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Factorization of Classical Characters Twisted by Roots of Unity: Extended Abstract

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Abstract. For a fixed integer $t \geq 2$, we consider the irreducible characters of representations of the classical groups of types A, B, C and D, namely $\mathrm{GL}_{tn}, \mathrm{SO}_{2tn+1}, \mathrm{Sp}_{2tn}$ and O_{2tn}, evaluated at elements $\omega^k x_i$ for $0 \leq k \leq t-1$ and $1 \leq i \leq n$, where ω is a primitive t'th root of unity. The case of GL_{tn} was considered by D. Prasad (Israel J. Math., 2016). In this article, we give a uniform approach for all cases. In each case, we characterize partitions for which the character value is nonzero in terms of what we call z-asymmetric partitions, where z is an integer that depends on the group. Moreover, if the character value is nonzero, we prove that it factorizes into characters of smaller classical groups. The proof uses Cauchy-type determinant formulas for these characters and involves a careful study of the beta sets of partitions. We also give product formulas for general z-asymmetric partitions and z-asymmetric t-cores. Lastly, we show that there are infinitely many z-asymmetric t-cores for $|z| \leq t-2$.

Saaransh (सारांश). इस शोध पत्र में, हम डी.प्रसाद के कार्य को, जिन्होंने केवल GL_{tn} के लिए किया था, सभी क्लासिकल समूहों के लिए करते है। एक निश्चित पूर्णांक $t \geq 2$ के लिए, हम A, B, C व D प्रकार के क्लासिकल समूहों, अर्थात् $GL_{tn}, SO_{2tn+1}, Sp_{2tn}$ तथा O_{2tn}, के अलघुकरणीय अभिलक्षणकों के $\omega^k x_i$; $0 \leq k \leq t-1$, $1 \leq i \leq n$ पर मूल्यांकन, जबकि ω इकाई का t-वाँ पूर्वग मूल है, पर विचार करते हैं। इस लेख में हम सभी प्रकार के समूहों के लिए प्रयोज्य एकमेव दृष्टिकोण प्रदान करते हैं। प्रत्येक प्रकार के लिए हम उन विभाजनों को वर्णित करते हैं जिनमें अभिलक्षणकों का मान शून्य नहीं है। यह वर्णन z-असममित विभाजनों, जबकि z समूह पर निर्भर करता हुआ एक पूर्णांक है, के माध्यम से किया गया है। अभिलक्षणकों का मान शून्य न होने की अवस्था में हम यह भी दर्शाते हैं कि इसे छोटे क्लासिकल समूहों के अभिलक्षणकों के गुणनफल के रूप में व्यक्त किया जा सकता है। हमारी उपपत्ति इन अभिलक्षणकों के कॉशी प्रकार के सारणिक सूत्रों तथा विभाजनों के बीटा समुच्चयों के अध्ययन पर आधारित है। इस लेख में हम z-असममित विभाजनों और z-असममित t-कोरों के गुणन सूत्रों की भी चर्चा करते हैं। अंत में हम यह भी सिद्ध करते हैं कि यदि $|z| \leq t-2$ हो तो z-असममित t-कोरों की संख्या अनंत होगी।

Keywords: Weyl character formula, classical groups, twisted characters, factorizations, generating functions

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1 Introduction

The characters of irreducible representations of the classical families of groups, namely the general linear, symplectic and orthogonal groups are amazing families of symmetric Laurent polynomials indexed by integer partitions. In particular, the character of the general linear groups are the *Schur polynomials*, which are extremely well-studied. They form one of the most natural bases of the ring of symmetric functions, and are orthonormal with respect to the standard Hall inner product. For background, see [13].

These families of Laurent polynomials also satisfy nontrivial relations, which are not well-understood from the point of view of representation theory. For instance, it was shown in [6] that the Schur polynomial for a rectangular partition in $2n$ variables specialized to the last n variables being reciprocals of the first n variables becomes a product of two other classical characters. In some cases, this is the product of a symplectic and an even orthogonal character, and in some others, the product of two odd orthogonal characters. Similar factorization results were obtained in [2] for so-called *double staircase partitions*, *i.e.* partitions of the form $(k,k,k-1,k-1,\dots,1,1)$ or $(k,k-1,k-1,\dots,1,1)$. This kind of factorization was generalized in [1] for a large class of partitions as follows: for any $\lambda = (\lambda_1,\dots,\lambda_n)$ and any positive integer $m \geq \lambda_1$, construct $\mu = (m+\lambda_1,\dots,m+\lambda_n,m-\lambda_n,\dots,m-\lambda_1)$. Then $s_\mu(x_1,\dots,x_n,1/x_1,\dots,1/x_n)$ factors. Such a result was further generalized to skew-Schur functions, *i.e.* induced characters, of similar shapes in [3].

