algebra was introduced in [13], combining the Hecke algebra and the Clifford algebra. The specialization $q = 0$ is called the 0-Hecke–Clifford algebra. The Grothendieck group of 0-Hecke–Clifford modules is isomorphic [4] to the *peak algebra* Peak, a subalgebra of QSym introduced in [16]. Under this isomorphism, which

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we call the *peak characteristic*, the images of the simple 0-Hecke–Clifford (super)modules are (certain scalings of) the *peak functions*, a basis of Peak introduced in [16] as enumerating functions of enriched P-partitions.

The representation theory of 0-Hecke–Clifford algebras has been further developed in [11]. A motivation was the question of finding 0-Hecke–Clifford modules whose peak characteristics are the *quasisymmetric Schur Q-functions*, a basis of Peak introduced in [8] analogous to the dual immaculate basis [2] of QSym.

In this extended abstract, we summarize results from [15]. We introduce a general method for constructing 0-Hecke modules and 0-Hecke–Clifford modules from certain standard fillings of box diagrams in the plane. The images of these modules under the quasisymmetric (respectively, peak) characteristic map are a sum of fundamental quasisymmetric (respectively, peak) functions determined by the descent sets of the standard fillings. An advantage of this framework is that it can be applied to families of standard tableaux of various shapes (*e.g.*, straight-shape, skew, shifted, *etc.*), from which many important families of symmetric and quasisymmetric functions are naturally generated.

We apply this construction method to produce 0-Hecke–Clifford modules whose peak characteristics are the quasisymmetric Schur Q-functions, answering the question raised in [8]. We moreover use this method to construct modules whose peak characteristics form a new basis of Peak analogous to the quasisymmetric Schur basis [7] of QSym. As a further application, we establish a new connection between the quasisymmetric Schur functions and the *Schur Q-functions*: an important family of symmetric functions related to projective representations of symmetric groups.

1.2 Quasisymmetric functions and peak functions

A *composition* is a finite sequence $\alpha = (\alpha_1, \ldots, \alpha_k)$ of positive integers. The integers α_i are called the *parts* of α. When the parts of α sum to n, α is called a *composition of* n and we write $\alpha \vDash n$.

Let $[n]$ denote the set $\{1, \ldots, n\}$. The *descent set* $\mathrm{Des}(\alpha)$ of $\alpha = (\alpha_1, \ldots, \alpha_k) \vDash n$ is the subset $\{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}$ of $[n-1]$. The map Des taking a composition to its descent set is a bijection between compositions of n and subsets of $[n-1]$. Given $X \subseteq [n-1]$, let $\mathrm{comp}_n(X)$ denote the composition of n whose descent set is X. For example, $\mathrm{Des}(2,2,1,2) = \{2,4,5\} \subseteq [7]$ and $\mathrm{comp}_8(\{1,3,6\}) = (1,2,3,2) \vDash 8$.

Denote by $\mathbb{C}[[x_1, x_2, \ldots]]$ the Hopf algebra of formal power series of bounded degree in infinitely many commuting variables. The Hopf algebras Sym of symmetric functions and QSym of quasisymmetric functions are subalgebras of $\mathbb{C}[[x_1, x_2, \ldots]]$, and Sym is a subalgebra of QSym. For $\alpha \vDash n$, the *fundamental quasisymmetric function* F_α is defined by

$$F_\alpha = \sum x_{i_1} x_{i_2} \cdots x_{i_n},$$

where the sum is over sequences $i_1, \ldots, i_n$ of integers satisfying $1 \leq i_1 \leq \cdots \leq i_n$ and

$i_j < i_{j+1}$ whenever $j \in \mathrm{Des}(\alpha)$. The functions F_α form a basis for QSym.

Example 1.1. If $\alpha = (1,2,1)$ then $\mathrm{Des}(\alpha) = \{1,3\}$, and we have

$$F_{(1,2,1)} = \sum_{1 \leq i < j < k} x_i x_j^2 x_k + \sum_{1 \leq i < j < k < \ell} x_i x_j x_k x_\ell.$$

A *peak composition* is a composition whose parts (except possibly the last part) are all greater than 1. Under the map Des, peak compositions of n are in bijection with subsets of $[n-1]$ that do not contain 1 and do not contain any pair of consecutive integers. Subsets satisfying this condition are called *peak sets*. For a set X of positive integers, the *peak set of* X, denoted $\mathrm{Peak}(X)$, is the set $\{i : i > 1, i \in X \text{ and } i-1 \notin X\}$. For any composition α, the *peak set of* α is the set $\mathrm{Peak}(\mathrm{Des}(\alpha))$, which we write as $\mathrm{Peak}(\alpha)$ for short. The *peak functions* K_α [16] are defined by

$$K_\alpha = 2^{|\mathrm{Peak}(\alpha)|+1} \sum_{\beta : \mathrm{Peak}(\alpha) \subseteq \mathrm{Des}(\beta) \Delta (\mathrm{Des}(\beta)+1)} F_\beta,$$

where $X \Delta Y$ denotes the symmetric difference of sets X and Y.