In a different direction, Prasad [14] considered factorizations of Schur polynomials in $2n$ variables where the last n variables were negatives of the first n variables motivated by a celebrated result of Kostant [10]. He showed that such a factorization is nonzero if and only if the corresponding 2-core is empty, and if it is nonzero, it factors into characters for the 2-quotients; see Section 2 for the definitions. He further generalized this result to tn variables, for $t \geq 2$ a fixed positive integer, specialized to $(\exp(2\pi \imath k/t)x_j)_{0\leq k\leq t-1, 1\leq j\leq n}$, obtaining similar results. We will think of these as twisted characters, where the twists are by all the t'th roots of unity.

We note in passing that Schur polynomials evaluated at roots of unity and their powers have been considered in [13, 15].

In this work, we generalize Prasad's results to other classical groups. We consider the classical groups $\mathrm{Sp}_{2tn}, \mathrm{O}_{2tn}$ and SO_{2tn+1} and obtain factorizations for their characters under the same specialization as that of Prasad. These are stated as Theorem 5, Theorem 8 and Theorem 9 respectively. Our proofs are more involved for the following reason. For the general linear group, there is only one possible value of the t-core for which the twisted character is nonzero, namely the empty partition. For the other classical characters, there are many possible values of the t-core for which the character is nonzero. We will show that these are t-cores which can be written in Frobenius coordinates as $(\alpha|\alpha+z)$, where the value of z depends on the group, and which we call z-asymmetric partitions. For the study here, $z \in \{-1,0,1\}$.

2 Summary of results

Throughout, we fix $t \geq 2$. Let ω be a primitive t'th root of unity. We also use n for a fixed positive integer and let $X = (x_1, \ldots, x_n)$ be a tuple of commuting indeterminates. For any integer j, we set $X^j = (x_1^j, \ldots, x_n^j)$, and for $a \in \mathbb{R}$, set $aX = (ax_1, \ldots, ax_n)$. Define $\overline{x} = 1/x$ for an indeterminate x and write $\overline{X} = (\overline{x}_1, \ldots, \overline{x}_n)$.

Recall that a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a sequence of weakly decreasing nonnegative integers. Any entry of the partition is called a *part*. The *length* of a partition λ, which is the number of positive parts, is denoted by $\ell(\lambda)$. By $a + \lambda$, we will mean the partition $(a + \lambda_1, \ldots, a + \lambda_m)$. For a partition λ and an integer m such that $\ell(\lambda) \leq m$, define the *beta-set of* λ by $\beta(\lambda) \equiv \beta(\lambda, m) = (\beta_1(\lambda, m), \ldots, \beta_m(\lambda, m))$ by $\beta_i(\lambda) = \lambda_i + m - i$. We will use the convention that we will write $\beta(\lambda)$ whenever m is clear from the context.

Macdonald [13] defines the t-core and t-quotient of a partition using the beta-set and we recall this construction. For a partition λ of length at most m, let $n_i(\lambda) \equiv n_i(\lambda, m)$, $0 \leq i \leq t-1$, be the number of parts of $\beta(\lambda)$ congruent to $i \pmod t$ and $\beta_j^{(i)}(\lambda)$, $1 \leq j \leq n_i(\lambda)$ be the $n_i(\lambda)$ parts of $\beta(\lambda)$ congruent to $i \pmod t$ in decreasing order.

Definition 1 ([13, Example I.1.8])**.** Let λ be a partition with $\ell(\lambda) \leq m$.

1. The m numbers $tj + i$, where $0 \leq j \leq n_i(\lambda) - 1$ and $0 \leq i \leq t-1$, are all distinct. Arrange them in descending order, say $\widetilde{\beta}_1 > \cdots > \widetilde{\beta}_m$. Then the t-core of λ has parts $(\mathrm{core}_t(\lambda))_i = \widetilde{\beta}_i - m + i$.
2. The parts $\beta_j^{(i)}(\lambda)$ may be written in the form $t\widetilde{\beta}_j^{(i)} + i$, $1 \leq j \leq n_i(\lambda)$, where $\widetilde{\beta}_1^{(i)} > \cdots > \widetilde{\beta}_{n_i(\lambda)}^{(i)} \geq 0$. Let $\lambda_j^{(i)} = \widetilde{\beta}_j^{(i)} - n_i(\lambda) + j$, so that $\lambda^{(i)} = (\lambda_1^{(i)}, \ldots, \lambda_{n_i(\lambda)}^{(i)})$ is a partition. Then the t-quotient $\mathrm{quo}_t(\lambda)$ of λ is $\lambda^\star = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(t-1)})$.

For example, the 3-core and 3-quotient of $\lambda = (4, 2, 2, 1)$ with $m = 4$ are $(4, 2)$ and $((1), \emptyset, \emptyset)$ respectively. We note that there is another and slightly different definition of cores and quotients arising from modular representation theory; see James and Kerber [9, Chapter 2], for instance. This difference turns out not to be important for us; see Remark 2.

Remark 2. Notice that Macdonald's definition of the t-quotient depends on m. In particular, if $\mathrm{quo}_t(\lambda) = (\lambda^{(0)}, \ldots, \lambda^{(t-1)})$ and m increases by 1 in Definition 1, the new t-quotient will be $(\lambda^{(t-1)}, \lambda^{(0)}, \ldots, \lambda^{(t-2)})$. So, Macdonald suggests that "$\lambda^\star$ should perhaps be thought of as a 'necklace' of partitions."

The *(Frobenius) rank* of a partition λ, denoted $\mathrm{rk}(\lambda)$, is the largest integer k such that $\lambda_k \geq k$. The *Frobenius coordinates* of λ are a pair of strict partitions, denoted $(\alpha|\beta)$, of length $\mathrm{rk}(\lambda)$ given by $\alpha_i = \lambda_i - i$ and $\beta_j = \lambda'_j - j$. For example, the Frobenius coordinates of $(4, 2, 2, 1)$ are $(3, 0|3, 1)$.

We write down the explicit Weyl character formulas for the infinite families of classical groups at the representation indexed by $\lambda = (\lambda_1, \dots, \lambda_n)$. See [7] for more details and background. The *Schur polynomial* or *general linear (type A) character* of GL_n is given by

$$s_\lambda(X) = \frac{\det\limits_{1 \le i,j \le n} \left(x_i^{\beta_j(\lambda)} \right)}{\det\limits_{1 \le i,j \le n} \left(x_i^{n-j} \right)}. \tag{2.1}$$

The *odd orthogonal (type B) character* of the group $\mathrm{SO}(2n+1)$ is given by

$$\mathrm{so}_\lambda(X) = \frac{\det\limits_{1 \le i,j \le n} \left(x_i^{\beta_j(\lambda)+1/2} - \overline{x}_i^{\beta_j(\lambda)+1/2} \right)}{\det\limits_{1 \le i,j \le n} \left(x_i^{n-j+1/2} - \overline{x}_i^{n-j+1/2} \right)} = \frac{\det\limits_{1 \le i,j \le n} \left(x_i^{\beta_j(\lambda)+1} - \overline{x}_i^{\beta_j(\lambda)} \right)}{\det\limits_{1 \le i,j \le n} \left(x_i^{n-j+1} - \overline{x}_i^{n-j} \right)}. \tag{2.2}$$

The *symplectic (type C) character* of the group $\mathrm{Sp}(2n)$ is given by

$$\mathrm{sp}_\lambda(X) = \frac{\det\limits_{1 \le i,j \le n} \left(x_i^{\beta_j(\lambda)+1} - \overline{x}_i^{\beta_j(\lambda)+1} \right)}{\det\limits_{1 \le i,j \le n} \left(x_i^{n-j+1} - \overline{x}_i^{n-j+1} \right)}. \tag{2.3}$$

Lastly, the *even orthogonal (type D) character* of the group $\mathrm{O}(2n)$ is given by

$$\mathrm{o}_\lambda^{\mathrm{even}}(X) = \frac{\det\limits_{1 \le i,j \le n} \left(x_i^{\beta_j(\lambda)} + \overline{x}_i^{\beta_j(\lambda)} \right)}{(1 + \delta_{\lambda_n,0}) \det\limits_{1 \le i,j \le n} \left(x_i^{n-j} + \overline{x}_i^{n-j} \right)}, \tag{2.4}$$

where δ is the Kronecker delta. There is an extra factor in the denominator because the last column becomes 2 if $\lambda_n = 0$. Notice that

$$s_\lambda(x_1, \dots, x_n) = \mathrm{sp}_\lambda(x_1, \dots, x_n) = \mathrm{so}_\lambda(x_1, \dots, x_n) = \mathrm{o}_\lambda^{\mathrm{even}}(x_1, \dots, x_n) = 0, \quad \text{if } n < \ell(\lambda).$$

We will consider classical characters evaluated at elements twisted by all the t'th roots of unity. The first result in this direction is due to D. Prasad for GL_{tn} [14, Theorem 2]. We generalize [14, Theorem 2] to other classical characters. We first need some definitions.

Definition 3. Let z be an integer. We say that a partition λ is *z-asymmetric* if $\lambda = (\alpha \,|\, \alpha + z)$, in Frobenius coordinates for some strict partition α. More precisely, $\lambda = (\alpha|\beta)$ where $\beta_i = \alpha_i + z$ for $1 \le i \le \mathrm{rk}(\lambda)$.

Definition 4. A 1-asymmetric partition is said to be *symplectic*. In addition, if a symplectic partition is also a t-core, we call it a *symplectic t-core*.