Since K_α depends only on $\mathrm{Peak}(\alpha)$, the peak functions can be indexed by peak compositions. As α ranges over peak compositions, the K_α form a basis for Peak.

1.3 0-Hecke algebras and 0-Hecke–Clifford algebras

The 0-Hecke algebra $H_n(0)$ is the $\mathbb{C}$-algebra with generators $\pi_1, \ldots, \pi_{n-1}$ and relations

$$\begin{aligned} \pi_i^2 &= -\pi_i && \text{for all } 1 \leq i \leq n-1, \\ \pi_i \pi_j &= \pi_j \pi_i && \text{for all } i, j \text{ such that } |i-j| \geq 2, \\ \pi_i \pi_{i+1} \pi_i &= \pi_{i+1} \pi_i \pi_{i+1} && \text{for all } 1 \leq i \leq n-2. \end{aligned} \tag{1.1}$$

For an $H_n(0)$-module $\mathbf{N}$, let $[\mathbf{N}]$ denote its isomorphism class. Let $G_0(H_n(0))$ denote the Grothendieck group of the category of finite-dimensional $H_n(0)$-modules, and let $\mathcal{G} = \bigoplus_{n \geq 0} \mathcal{G}_0(H_n(0))$. There are 2^{n-1} simple $H_n(0)$-modules; these may be indexed by the compositions of n. Let $\mathbf{F}_\alpha$ denote the simple $H_n(0)$-module corresponding to the composition α. In [5], an algebra isomorphism $ch \colon \mathcal{G} \to \mathrm{QSym}$ is defined by

$$ch([\mathbf{F}_\alpha]) = F_\alpha.$$

The quasisymmetric function $ch([\mathbf{N}])$ is called the *quasisymmetric characteristic* of $\mathbf{N}$.

The 0-*Hecke–Clifford algebra* $HCl_n(0)$ is the algebra generated by $\pi_1, \ldots, \pi_{n-1}$ and $c_1, \ldots, c_n$, where the π_i generate the 0-Hecke algebra $H_n(0)$ and the c_j generate the *Clifford algebra* Cl_n, *i.e.*, satisfy the relations

$$c_i^2 = -1 \quad \text{for all } 1 \leq i \leq n \qquad \text{and} \qquad c_i c_j = -c_j c_i \quad \text{for all } i, j \text{ such that } i \neq j \tag{1.2}$$

and the π_i and c_j additionally satisfy the cross-relations

$$\begin{aligned} \pi_i c_j &= c_j \pi_i && \text{for } j \neq i, i+1, \\ \pi_i c_i &= c_{i+1} \pi_i && \text{for all } 1 \leq i \leq n-1, \\ (\pi_i + 1) c_{i+1} &= c_i(\pi_i + 1) && \text{for all } 1 \leq i \leq n-1. \end{aligned} \tag{1.3}$$

The 0-Hecke–Clifford algebra is moreover a superalgebra: it is graded by $\mathbb{Z}_2$ with each π_i having degree 0 and each c_j having degree 1.

Bergeron, Hivert and Thibon [4] defined a family of $HCl_n(0)$-modules $\mathbf{M}_\alpha$, and extracted the simple $HCl_n(0)$-modules from these. Letting $\widetilde{\mathcal{G}}$ denote $\bigoplus_{n\geq 0} \mathcal{G}_0(HCl_n(0))$, they defined an algebra isomorphism $\widetilde{ch}\colon \widetilde{\mathcal{G}} \to \mathrm{Peak}$, which we call the *peak characteristic*. Under this map, one has

$$\widetilde{ch}([\mathbf{M}_\alpha]) = K_{\mathrm{comp}_n(\mathrm{Peak}(\alpha))}. \tag{1.4}$$

2 Diagram modules

A *diagram* D is a collection of n boxes in the coordinate plane. A *standard filling* of D is a bijective assignment of the numbers $1, \ldots, n$ to the boxes of D, typically depicted by writing each number inside its corresponding box. The numbers in the boxes are called the *entries* of a standard filling. Let $\mathsf{StdFill}(D)$ denote the set of all $n!$ standard fillings of D. Fix any total ordering on the boxes of D and call this the *reading order*. For $T \in \mathsf{StdFill}(D)$, define the *reading word* $\mathrm{rw}(T)$ to be the word obtained by listing the entries of T in reading order from left to right.