Note that the empty partition is vacuously symplectic. For example, the only symplectic partitions of 6 are $(3,1,1,1)$ and $(2,2,2)$, and the first few symplectic 3-cores are $(1,1)$, $(2,1,1),(4,2,2,1,1)$ and $(5,3,2,2,1,1)$.

For a partition of length at most tn, let $\sigma_\lambda \in S_{tn}$ be the permutation that rearranges the parts of $\beta(\lambda)$ such that

$$\beta_{\sigma_\lambda(j)}(\lambda) \equiv q \pmod t, \quad \sum_{i=0}^{q-1} n_i(\lambda) + 1 \le j \le \sum_{i=0}^{q} n_i(\lambda), \tag{2.5}$$

arranged in decreasing order for each $q \in \{0,1,\dots,t-1\}$. We will denote our indeterminates by $(X,\omega X,\omega^2 X,\dots,\omega^{t-1}X)$, where we recall that $X=(x_1,\dots,x_n)$. To state our results, it will be convenient to define, for $\lambda=(\lambda_1,\dots,\lambda_k)$, the reverse of λ as $\operatorname{rev}(\lambda)=(\lambda_k,\dots,\lambda_1)$. Further, if $\mu=(\mu_1,\dots,\mu_j)$ is another partition such that $\mu_1 \le \lambda_k$, then we write the concatenated partition as $(\lambda,\mu)=(\lambda_1,\dots,\lambda_k,\mu_1,\dots,\mu_j)$.

Theorem 5. *Let λ be a partition of length at most tn indexing an irreducible representation of Sp_{2tn} and $\operatorname{quo}_t(\lambda)=(\lambda^{(0)},\dots,\lambda^{(t-1)})$. Then the Sp_{2tn}-character $\mathrm{sp}_\lambda(X,\omega X,\dots,\omega^{t-1}X)$ is given as follows.*

1. *If $\operatorname{core}_t(\lambda)$ is not a symplectic t-core, then $\mathrm{sp}_\lambda(X,\omega X,\dots,\omega^{t-1}X)=0$.*
2. *If $\operatorname{core}_t(\lambda)$ is a symplectic t-core with rank r, then*

$$\begin{aligned}\mathrm{sp}_\lambda(X,\omega X,\dots,\omega^{t-1}X) = (-1)^{\epsilon_1}\operatorname{sgn}(\sigma_\lambda)\,\mathrm{sp}_{\lambda^{(t-1)}}(X^t)\prod_{i=0}^{\lfloor\frac{t-3}{2}\rfloor} s_{\mu_i^{(1)}}(X^t,\overline{X}^t)\\ \times\begin{cases}\mathrm{so}_{\lambda^{(\frac{t}{2}-1)}}(X^t) & t \text{ even},\\ 1 & t \text{ odd},\end{cases}\end{aligned} \tag{2.6}$$

where

$$\epsilon_1 = -\sum_{i=\lfloor\frac{t}{2}\rfloor}^{t-2}\binom{n_i(\lambda)+1}{2} + \begin{cases}\frac{n(n+1)}{2}+nr & t \text{ even},\\ 0 & t \text{ odd},\end{cases}$$

and $\mu_i^{(1)} = \lambda_1^{(t-2-i)} + \left(\lambda^{(i)},0,-\operatorname{rev}(\lambda^{(t-2-i)})\right)$ has $2n$ parts for $0 \le i \le \lfloor\frac{t-3}{2}\rfloor$.

So, nonzero Sp_{2tn} characters are a product of characters of smaller groups, of which there are $\lfloor (t-1)/2 \rfloor$ GL_{2n} characters, one Sp_{2n} character and, if t is even, one additional SO_{2n+1} character. One can show that the only 2-cores are self-conjugate. Therefore, this character when $t=2$ is nonzero if and only if $\operatorname{core}_2(\lambda)=\emptyset$.

Example 6. For $t = 2$, Theorem 5 says that the character of the group $\mathrm{Sp}(4)$ $(n = 1)$ of the representation indexed by the partition (a, b), $a \geq b \geq 0$, evaluated at $(x, -x)$ is nonzero if and only if a and b have the same parity. If a and b are both odd, then

$$\mathrm{sp}_{(a,b)}(x, -x) = -\mathrm{sp}_{(\frac{b-1}{2})}(x^2)\, \mathrm{so}_{(\frac{a+1}{2})}(x^2),$$

and if a and b are both even, then

$$\mathrm{sp}_{(a,b)}(x, -x) = \mathrm{sp}_{(\frac{a}{2})}(x^2)\, \mathrm{so}_{(\frac{b}{2})}(x^2).$$

Notice that all the characters on the right-hand side are for the groups $\mathrm{Sp}(2)$ and $\mathrm{SO}(3)$, and in both cases, the partitions indexing them are the 2-quotients and of length 1.

Definition 7. A (-1)-asymmetric partition is said to be *orthogonal*. In addition, if an orthogonal partition is also a t-core, we call it an *orthogonal t-core*.