Let $\mathsf{StdTab}(D)$ denote an arbitrary subset of $\mathsf{StdFill}(D)$. Next we define a function $\mathrm{Des}\colon \mathsf{StdTab}(D) \to [n-1]$ by

$$\mathrm{Des}(T) = \{i \in T : i \text{ is to the right of } i+1 \text{ in } \mathrm{rw}(T)\}.$$

We call $\mathrm{Des}(T)$ the *descent set* of T, and call its elements the *descents* of T. The elements of $[n-1]$ not in $\mathrm{Des}(T)$ are called the *ascents* of T. Given $1 \leq i \leq n-1$ and $T \in \mathsf{StdFill}(D)$, let $s_i T$ be the standard filling of D obtained by exchanging the entries i and $i+1$ of T. We define an ascent of T to be *attacking* if $s_i T \notin \mathsf{StdTab}(D)$, and *nonattacking* otherwise. We also say T has an *ascent in positions* (r, s) if $1 \leq r < s \leq n$ and $\mathrm{rw}(T)_s - \mathrm{rw}(T)_r = 1$.

Define $\mathsf{StdTab}(D)$ to be *ascent-compatible* if whenever $T, T' \in \mathsf{StdTab}(D)$ both have an ascent in positions (r, s), this ascent is attacking in T if and only if it is attacking in T'.

Example 2.1. Let $n = 3$ and let D be a row of three boxes, with reading order from left to right. Let $\mathsf{StdTab}(D) = \left\{ Q = \boxed{1}\boxed{2}\boxed{3}, R = \boxed{1}\boxed{3}\boxed{2}, S = \boxed{2}\boxed{3}\boxed{1}, T = \boxed{3}\boxed{2}\boxed{1} \right\}$. Here $\mathsf{StdTab}(D)$ is not ascent-compatible: we have $\mathrm{rw}(Q) = 123$ and $\mathrm{rw}(S) = 231$, but the ascent of Q in positions $(1,2)$ is attacking while the ascent of S in positions $(1,2)$ is nonattacking.

Ascent-compatibility allows us to define a family of $H_n(0)$-modules. Let $\mathbf{N}_{\mathsf{StdTab}(D)}$ denote the complex span of $\mathsf{StdTab}(D)$. Define operators $\pi_1, \ldots, \pi_{n-1}$ on $\mathbf{N}_{\mathsf{StdTab}(D)}$ by

$$\pi_i T = \begin{cases} -T & \text{if } i \in \mathrm{Des}(T), \\ 0 & \text{if } i \notin \mathrm{Des}(T) \text{ and } i \text{ is attacking}, \\ s_i T & \text{if } i \notin \mathrm{Des}(T) \text{ and } i \text{ is nonattacking}. \end{cases}$$

Theorem 2.2. *Let* $\mathsf{StdTab}(D)$ *be ascent-compatible. Then the operators* $\{\pi_i : 1 \leq i \leq n-1\}$ *define a* 0*-Hecke action on* $\mathbf{N}_{\mathsf{StdTab}(D)}$.

Define $\widetilde{\mathbf{N}}_{\mathsf{StdTab}(D)}$ to be the $HCl_n(0)$-(super)module induced from $\mathbf{N}_{\mathsf{StdTab}(D)}$:

$$\widetilde{\mathbf{N}}_{\mathsf{StdTab}(D)} = \mathrm{Ind}_{H_n(0)}^{HCl_n(0)} \mathbf{N}_{\mathsf{StdTab}(D)}.$$

A basis for $\widetilde{\mathbf{N}}_{\mathsf{StdTab}(D)}$ is given by $\{c_X T : X \subseteq [n], T \in \mathsf{StdTab}(D)\}$.

We now determine the peak characteristic $\widetilde{ch}([\widetilde{\mathbf{N}}_{\mathsf{StdTab}(D)}])$ in terms of $\mathsf{StdTab}(D)$. Define a partial ordering $\preceq$ on $\mathsf{StdTab}(D)$ by $S \preceq T$ if S can be obtained from T by applying a (possibly empty) sequence of the π_i operators, and extend arbitrarily to a total ordering $\preceq'$. If the elements of $\mathsf{StdTab}(D) = \{T_1, \ldots, T_m\}$ are ordered $T_1 \preceq' T_2 \preceq' \cdots \preceq' T_m$, then for each $1 \leq k \leq m$, define $HCl_n(0)$-submodules $\widetilde{\mathbf{N}}_k$ of $\widetilde{\mathbf{N}}_{\mathsf{StdTab}(D)}$ by

$$\widetilde{\mathbf{N}}_k = \mathrm{span}\{\mathrm{c_X T_j : X \subseteq [n], j \leq k}\}.$$

Also define $\widetilde{\mathbf{N}}_0 = \{0\}$. We thus have a filtration of $HCl_n(0)$-modules

$$\{0\} = \widetilde{\mathbf{N}}_0 \subset \widetilde{\mathbf{N}}_1 \subset \cdots \subset \widetilde{\mathbf{N}}_m = \widetilde{\mathbf{N}}_{\mathsf{StdTab}(D)}.$$

For each $1 \leq k \leq m$, the quotient module $\widetilde{\mathbf{N}}_k / \widetilde{\mathbf{N}}_{k-1}$ has basis $\{c_X T_k : X \subseteq [n]\}$. Define the *peak set* $\mathrm{Peak}(T)$ of $T \in \mathsf{StdTab}(D)$ to be $\mathrm{Peak}(\mathrm{Des}(T))$.

Lemma 2.3. *Let* $T_k \in \mathsf{StdTab}(D)$*. Then* $\widetilde{\mathbf{N}}_k / \widetilde{\mathbf{N}}_{k-1}$ *is isomorphic to* $\mathbf{M}_{\mathrm{comp}_n(\mathrm{Peak}(T_k))}$ *as* $HCl_n(0)$*-modules.*

This lemma is proved by first establishing explicit formulas for $\pi_i c_X T$ in $\widetilde{\mathbf{N}}_{\mathsf{StdTab}(D)}$ using the cross-relations (1.3). One can then show that in the quotient module $\widetilde{\mathbf{N}}_k / \widetilde{\mathbf{N}}_{k-1}$, these formulas match the corresponding formulas for $\pi_i c_X \varepsilon_\alpha$ in $\mathbf{M}_\alpha$ [4], where $\alpha = \mathrm{comp}_n(\mathrm{Peak}(T_k))$ and $\{c_X \varepsilon_\alpha : X \subseteq [n]\}$ is a basis for $\mathbf{M}_\alpha$. Theorem 2.4 below follows from (1.4) and Lemma 2.3.

Theorem 2.4. *Let D be a diagram in the plane with n boxes and* $\mathsf{StdTab}(D)$ *an ascent-compatible subset of* $\mathsf{StdFill}(D)$*. Then*

$$\widetilde{ch}([\widetilde{\mathbf{N}}_{\mathsf{StdTab}(D)}]) = \sum_{T \in \mathsf{StdTab}(D)} K_{\mathrm{comp}_n(\mathrm{Peak}(T))}.$$

One similarly obtains a formula for the quasisymmetric characteristics of the $H_n(0)$-modules $\mathbf{N}_{\mathsf{StdTab}(D)}$.

Theorem 2.5. *Let D be a diagram in the plane with n boxes and* $\mathsf{StdTab}(D)$ *an ascent-compatible subset of* $\mathsf{StdFill}(D)$*. Then*

$$ch([\mathbf{N}_{\mathsf{StdTab}(D)}]) = \sum_{T \in \mathsf{StdTab}(D)} F_{\mathrm{comp}_n(\mathrm{Des}(T))}.$$

3 Applications

3.1 0-Hecke modules for bases of QSym

Recently, $H_n(0)$-modules have been constructed whose images under the quasisymmetric characteristic map are the dual immaculate functions [3], quasisymmetric Schur functions [17], extended Schur functions [14] and Young row-strict quasisymmetric Schur functions [1]. All of these functions can be defined as sums of fundamental quasisymmetric functions indexed by the descent sets of certain families of tableaux of composition shape. Letting α be a composition of n, these are respectively the *standard immaculate tableaux* $\mathrm{SIT}(\alpha)$, *standard reverse composition tableaux* $\mathrm{SRCT}(\alpha)$, *standard extended tableaux* $\mathrm{SET}(\alpha)$, and *standard Young row-strict composition tableaux* $\mathrm{SYRT}(\alpha)$.

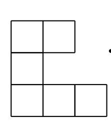
In each of [1, 3, 14, 17], generators $\pi_i + 1$ of $H_n(0)$ are used instead of π_i. One can define a variant $\widehat{\mathbf{N}}_{\mathsf{StdTab}(D)}$ of $\mathbf{N}_{\mathsf{StdTab}(D)}$ using generators $\pi_i + 1$, for which ascent-compatibility is replaced by an analogous *descent-compatibility* condition. All the aforementioned families of tableaux can straightforwardly be checked to be descent-compatible. Accordingly, we have

Theorem 3.1. *Let $\alpha \vDash n$. For* $\mathsf{StdTab}(D)$ *equal to, respectively,* $\mathrm{SIT}(\alpha)$, $\mathrm{SRCT}(\alpha)$, $\mathrm{SET}(\alpha)$, $\mathrm{SYRT}(\alpha)$*, the $H_n(0)$-module defined in, respectively, [3] for dual immaculate functions, [17] for quasisymmetric Schur functions, [14] for extended Schur functions, [1] for Young row strict quasisymmetric Schur functions, is precisely* $\widehat{\mathbf{N}}_{\mathsf{StdTab}(D)}$.