Our notion of an orthogonal partition is same as Macdonald's *double of α* [13, p. 14], and Garvan–Kim–Stanton's *doubled partition of α*, denoted $\alpha\alpha$ [8, Sec. 8]. The first few orthogonal 3-cores are (2), $(3, 1), (5, 3, 1, 1)$ and $(6, 4, 2, 1, 1)$, which are precisely the conjugates of the symplectic 3-cores listed earlier. Then our result for factorization of even orthogonal characters is as follows.

Theorem 8. *Let λ be a partition of length at most tn indexing an irreducible representation of O_{2tn} and $\mathrm{quo}_t(\lambda) = (\lambda^{(0)}, \dots, \lambda^{(t-1)})$. Then the O_{2tn} character $\mathrm{o}^{even}_{\lambda}(X, \omega X, \dots, \omega^{t-1} X)$ is given as follows.*

1. *If $\mathrm{core}_t(\lambda)$ is not an orthogonal t-core, then $\mathrm{o}^{even}_{\lambda}(X, \omega X, \dots, \omega^{t-1} X) = 0$.*
2. *If $\mathrm{core}_t(\lambda)$ is an orthogonal t-core with rank r, then*

$$\begin{aligned}\mathrm{o}^{even}_{\lambda}(X, \omega X, \dots, \omega^{t-1} X) &= (-1)^{\epsilon_2}\, \mathrm{sgn}(\sigma_\lambda)\, \mathrm{o}^{even}_{\lambda^{(0)}}(X^t) \prod_{i=1}^{\lfloor \frac{t-1}{2} \rfloor} s_{\mu_i^{(2)}}(X^t, \overline{X}^t) \\ &\quad \times \begin{cases} (-1)^{\sum_{i=1}^{n} \lambda_i^{(t/2)}} \mathrm{so}_{\lambda^{(t/2)}}(-X^t) & t \text{ even}, \\ 1 & t \text{ odd}, \end{cases}\end{aligned} \tag{2.7}$$

where

$$\epsilon_2 = -\sum_{i=\lfloor \frac{t+2}{2} \rfloor}^{t-1} \binom{n_i(\lambda)}{2} + \begin{cases} \frac{n(n+t-1)}{2} + nr & t \text{ even}, \\ \frac{(t-1)n}{2} & t \text{ odd}, \end{cases}$$

and $\mu_i^{(2)} = \lambda_1^{(t-i)} + \left(\lambda^{(i)}, 0, -\mathrm{rev}(\lambda^{(t-i)})\right)$ has $2n$ parts for $0 \leq i \leq \left\lfloor \frac{t-1}{2} \right\rfloor$.

Lastly, we consider the odd orthogonal case. It will turn out that the notion of an 'odd-orthogonal partition' is the same as being self-conjugate, or equivalenty, 0-asymmetric. The first few self-conjugate 3-cores are (1), $(3,1,1)$, $(4,2,1,1)$ and $(6,4,2,2,1,1)$. Our result for factorization of odd orthogonal characters is as follows.

Theorem 9. *Let λ be a partition of length at most tn indexing an irreducible representation of SO_{2tn+1}. Then the SO_{2tn+1} character $\mathrm{so}_\lambda(X,\omega X,\dots,\omega^{t-1}X)$ is given as follows.*

1. *If $\mathrm{core}_t(\lambda)$ is not self-conjugate, then $\mathrm{so}_\lambda(X,\omega X,\dots,\omega^{t-1}X)=0$.*
2. *If $\mathrm{core}_t(\lambda)$ is self-conjugate with rank r, then*

$$\begin{aligned}\mathrm{so}_\lambda(X,\omega X,\dots,\omega^{t-1}X) = (-1)^{\epsilon_3}\,\mathrm{sgn}(\sigma_\lambda) \prod_{i=0}^{\lfloor\frac{t-2}{2}\rfloor} s_{\mu_i^{(3)}}(X^t,\overline{X}^t) \\ \times \begin{cases}\mathrm{so}_{\lambda^{\left(\frac{t-1}{2}\right)}}(X^t) & t \text{ odd,}\\ 1 & t \text{ even,}\end{cases}\end{aligned} \tag{2.8}$$

where

$$\epsilon_3 = -\sum_{i=\lfloor\frac{t}{2}\rfloor}^{t-1}\binom{n_i(\lambda)+1}{2} + \begin{cases} nr & t \text{ odd,}\\ 0 & t \text{ even,}\end{cases}$$

and $\mu_i^{(3)} = \lambda_1^{(t-1-i)} + \left(\lambda^{(i)},0,-\mathrm{rev}(\lambda^{(t-1-i)})\right)$ has $2n$ parts for $0\le i\le\lfloor\frac{t-2}{2}\rfloor$.