3.2 0-Hecke–Clifford modules for quasisymmetric Schur Q-functions

Jing and Li [8] introduced the quasisymmetric Schur Q-functions and asked whether a representation-theoretic interpretation in terms of 0-Hecke–Clifford modules can be given to these functions. We use the construction method developed in Section 2 to provide an answer.

The *diagram* of a composition α, denoted $D(\alpha)$, is the array of boxes with α_i boxes in row i, left-justified. We number rows from bottom to top, *e.g.*, if $\alpha = (3, 1, 2)$ then

$D(\alpha) =$  .

Let α be a peak composition of n. Define a *standard peak composition tableau* (SPCT) of *shape* α [8] to be a standard filling of $D(\alpha)$ satisfying the following conditions.

(SPCT1) Entries increase from left to right along each row.

(SPCT2) Entries increase from bottom to top in the first column.

(SPCT3) For every $1 \leq k \leq n$, the subdiagram of α consisting of the boxes with entries at most k is the diagram of a peak composition.

Let $\mathsf{SPCT}(\alpha)$ denote the set of all standard peak composition tableaux of shape α.

Given $T \in \mathsf{SPCT}(\alpha)$, the reading word $\mathrm{rw}(T)$ of T is obtained by reading the entries in the columns of T from bottom to top, starting with the leftmost column and proceeding rightwards [10].

Example 3.2. Let $\alpha = (3,4)$. The elements of $\mathsf{SPCT}(\alpha)$, along with their descent sets, are

$$\begin{array}{ccccc}
\begin{array}{|c|c|c|c|}\hline 4&5&6&7\\\hline 1&2&3&\\\cline{1-3}\end{array}, &
\begin{array}{|c|c|c|c|}\hline 3&5&6&7\\\hline 1&2&4&\\\cline{1-3}\end{array}, &
\begin{array}{|c|c|c|c|}\hline 3&4&6&7\\\hline 1&2&5&\\\cline{1-3}\end{array}, &
\begin{array}{|c|c|c|c|}\hline 3&4&5&7\\\hline 1&2&6&\\\cline{1-3}\end{array}, &
\begin{array}{|c|c|c|c|}\hline 3&4&5&6\\\hline 1&2&7&\\\cline{1-3}\end{array}, \\
\{3\}, & \{2,4\}, & \{2\}, & \{2,5\}, & \{2,6\}.
\end{array}$$

Jing and Li conjectured [8] that quasisymmetric Schur Q-functions $\widetilde{Q}_\alpha$ expand positively in the peak basis, and this conjecture was resolved by Kantarcı Oğuz [10] with the following formula for this expansion. We use this formula as our definition.

Theorem 3.3. *[10] Let α be a peak composition of n. Then*

$$\widetilde{Q}_\alpha = \sum_{T \in \mathsf{SPCT}(\alpha)} K_{\mathrm{comp}_n(\mathrm{Peak}(T))}.$$

Example 3.4. Let $\alpha = (3,4)$. Noting that $\mathrm{Peak}(T) = \mathrm{Des}(T)$ for each $T \in \mathsf{SPCT}(\alpha)$ (see Example 3.2), we have

$$\widetilde{Q}_{(3,4)} = K_{(3,4)} + K_{(2,2,3)} + K_{(2,5)} + K_{(2,3,2)} + K_{(2,4,1)}.$$

To construct modules whose peak characteristics are the $\widetilde{Q}_\alpha$, by Theorems 2.4 and 3.3 it suffices to verify the following lemma. This can be done by showing that whether an ascent is attacking depends only on the relative positions of the boxes involved.

Lemma 3.5. *Let α be a peak composition of n. Then $\mathsf{SPCT}(\alpha)$ is ascent-compatible.*

Therefore, we can answer the question of Jing and Li as follows.

Theorem 3.6. *Let α be a peak composition of n, and let $D = D(\alpha)$ and $\mathsf{StdTab}(D) = \mathsf{SPCT}(\alpha)$. Then*

$$\widetilde{ch}([\widetilde{\mathbf{N}}_{\mathsf{SPCT}(\alpha)}]) = \widetilde{Q}_\alpha.$$

Define $\mathbf{N}_{\mathsf{StdTab}(D)}$ or $\widetilde{\mathbf{N}}_{\mathsf{StdTab}(D)}$ to be *tableau-cyclic* if it is generated by a single tableau.

Theorem 3.7. *Let α be a peak composition of n. Then $\mathbf{N}_{\mathsf{SPCT}(\alpha)}$ and $\widetilde{\mathbf{N}}_{\mathsf{SPCT}(\alpha)}$ are tableau-cyclic.*

Theorem 3.7 is proved by explicitly constructing the generator T. This theorem provides a contrast with families of modules we consider in the next section that are not tableau-cyclic.

3.3 A new basis of the peak algebra

The quasisymmetric Schur Q-function basis of Peak can be considered, by its construction in [8], as an analogue of the dual immaculate basis [2] of QSym. It is natural to ask whether we can use the methods of Section 2 to find analogues in Peak of other important bases of QSym.

An obvious next candidate is the quasisymmetric Schur functions [7]. For ease of comparison with the Schur Q-functions later, we choose instead to work with a variant called the *Young quasisymmetric Schur functions* [12], which are obtained from quasisymmetric Schur functions by reversing both the indexing composition and the variable set.

Let $\alpha \vDash n$. A *standard Young composition tableau* [12] of shape α is a bijective assignment T of the cells of $D(\alpha)$ to entries $1, \dots, n$ satisfying the following conditions:

(SYCT1) Entries increase from left to right along rows.

(SYCT2) Entries increase from bottom to top in the first column.

(SYCT3) Let (c, r) denote the box in column c and row r. If boxes (c, r) and $(c+1, r')$ for $r' < r$ are in $D(\alpha)$ and $T(c, r) < T(c+1, r')$, then $T(c+1, r) < T(c+1, r')$, where $T(c+1, r)$ is defined to be ∞ if $(c+1, r) \notin D(\alpha)$.

Denote the set of all standard Young row-strict composition tableaux of shape α by $\mathsf{SYCT}(\alpha)$. Then the Young quasisymmetric Schur function $\mathcal{S}_\alpha$ [12] is defined by

$$\mathcal{S}_\alpha = \sum_{T \in \mathsf{SYCT}(\alpha)} F_{\mathrm{Des}(T)}, \tag{3.1}$$

where $\mathrm{Des}(T)$ arises from the reading word $\mathrm{rw}(T)$ obtained by reading the entries of T from top to bottom in the first column and then bottom to top in subsequent columns, reading the columns in order from left to right.

Example 3.8. Let $\alpha = (3, 3)$. Then $\mathcal{S}_\alpha = F_{(3,3)} + F_{(2,2,2)} + F_{(1,2,2,1)} + F_{(1,3,2)}$. This is obtained from the elements of $\mathsf{SYCT}(3, 3)$, which are shown below with their descent sets:

$$\begin{array}{|c|c|c|}\hline 4&5&6\\\hline 1&2&3\\\hline\end{array}, \quad \begin{array}{|c|c|c|}\hline 3&5&6\\\hline 1&2&4\\\hline\end{array}, \quad \begin{array}{|c|c|c|}\hline 2&3&5\\\hline 1&4&6\\\hline\end{array}, \quad \begin{array}{|c|c|c|}\hline 2&3&4\\\hline 1&5&6\\\hline\end{array},$$

$$\{3\}, \qquad \{2,4\}, \qquad \{1,3,5\}, \qquad \{1,4\}.$$

Lemma 3.9. *Let α be a composition of n. Then $\mathsf{SYCT}(\alpha)$ is ascent-compatible.*

The next result follows from (3.1), Lemma 3.9 and Theorems 2.4 and 2.5.

Proposition 3.10. *Let α be a composition of n, and let $D = D(\alpha)$ and $\mathsf{StdTab}(D) = \mathsf{SYCT}(\alpha)$. Then*

$$ch([\mathbf{N}_{\mathsf{SYCT}(\alpha)}]) = \mathcal{S}_\alpha \qquad \textit{and} \qquad \widetilde{ch}([\widetilde{\mathbf{N}}_{\mathsf{SYCT}(\alpha)}]) = \sum_{T \in \mathsf{SYCT}(\alpha)} K_{\mathrm{comp}_n(\mathrm{Peak}(T))}.$$

Even when considering only peak compositions α, the functions $\widetilde{ch}([\widetilde{\mathbf{N}}_{\mathsf{SYCT}(\alpha)}])$ are not linearly independent. However, it turns out that imposing an extra condition on $\mathsf{SYCT}(\alpha)$ yields $HCl_n(0)$-modules whose peak characteristics are in fact a basis of Peak. This condition is precisely the condition (SPCT3) required for $\mathsf{SPCT}(\alpha)$: that all subdiagrams of $D(\alpha)$ consisting of boxes with entries smaller than k is the diagram of a peak composition.

Define $\mathrm{SPYCT}(\alpha)$ to be the elements of $\mathsf{SYCT}(\alpha)$ that additionally satisfy (SPCT3).