We give sketch of the proof of Theorem 5 in Section 4. The proofs of Theorems 8 and 9 follow similar ideas and are skipped. The details can be seen in [4].

Remark 10. It might seem that the results of Theorems 5, 8 and 9 are not well-defined because of Remark 2. More precisely, the lack of symmetry of the t-quotients on the right hand sides of these theorems might cause some worry. However, since changing $n\to n+1$ will change the length of the partition λ by tn, the order of the quotients remains unchanged.

Remark 11. In some cases, the Schur functions $s_{\mu_i^{(j)}}(X^t,\overline{X}^t)$ appearing on the right hand sides of Theorems 5, 8 and 9 for $j\in[3]$ respectively factorize further into characters of other classical groups, but we do not understand this behavior fully. Whenever μ_i can be written as $\rho_1+(\rho,-\mathrm{rev}(\rho))$ or $\rho_1+(1+\rho,-\mathrm{rev}(\rho))$ for a partition ρ of length at most n, such a factorization occurs by the results in [1]. In that case s_{μ_i} is either a product of two odd orthogonal characters or an even orthogonal and a symplectic character.

It is natural to ask if there are infinitely many symplectic, orthogonal and self-conjugate t-cores. As we have seen, there are no symplectic or orthogonal 2-cores and all 2-cores are self-conjugate. For $t\ge 3$, it has been proved [8] that there are infinitely many self-conjugate t-cores. Our last result gives a generalisation.

Theorem 12. *There are infinitely many symplectic and orthogonal t-cores for $t\ge 3$.*

We give sketch of the proof of Theorem 12 in Section 5.

3 Background results

We now state the main results we will need for proving our theorems. See [4] for detailed proofs. For a partition λ of length at most m, we see that

$$n_i(\lambda, m) = n_i(\mathrm{core}_t(\lambda), m), \quad 0 \le i \le t-1. \tag{3.1}$$

We now use the beta set of partitions to classify z-asymmetric partitions; see Definition 3. Let $\mathcal{P}_z$ be the set of z-asymmetric partitions and $\mathcal{P}_{z,t}$ be the set of z-asymmetric t-cores.

Lemma 13. *Let $\lambda = (\alpha|\beta)$ be a partition of length at most m and rank r. Then the following statements are equivalent.*

1. $\lambda \in \mathcal{P}_z$.
2. *an integer ξ between 0 and $m-z-1$ occurs in $\beta(\lambda)$ if and only if $2m-z-1-\xi$ does not.*
3. *$\beta(\lambda)$ is obtained from the sequence $(\alpha_1+m,\dots,\alpha_r+m, m-1,\dots,1,0)$ by deleting the numbers $m-z-1-\alpha_r > m-z-1-\alpha_{r-1} > \cdots > m-z-1-\alpha_1$ lying between 0 and $m-1$.*

Lemma 14. *For $|z| \ge t-1$, the empty partition is the only t-core in $\mathcal{P}_{z,t}$.*

Now we state the constraints satisfied by $n_i(\lambda), 0 \le i \le t-1$, for a z-asymmetric t-core λ of length at most tn.

Lemma 15. *Let λ be a t-core of length at most tn and $0 \le z \le t-2$. Then $\lambda \in \mathcal{P}_{z,t}$ if and only if*

$$\begin{aligned} n_i(\lambda) + n_{t-z-1-i}(\lambda) =& 2n \quad \textit{for} \quad 0 \le i \le t-z-1, \\ \textit{and} \quad n_i(\lambda) =& n, \quad t-z \le i \le t-1. \end{aligned} \tag{3.2}$$

Since $\mathrm{core}_t(\lambda)' = \mathrm{core}_t(\lambda')$ [13, Example I.1 8(e)], we have the following corollary.

Corollary 16. *Let λ be a t-core of length at most tn and $2-t \le z \le -1$. Then $\lambda \in \mathcal{P}_{z,t}$ if and only if*

$$\begin{aligned} n_i(\lambda) + n_{t-z-1-i}(\lambda) =& 2n \quad \textit{for} \quad -z \le i \le t-1, \\ \textit{and} \quad n_i(\lambda) =& n, \quad 0 \le i \le -z-1. \end{aligned} \tag{3.3}$$

We now see how to compute the rank of a t-core from its beta-set.

Lemma 17. *Let λ be a partition of length at most tn. For $z \in \{-1,0,1\}$, if $\mathrm{core}_t(\lambda)$ is a z-asymmetric t-core, then*

$$\mathrm{rk}(\mathrm{core}_t(\lambda)) = \sum_{i=0}^{\left\lfloor \frac{t-z-2}{2} \right\rfloor} |n_i(\lambda) - n|.$$

We now state the determinant identities for block matrices, which we need to prove our character identities. We note that we have not found our identities in Krattenthaler's treatises [11, 12]. The first is an elementary computation.