Lemma 3.11. *Let α be a composition of n. Then $\mathrm{SPYCT}(\alpha)$ is ascent-compatible.*

Given a peak composition α of n, we now define the *peak Young quasisymmetric Schur function* $\widetilde{\mathcal{S}}_\alpha$ to be the peak characteristic of $\widetilde{\mathbf{N}}_{\mathrm{SPYCT}(\alpha)}$. Then by Theorem 2.4 and Lemma 3.11, we have the following.

Proposition 3.12. *Let α be a peak composition of n. Then*

$$\widetilde{\mathcal{S}}_\alpha = \widetilde{ch}([\widetilde{\mathbf{N}}_{\mathrm{SPYCT}(\alpha)}]) = \sum_{T \in \mathrm{SPYCT}(\alpha)} K_{\mathrm{comp}_n(\mathrm{Peak}(T))}.$$

Theorem 3.13. *The set $\{\widetilde{\mathcal{S}}_\alpha : \alpha$ is a peak composition$\}$ forms a basis of* Peak.

Computations suggest the quasisymmetric Schur Q-functions might expand positively in the peak Young quasisymmetric Schur functions. As further evidence for this, the functions in the peak expansion of $\widetilde{\mathcal{S}}_\alpha$ are a subset of those in the peak expansion of $\widetilde{Q}_\alpha$, for any composition α.

Proposition 3.14. *Let α be a peak composition. Then $\widetilde{Q}_\alpha - \widetilde{\mathcal{S}}_\alpha$ expands positively in the peak basis.*

Remark 3.15. In contrast to Theorem 3.7, none of the modules $\mathbf{N}_{\mathsf{SYCT}(\alpha)}$, $\mathbf{N}_{\mathrm{SPYCT}(\alpha)}$, $\widetilde{\mathbf{N}}_{\mathsf{SYCT}(\alpha)}$ and $\widetilde{\mathbf{N}}_{\mathrm{SPYCT}(\alpha)}$ are tableau-cyclic in general.

3.4 Connections to Schur Q-functions

The (Young) quasisymmetric Schur functions provide an elegant refinement of Schur functions. This raises the question of how our new peak analogue of Young quasisymmetric Schur functions relates to the Schur Q-functions (the peak analogue of Schur functions). We construct $HCl_n(0)$-modules whose peak characteristics are the Schur Q-functions, and then establish an isomorphism of $HCl_n(0)$-modules that proves the peak Young quasisymmetric Schur actually contain the Schur Q-functions. Moreover, the quasisymmetric characteristics of the $H_n(0)$-modules associated to the $HCl_n(0)$-modules for Schur Q-functions, are in fact Young quasisymmetric Schur functions.

Let λ denote a *strict partition*, *i.e.*, a strictly decreasing sequence of positive integers. If the parts of λ sum to n we write $\lambda \vdash n$. The *shifted diagram* of λ is the box diagram $ShD(\lambda)$ obtained from $D(\lambda)$ by shifting all boxes in the ith row $i-1$ units to the right.

If $\lambda \vdash n$, then a *standard shifted tableau* of shape λ is a bijective filling of $ShD(\lambda)$ with $1, \ldots, n$ such that entries increase from left to right along each row and from bottom to top in each column. Let $\mathsf{SShT}(\lambda)$ denote the set of all standard shifted tableaux of shape λ. Define the reading word $\mathrm{rw}(S)$ of $S \in \mathsf{SShT}(\lambda)$ by reading the entries of S from left to right along rows, starting at the top row and proceeding downwards.

Example 3.16. Let $\lambda = (4,2,1)$. The elements of $\mathsf{SShT}(\lambda)$, along with their descent sets, are shown below:

$$
\begin{array}{|c|c|c|c|}\cline{3-3} \multicolumn{2}{c|}{} & 7 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c|}{} & 5 & 6 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 3 & 4 \\ \hline \end{array}\ , \quad
\begin{array}{|c|c|c|c|}\cline{3-3} \multicolumn{2}{c|}{} & 7 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c|}{} & 4 & 6 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 3 & 5 \\ \hline \end{array}\ , \quad
\begin{array}{|c|c|c|c|}\cline{3-3} \multicolumn{2}{c|}{} & 7 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c|}{} & 4 & 5 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 3 & 6 \\ \hline \end{array}\ , \quad
\begin{array}{|c|c|c|c|}\cline{3-3} \multicolumn{2}{c|}{} & 6 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c|}{} & 4 & 5 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 3 & 7 \\ \hline \end{array}\ , \quad
\begin{array}{|c|c|c|c|}\cline{3-3} \multicolumn{2}{c|}{} & 7 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c|}{} & 3 & 6 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 4 & 5 \\ \hline \end{array}\ , \quad
\begin{array}{|c|c|c|c|}\cline{3-3} \multicolumn{2}{c|}{} & 7 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c|}{} & 3 & 5 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 4 & 6 \\ \hline \end{array}\ , \quad
\begin{array}{|c|c|c|c|}\cline{3-3} \multicolumn{2}{c|}{} & 6 & \multicolumn{1}{c}{} \\ \cline{2-3} \multicolumn{1}{c|}{} & 3 & 5 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 4 & 7 \\ \hline \end{array}\ ,
$$

$$
\{4,6\}, \qquad \{3,5,6\}, \qquad \{3,6\}, \qquad \{3,5\}, \qquad \{2,5,6\}, \qquad \{2,4,6\}, \qquad \{2,4,5\}.
$$