Lemma 18. *For $i = 1, \dots, k$, let T_i be matrices of order $\ell_i \times m_i$ such that $\ell_1 + \cdots + \ell_k = m_1 + \cdots + m_k = d$. Define block-diagonal and block-antidiagonal matrices*

$$U := \begin{pmatrix} T_1 & & & \\ & T_2 & & \mathit{0} \\ & & \ddots & \\ \mathit{0} & & & T_k \end{pmatrix} \quad \text{and} \quad V := \begin{pmatrix} & & & T_1 \\ \mathit{0} & & T_2 & \\ & \iddots & & \\ T_k & & & \mathit{0} \end{pmatrix}.$$

Then

$$\det(U) = (-1)^{\sum_{1 \le i < j \le k} m_i m_j} \det(V) = \begin{cases} 0 & \text{if } \ell_i \neq m_i \text{ for some } i, \\ \prod_{i=1}^{k} \det(T_i) & \text{otherwise.} \end{cases}$$

Lemma 18 is used in the proof of the following result, which is the common ingredient in the proofs of Theorems 5, 8 and 9.

Lemma 19. *Suppose $u_1, \dots, u_k$ are positive integers summing up to kn. Further, let $(\gamma_{i,j})_{1 \le i \le k, 1 \le j \le k+1}$ be a matrix of parameters such that $\gamma_{i,k+1} = \gamma_{i,k}$, $1 \le i \le k$ and Γ be the square matrix consisting of its first k columns. Let U_j and V_j be matrices of order $n \times u_j$ for $j \in [k]$. Finally, define a $kn \times kn$ matrix with $k \times k$ blocks as*

$$\Pi := \left(\left(\gamma_{i,2j-1} U_j - \gamma_{i,2j} V_j\right)_{\substack{1 \le i \le k \\ 1 \le j \le \left\lfloor \frac{k+1}{2} \right\rfloor}} \middle| \left(\gamma_{i,2k+2-2j} U_j - \gamma_{i,2k+1-2j} V_j\right)_{\substack{1 \le i \le k \\ \left\lfloor \frac{k+3}{2} \right\rfloor \le j \le k}} \right).$$

1. *If $u_p + u_{k+1-p} \neq 2n$ for some $p \in [k]$, then $\det \Pi = 0$.*
2. *If $u_p + u_{k+1-p} = 2n$ for all $p \in [k]$, then*

$$\det \Pi = (-1)^{\Sigma} (\det \Gamma)^n \prod_{i=1}^{\left\lfloor \frac{k+1}{2} \right\rfloor} \det W_i, \tag{3.4}$$

where

$$W_i = \begin{cases} \left(\begin{array}{c|c} U_i & -V_{k+1-i} \\ \hline -V_i & U_{k+1-i} \end{array} \right) & 1 \le i \le \left\lfloor \frac{k}{2} \right\rfloor, \\ \left(U_{\frac{k+1}{2}} - V_{\frac{k+1}{2}} \right) & k \text{ odd and } i = \frac{k+1}{2}, \end{cases}$$

and

$$\Sigma = \sum_{i=1}^{\left\lfloor \frac{k}{2} \right\rfloor} (n + u_i) + \begin{cases} 0 & k \text{ even,} \\ n \sum_{i=1}^{\frac{k-1}{2}} u_i & k \text{ odd.} \end{cases}$$

4 Sketch of proof of Theorem 5

Using the formula in (2.3), we consider the symplectic polynomial $\mathrm{sp}_\lambda(X, \omega X, \dots, \omega^{t-1} X)$. We first compute the factorization for the numerator. Permuting the columns of the determinant in the numerator by σ_λ from (2.5) and applying certain blockwise row operations, the numerator becomes

$$\mathrm{sgn}(\sigma_\lambda) \det \left(\begin{array}{c|c} 0 & t\left(A^\lambda_{t-1,1} - \overline{A}^\lambda_{t-1,1}\right) \\ \hline \left(\omega^{p(q+1)} A^\lambda_{q,1} - \overline{\omega}^{p(q+1)} \overline{A}^\lambda_{q,1}\right)_{\substack{1 \le p \le t-1 \\ 0 \le q \le t-2}} & 0 \end{array} \right), \tag{4.1}$$

where $A^\lambda_{q,1} = \left(x_i^{\beta_j^{(q)}(\lambda)+1} \right)_{\substack{1 \le i \le n \\ 1 \le j \le n_q(\lambda)}}$ and $\overline{A}^\lambda_{q,1} = \left(\overline{x}_i^{\beta_j^{(q)}(\lambda)+1} \right)_{\substack{1 \le i \le n \\ 1 \le j \le n_q(\lambda)}}$.