Given $\lambda \vdash n$, the *Schur Q-function* Q_λ is defined by

$$
Q_\lambda = \sum_{S \in \mathsf{SShT}(\lambda)} K_{\mathrm{comp}_n(\mathrm{Peak}(S))}. \tag{3.2}
$$

Example 3.17. Let $\lambda = (4,2,1)$. Using Example 3.16 and noting that $\mathrm{Peak}(\{3,5,6\}) = \{3,5\}$, $\mathrm{Peak}(\{2,5,6\}) = \{2,5\}$ and $\mathrm{Peak}(\{2,4,5\}) = \{2,4\}$, we have

$$
Q_{(4,2,1)} = K_{(4,2,1)} + 2K_{(3,2,2)} + K_{(3,3,1)} + K_{(2,3,2)} + K_{(2,2,2,1)} + K_{(2,2,3)}.
$$

Using the methods of Section 2, the formula (3.2) can be naturally realised in terms of peak characteristics of $HCl_n(0)$-modules.

Lemma 3.18. *Let $\lambda \vdash n$. Then $\mathsf{SShT}(\lambda)$ is ascent-compatible.*

Theorem 3.19. *Let $\lambda \vdash n$. Then*

$$
ch([\mathbf{N}_{\mathsf{SShT}(\lambda)}]) = \sum_{S \in \mathsf{SShT}(\lambda)} F_{\mathrm{comp}_n(\mathrm{Des}(S))} \qquad \textit{and} \qquad \widetilde{ch}([\widetilde{\mathbf{N}}_{\mathsf{SShT}(\lambda)}]) = Q_\lambda.
$$

Remark 3.20. For $\lambda \vdash n$, $\mathsf{SPYCT}(\lambda)$ is actually equal to $\mathsf{SYCT}(\lambda)$. Therefore, the statements below concerning $\mathsf{SPYCT}(\lambda)$ also apply to $\mathsf{SYCT}(\lambda)$.

Theorem 3.21. *Let $\lambda \vdash n$. Then $\widetilde{\mathbf{N}}_{\mathsf{SShT}(\lambda)}$ is isomorphic to $\widetilde{\mathbf{N}}_{\mathsf{SPYCT}(\lambda)}$ as $HCl_n(0)$-modules.*

Central to the proof of Theorem 3.21 is a map from $\mathsf{SShT}(\lambda)$ to $\mathsf{StdFill}(D(\lambda))$ defined by moving all boxes in each row i of $S \in \mathsf{SShT}(\lambda)$ $i-1$ units leftwards, and showing that this map is a descent-preserving bijection from $\mathsf{SShT}(\lambda)$ to $\mathsf{SPYCT}(\lambda)$.

Corollary 3.22. *The basis $\{\widetilde{\mathcal{S}}_\alpha\}$ of* Peak *contains the Schur Q-functions.*

The isomorphism from Theorem 3.21 descends to an isomorphism of $H_n(0)$-modules between $\mathbf{N}_{\mathsf{SShT}(\lambda)}$ and $\mathbf{N}_{\mathsf{SPYCT}(\lambda)}$ (or $\mathbf{N}_{\mathsf{SYCT}(\lambda)}$). Therefore, for the $HCl_n(0)$-modules whose peak characteristics are the Schur Q-functions, the quasisymmetric characteristics of the corresponding $H_n(0)$-modules are Young quasisymmetric Schur functions.

Corollary 3.23. *For λ a strict partition of n, the quasisymmetric characteristics of the $H_n(0)$-modules $\mathbf{N}_{\mathsf{SShT}(\lambda)}$ are the Young quasisymmetric Schur functions $\mathcal{S}_\lambda$.*

4 Future directions

The construction methods in Section 2 raise a number of natural questions and avenues for future work. We present a few here.

- Can we obtain yet more bases of Peak using these constructions? If so, how do these bases relate to other bases of Peak, and what properties do they have?
- Do the quasisymmetric Schur Q-functions expand positively in the basis of peak Young quasisymmetric Schur functions?
- What can be said about the structure of the $H_n(0)$-modules and $HCl_n(0)$-modules on diagrams constructed in Section 2? Under what conditions are these modules tableau-cyclic, or indecomposable?

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