If $\mathrm{core}_t(\lambda)$ is not a symplectic t-core, then by Lemma 15 for $z = 1$ and (3.1), either $n_{t-1}(\lambda) \neq n$ or $n_i(\lambda) + n_{t-2-i}(\lambda) \neq 2n$ for some $i \in \left\{0, 1, \dots, \left\lfloor \frac{t-2}{2} \right\rfloor \right\}$. In the first case, using Lemma 18, the determinant in (4.1) is 0. In the second case the determinant is 0 by Lemma 19 and therefore, in both cases,

$$\mathrm{sp}_\lambda(X, \omega X, \dots, \omega^{t-1} X) = 0.$$

If $\mathrm{core}_t(\lambda)$ is a symplectic t-core, then by Lemma 15 for $z = 1$ and (3.1), $n_{t-1}(\lambda) = n$ and $n_i(\lambda) + n_{t-2-i}(\lambda) = 2n$, $i \in \left\{0, 1, \dots, \left\lfloor \frac{t-2}{2} \right\rfloor \right\}$. Using Lemma 19, we get the factorization for the determinant in the numerator. Evaluating at the empty partition, we get the factorization for the determinant in the denominator. The symplectic character is thus given by

$$(-1)^{\epsilon_1} \mathrm{sgn}(\sigma_\lambda) \mathrm{sp}_{\lambda^{(t-1)}}(X^t) \prod_{i=0}^{\left\lfloor \frac{t-3}{2} \right\rfloor} s_{\mu_i^{(1)}}(X^t, \overline{X}^t) \times \begin{cases} \mathrm{so}_{\lambda^{\left(\frac{t}{2}-1\right)}}(X^t) & t \text{ even}, \\ 1 & t \text{ odd}. \end{cases}$$

Here ϵ_1 has the same parity as

$$-\sum_{i=\left\lfloor \frac{t}{2} \right\rfloor}^{t-2} \binom{n_i(\lambda)+1}{2} + \begin{cases} \frac{n(n+1)}{2} + nr & t \text{ even}, \\ 0 & t \text{ odd}, \end{cases}$$

where we have used Lemma 17 with $z = 1$ for the rank r. This completes the proof.

5 Generating functions

We now state the enumerative results for z-asymmetric partitions defined in Definition 3. See [4] for the proofs. We first recall that the limiting infinite product is given by

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1 - aq^j). \tag{5.1}$$

Proposition 20. *Let $z \geq 0$. The number of z-asymmetric partitions of m is equal to the number of partitions of m with distinct parts of the form $2k + 1 + z$, $k \geq 0$.*

As a corollary, we have an expression of the generating function for z-asymmetric partitions. Since the number of z-asymmetric partitions of m is equal to the number of $-z$-asymmetric partitions of m, we have the following corollary.

Corollary 21. *For $z \in \mathbb{Z}$, $\sum_{\lambda \in \mathcal{P}_z} q^{|\lambda|} = \prod_{k \geq 0}(1 + q^{|z|+1+2k}) = (-q^{|z|+1}; q^2)_\infty$.*

We now move on to enumerating z-asymmetric partitions which are also t-cores. Recall from Lemma 14 that there are no nontrivial partitions if $|z| > t - 2$.

Theorem 22. *Let $0 \leq z \leq t - 2$. Represent elements of $\mathbb{Z}^{\lfloor \frac{t-z}{2} \rfloor}$ by $\left(z_0, \ldots, z_{\lfloor \frac{t-z-2}{2} \rfloor}\right)$ and define $b \in \mathbb{Z}^{\lfloor \frac{t-z}{2} \rfloor}$ by $\vec{b}_i = t - z - 1 - 2i$. Then there exists a bijection $\phi \colon \mathcal{P}_{z,t} \to \mathbb{Z}^{\lfloor \frac{t-z}{2} \rfloor}$ satisfying $|\lambda| = t||\phi(\vec{\lambda})||^2 - \vec{b} \cdot \phi(\vec{\lambda})$, where $\cdot$ represents the standard inner product.*

Define the *Ramanujan theta function* [5, Equation (18.1)], $f(a, b) = \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}$.

Let $p_{z,t}(m)$ be the cardinality of partitions in $\mathcal{P}_{z,t}$ of size m.

Corollary 23. *For $0 \leq z \leq t - 2$, we have*

$$\sum_{m \geq 0} p_{z,t}(m) q^m = \prod_{i=0}^{\lfloor (t-z-2)/2 \rfloor} f(q^{2i+z+1}, q^{2t-2i-z-1}).$$

We remark that the special case of $z = 0$ (*i.e.* self-conjugate t-cores) in Corollary 23 was obtained by Garvan–Kim–Stanton [8, Equations (7.1a) and (7.1b)]. Thus, our result can be viewed as a generalization of theirs for symplectic t-cores, leading to an immediate proof of Theorem 12 for symplectic t-cores. Since the number of symplectic t-cores is same as the number of orthogonal t-cores, there are infinitely many orthogonal t-cores as well.

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